

Moral-Hazard Credit Cycles with Risk-Averse Agents by Roger Myerson

Abstract: We consider a simple overlapping-generations model with risk-averse financial agents subject to moral hazard. Efficient contracts for such financial intermediaries involve back-loaded late-career rewards. Compared to the analogous model with risk-neutral agents, risk aversion tends to reduce the growth of agents' responsibilities over their careers. This moderation of career growth rates can reduce the amplitude of the widest credit cycles, but it also can cause small deviations from steady state to amplify over time in rational-expectations equilibria. We find equilibria in which fluctuations increase until the economy enters a boom/bust cycle where no financial agents are hired in booms.

These notes are available at

<http://home.uchicago.edu/~rmyerson/research/rabankers.pdf>

A spreadsheet for all computations in this model is available at

<http://home.uchicago.edu/~rmyerson/research/rabankers.xls>

See also the related paper "A model of moral-hazard credit cycles" at

<http://home.uchicago.edu/~rmyerson/research/bankers.pdf>

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The model.

Investments are managed each period by agents subject to moral hazard.

Agents live 3 periods (1=young, 2=old, 3=retired), manage investments in 1,2.

Consuming $c_t \geq 0$ in period t yields current utility $u_t = (c_t)^{0.5}$.

Utilities over a lifetime are summed with discount factor ρ : $U = u_1 + \rho u_2 + \rho^2 u_3$.

The agent is hired by risk-neutral investors with the same discount factor ρ .

Given investment h_t to manage in period $t \in \{1,2\}$, the agent can manage well or can mismanage it and divert γh_t into her current consumption, getting $(c_t + \gamma h_t)$.

The investment will either succeed or fail. The probability of the investment succeeding is α if the agent manages it well, but is β if the agent mismanages it.

The return next period is $h_t(1+r_t)/(\rho\alpha)$ if the investment succeeds, 0 if it fails.

So the expected t -discounted return from an investment h_t is $h_t(1+r_t)$ if it is well managed, but is $(\beta/\alpha)h_t(1+r_t)$ if it is mismanaged.

The expected return rate r_t is related to aggregate investment I_t in period t by a given decreasing investment-demand function $I_t = I(r_t) \geq 0$.

We assume that $\alpha > \beta$ and $\gamma < (\alpha - \beta) / \{ \alpha + 0.5\beta[1 + 4\alpha / (\rho(\alpha - \beta)^2)]^{0.5} - 0.5\beta \}$.

These assumptions will imply that $\gamma + (\beta/\alpha)(1+r_t) < 1$ for any r_t in equilibrium, and so mismanaged investments will never be worthwhile.

Example: $\rho=0.5$, $\alpha = 0.7$, $\beta = 0.4$, $\gamma = 0.1$, $I(r_t) = \max\{0.5 - r_t, 0\}$.

At any time, investors can hire young agents to manage investments over the next two periods, with two-period contracts that are designed optimally for investors.

The optimal contract can be recursively characterized as follows:

The agent will be promised some expected utility in the second-period problem that will depend on whether her first-period investment succeeds or fails.

Let v be the current-consumption certainty equivalent of the promised utility $v^{0.5}$.

In each of the agent's work periods $t \in \{1,2\}$, c_t is the agent's consumption, h_t is the investment managed by the agent, and the agent's promised future next-period certainty equivalent will be g_t if the project succeeds but b_t if it fails.

Given v , the old-agent incentive problem is to choose $(c_2, h_2, g_2, b_2) \geq 0$ to

$$\begin{aligned} & \text{minimize } c_2 + \rho(\alpha g_2 + (1-\alpha)b_2) - r_2 h_2 \\ & \text{subject to } c_2^{0.5} + \rho(\alpha g_2^{0.5} + (1-\alpha)b_2^{0.5}) \geq v^{0.5}, \\ & c_2^{0.5} + \rho(\alpha g_2^{0.5} + (1-\alpha)b_2^{0.5}) \geq (c_2 + \gamma h_2)^{0.5} + \rho(\beta g_2^{0.5} + (1-\beta)b_2^{0.5}). \end{aligned}$$

The period-2 promise v will be $v=g_1$ or $v=b_1$, depending on whether the period-1 investment succeeds or fails.

The optimal solution of the old-agent incentive problem is homogeneous in the promised certainty-equivalent payoff v , and so its optimal value can be written $vN(r_2)$, for some net promise-cost rate $N(r_2)$.

That is, $N(r_2)$ is the optimal value of the old-agent problem for $v=1$:

$$\begin{aligned} N(r_2) = \text{minimum}_{(c_2, h_2, g_2, b_2) \geq 0} & c_2 + \rho(\alpha g_2 + (1-\alpha)b_2) - r_2 h_2 \\ \text{subject to} & c_2^{0.5} + \rho(\alpha g_2^{0.5} + (1-\alpha)b_2^{0.5}) \geq 1, \\ & c_2^{0.5} + \rho(\alpha g_2^{0.5} + (1-\alpha)b_2^{0.5}) \geq (c_2 + \gamma h_2)^{0.5} + \rho(\beta g_2^{0.5} + (1-\beta)b_2^{0.5}). \end{aligned}$$

Notice that $N(r_2)$ is a decreasing function of r_2 .

Let $H_2(r_2) = h_2$ an optimal solution (c_2, h_2, g_2, b_2) to this problem.

That is, $H_2(r_2)$ is the amount of investment that an agent would be asked to manage in her second period, per unit of certainty equivalent payoff value that the agent has been promised, when r_2 is the rate of return on investments then.

By the envelope theorem, $N'(r_2) = -H(r_2)$.

The first period has no prior promised payoff.

So the employment terms (c_1, h_1, g_1, b_1) that are offered to a young agent in her first period must satisfy only nonnegativity and the incentive constraint

$$c_1^{0.5} + \rho(\alpha g_1^{0.5} + (1-\alpha)b_1^{0.5}) \geq (c_1 + \gamma h_1)^{0.5} + \rho(\beta g_1^{0.5} + (1-\beta)b_1^{0.5}).$$

Given that future promises have net cost $N(r_2)$ next period, per unit payoff promised, the investors' expected profit is $r_1 h_1 - c_1 - \rho N(r_2) (\alpha g_1 + (1-\alpha)b_1)$.

If investor could earn positive profits from a feasible contract this period, they would want to increase investment, driving rates of return down until investments in young agents earn zero expected profits.

Let $Y(r_2)$ be the maximal r_1 for which the young-agent optimal value is 0, that is the least cost of motivating a young agent to manage one unit investment $h_1=1$:

$$Y(r_2) = \text{minimum}_{(c_1, g_1, b_1) \geq 0} c_1 + \rho N(r_2) (\alpha g_1 + (1-\alpha)b_1) \\ \text{subject to } c_1^{0.5} + \rho(\alpha g_1^{0.5} + (1-\alpha)b_1^{0.5}) \geq (c_1 + \gamma)^{0.5} + \rho(\beta g_1^{0.5} + (1-\beta)b_1^{0.5}).$$

Notice that $Y(r_2)$ is a decreasing function of r_2 , because $N(r_2)$ is decreasing.

Let $G(r_2) = (\alpha g_1 + (1-\alpha)b_1)H(r_2)$ for an optimal solution (c_1, g_1, b_1) of this problem. So $G(r_2)$ is the expected investment managed by the agent next period per unit investment managed this period, given the return rate r_2 expected next period.

By the envelope theorem, $Y'(r_2) = \rho(\alpha g_1 + (1-\alpha)b_1)N'(r_2) = -\rho G(r_2)$.

Now let us consider these problems where agents can be hired in any period t .

Let J_t denote the total investments managed by young agents in any period t .

If returns r_t in a period t were greater than $Y(r_{t+1})$, then investors could expect positive surplus by hiring young agents to invest more, thus decreasing r_t .

So in equilibrium, in any period t , we must have $r_t \leq Y(r_{t+1})$,

and if $J_t > 0$ then we must have $r_t = Y(r_{t+1})$.

The aggregate investment responsibilities of young agents from period t are expected to grow to $J_t G(r_{t+1})$ in period $t+1$ when they are old agents.

Initial contracts may promise some payoffs V_0 to old agents in period 1.

Given this initial condition, an equilibrium is a sequence of return rates r_t and young investment responsibilities J_t for each period $t \in \{1, 2, 3, \dots\}$ such that

$$I(r_1) = V_0 H(r_1) + J_1, \quad I(r_{t+1}) = J_t G(r_{t+1}) + J_{t+1},$$

$$J_t \geq 0, \quad r_t \leq Y(r_{t+1}), \quad \text{and} \quad [r_t - Y(r_{t+1})]J_t = 0, \quad \forall t \in \{1, 2, 3, \dots\}.$$

General results

Lemma: $I(r)$, $N(r)$ and $Y(r)$ are decreasing in r ; $H(r)$ and $G(r)$ are increasing in r . So $I(Y(r))-I(r)/G(r)$ is an increasing function of r .

Fact: There exists a unique $r^* \in [0, \theta]$ such that $r^* = Y(r^*)$.

A steady state equilibrium in which $r_t = r^*$ every period exists from the initial condition $V_0^* = I(r^*)/[H(r^*)(1+1/G(r^*))]$.

Assumptions: Suppose that $I(r^*) > 0$ and $1 < G(r^*) < 1/\rho$.

$G(r^*) > 1$ and $Y(r^*) = r^*$ imply that $I(Y(r^*)) - I(r^*)/G(r^*) > 0$.

But $G(0) = 0$, so $I(Y(r)) - I(r)/G(r) < 0$ for small r near 0. Thus we get:

Fact: There exist (q_1, q_2) such that $q_2 < r^* < q_1 = Y(q_2)$ and $I(q_1) - I(q_2)/G(q_2) = 0$.

There exists an cyclical equilibrium in which return rates r_t alternate between q_1 and q_2 , and new young agents hired only in periods when the rate of return is q_1 .

Each new cohort of agents manages aggregate investments $J_1 = I(q_1)$ when young and manages $G(q_2)J_1 = I(q_2)$ when old. We have $G(q_2) > 1$ and $q_2 < Y(q_1)$.

With $G(r^*) < 1/\rho$, $-Y'(r^*) = \rho G(r^*) < 1$, and so an equilibrium path near the steady state must diverge away from the steady state in a cobweb pattern.

In our example with $\alpha=0.7$, $\beta=0.4$, $\gamma=0.1$, $\rho=0.5$, these assumptions are satisfied, and $1 < G(r) < 1/\rho$ for all r between q_1 and q_2 . So any equilibrium path with any initial condition V_0 other than V_0^* will converge to the (q_1, q_2) cycle.

Example: $\rho=0.5$, $\alpha = 0.7$, $\beta = 0.4$, $\gamma = 0.1$, $I(r_t) = \max\{0.5-r_t, 0\}$.

In the steady state, investments return $r^* = 0.191$. A young agent who manages $h_1=1$ when young would get $c_1=0.076$ and be promised $g_1 = 0.922$ if successful. After young success, the old agent (with $v_2=g_1$) would manage $h_2=2.019$, get $c_2=0.385$ and be promised $g_2=0.941$ in retirement if successful again; but $b_2=0$. The expected growth of responsibilities from young to old in any cohort would then be $G(r^*) = \alpha h_2 = 1.41$ in steady state.

In aggregate investment units, young agents manage $J_t = I(r^*)/(1+G(r^*)) = 0.128$ and old agents manage $G(r^*)J_t = 0.181$ each period in the steady state.

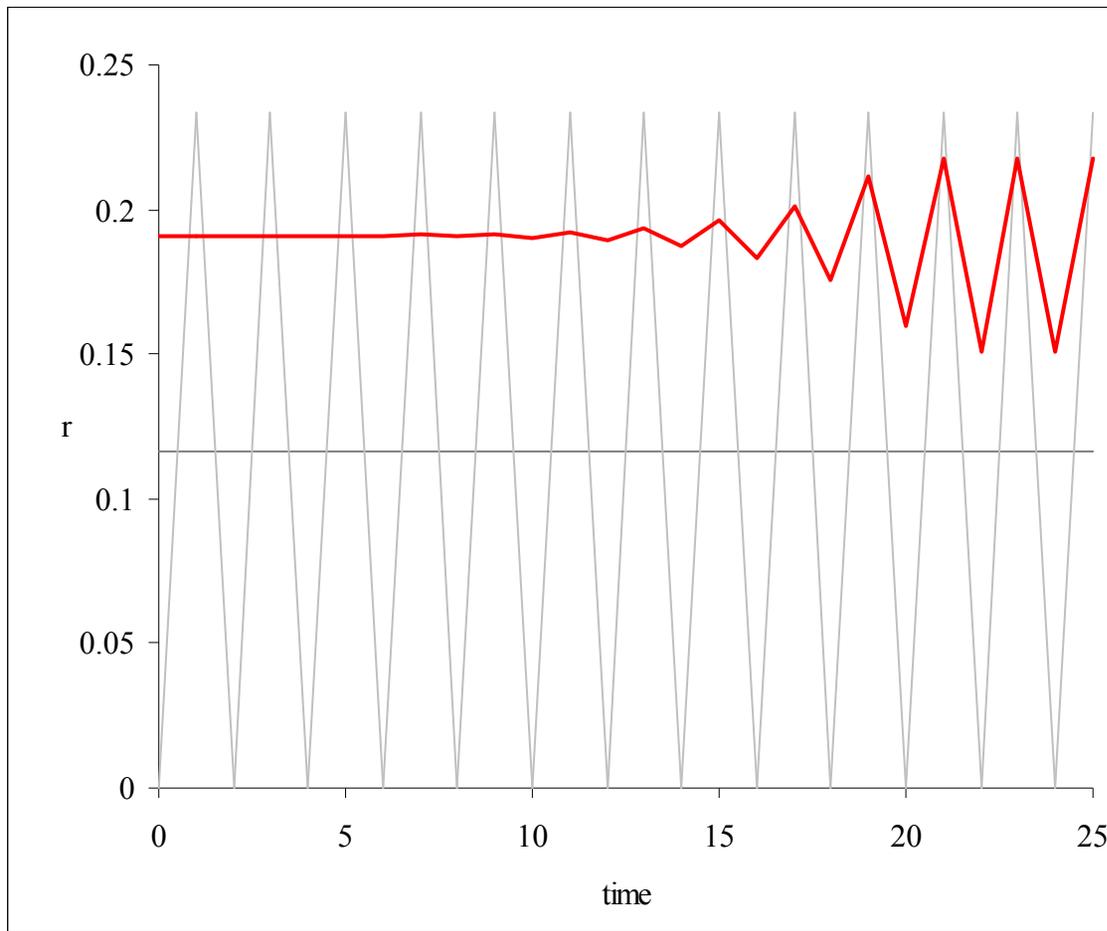
In the cyclical solution, recession periods t when $r_t = q_1 = 0.217$ alternative with boom periods $t+1$ when $r_{t+1} = q_2 = 0.151$.

All investments in booms are managed by old agents, who crowd out new hiring until the subsequent recession, when a new cohort of young agents takes over.

In aggregate units, we must have $J_t = I(q_1) = 0.282$ managed by young agents in the recession, $G(q_2)J_t = I(q_2) = 0.349$ managed by old agents in the boom.

A young agent who manages $h_1=1$ would get $c_1=0.088$, $g_1=0.831$ and, if successful when young, then $v_2=g_1$, $h_2=1.767$, $c_2=0.364$, $g_2 = 0.774$, and $b_2=0$. The expected growth of agents' responsibilities from young to old in any cohort would then be $G(q_2) = \alpha h_2 = 1.24 = I(q_2)/I(q_1)$.

With $\rho=0.5$, $\alpha = 0.7$, $\beta = 0.4$, $\gamma = 0.1$, $I(r_t) = \max\{0.5-r_t, 0\}$:



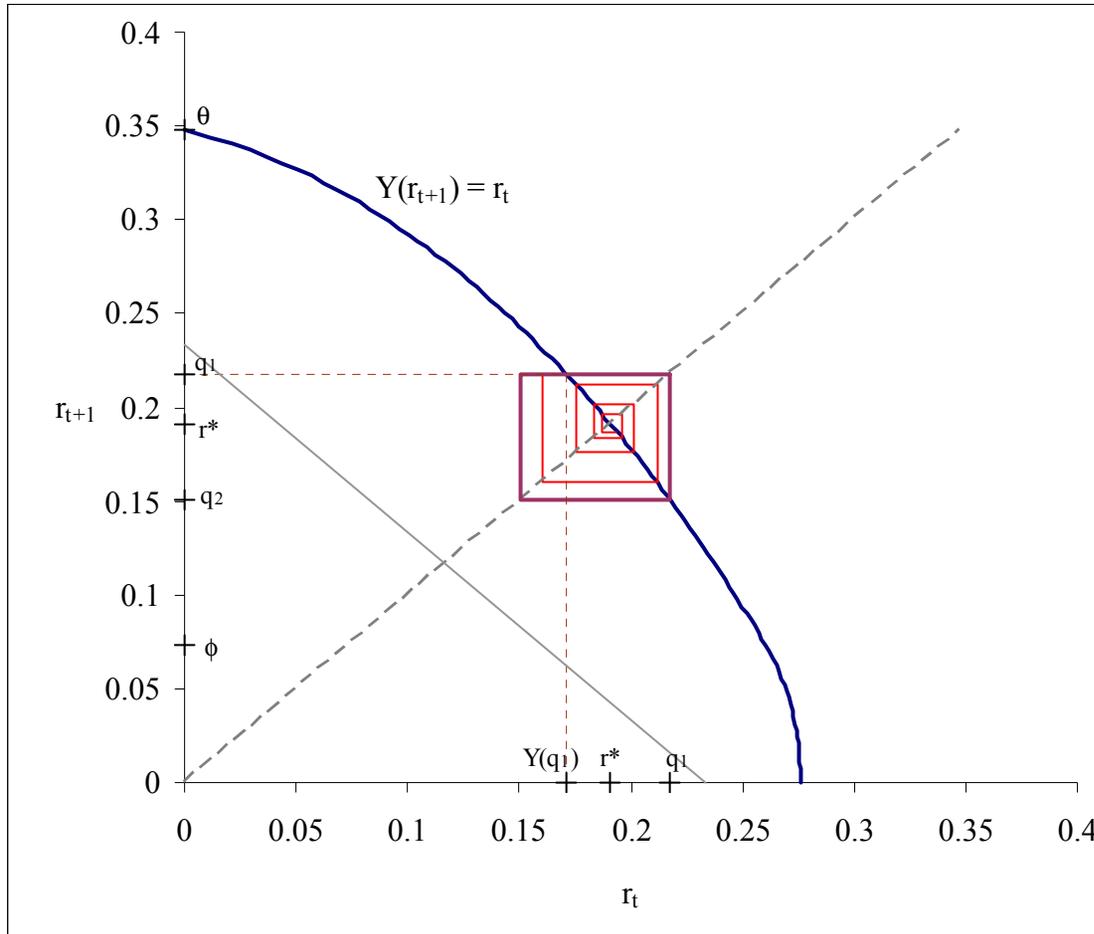
An equilibrium path for this model, with risk averse agents, is shown in red. From a V_0 at period 1 only 0.85% below steady state, the extreme cycle is reached in period 21.

The steady state and widest cycle for a model with risk-neutral agents (else same), are shown in gray. Risk neutrality can yield more extreme cycles, but its steady state is locally stable.

In the risk-neutral model, agents' rewards are fully backloaded, and so the growth of expected responsibilities is always $G=1/\rho$. Thus $Y'(r_{t+1}) = -1$ always, and so the steady state and the most

extreme cycle are both locally stable. See my companion paper: "A model of moral-hazard credit cycles." <http://home.uchicago.edu/~rmyerson/research/bankers.pdf>

With $\rho=0.5$, $\alpha = 0.7$, $\beta = 0.4$, $\gamma = 0.1$, $I(r_t) = \max\{0.5-r_t, 0\}$:



The red equilibrium is shown in (r_t, r_{t+1}) space as a cobweb cycle.

The blue curve shows $r_t = Y(r_{t+1})$.

Steady state r^* is where it meets the gray dashed line.

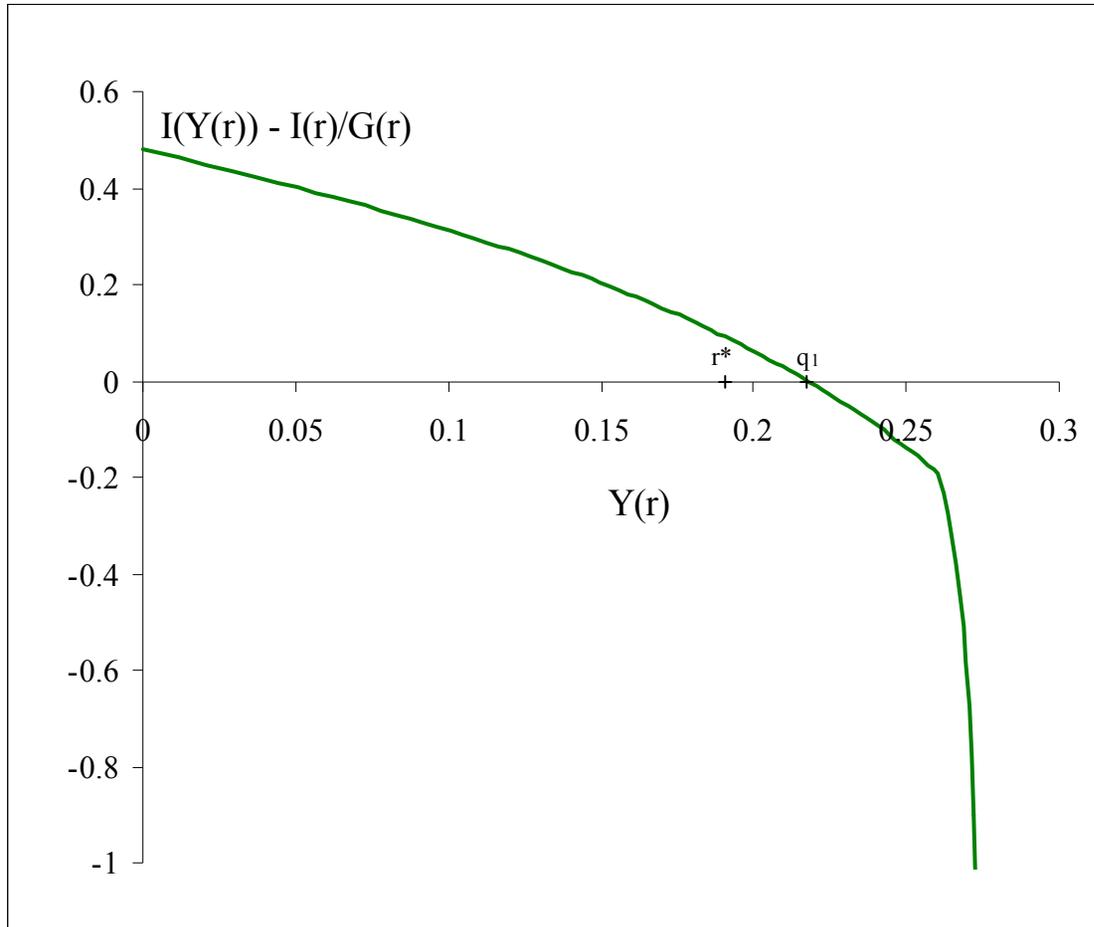
The limit cycle (q_1, q_2) is shown in purple.

The brown dotted line meets the horizontal axis at $Y(q_1)$. Any $r \leq Y(q_1)$ can be followed by the limit cycle, starting in q_1 .

Compared to risk neutrality ($Y =$ gray line of slope -1),

risk aversion tends to decrease $G(r_{t+1})$ so that $-Y'(r_{t+1}) = \rho G(r_{t+1}) < 1$ for all r_{t+1} between q_1 and q_2 here (which corresponds to a slope steeper than -1 here, with r_{t+1} on the vertical axis).

With $\rho=0.5$, $\alpha = 0.7$, $\beta = 0.4$, $\gamma = 0.1$, $I(r_t) = \max\{0.5-r_t, 0\}$:



The green curve graphs $I(Y(r)) - I(r)/G(r)$ on the vertical axis against $Y(r)$ on the horizontal axis. So for each current rate of return on the horizontal axis, the vertical quantity denotes the amount of investment that current old agents must manage if the current young cohort is to manage all investments next period, with no new young agents next period. This quantity is 0 at q_1 .

Given the other parameters, we get $G(r^*) > 1$ and $r^* < q_1$ when $\rho < 0.66$.

Detailed analysis of old-agent incentive problem

$$\begin{aligned} N(r) = \text{minimum}_{(c,h,g,b) \geq 0} & \quad c + \rho(\alpha g + (1-\alpha)b) - rh \\ \text{subject to} & \quad c^{0.5} + \rho(\alpha g^{0.5} + (1-\alpha)b^{0.5}) \geq 1, \\ & \quad c^{0.5} + \rho(\alpha g^{0.5} + (1-\alpha)b^{0.5}) \geq (c + \gamma h)^{0.5} + \rho(\beta g^{0.5} + (1-\beta)b^{0.5}). \end{aligned}$$

$H(r) = h$ at an optimal solution.

$$\text{Let } B = \rho b^{0.5}, \quad D = [(c + \gamma h)^{0.5} - c^{0.5}] / (\alpha - \beta).$$

Then with the participation and incentive constraints binding, we have
 $b = B^2 / \rho^2$, $g = (B + D)^2 / \rho^2$, $c = (1 - B - \alpha D)^2$, $h = [2D - 2BD - (\alpha + \beta)D^2] / (\alpha - \beta) / \gamma$.

Then the cost $N(r)$ can be rewritten as a quadratic form in terms of B and D :

$$\begin{aligned} N = n(B, D) = & \quad 1 - 2B - 2[\alpha + r(\alpha - \beta) / \gamma]D + [1 + 1 / \rho]B^2 + \\ & \quad + 2[\alpha / \rho + \alpha + r(\alpha - \beta) / \gamma]BD + [\alpha / \rho + \alpha^2 + r(\alpha^2 - \beta^2) / \gamma]D^2. \end{aligned}$$

With $r > 0$, the only nonnegativity constraint that can bind at a solution is $b \geq 0$.

This binds in *case 1*, where the solution is determined by $B = 0$ and $\partial n / \partial D = 0$:

$$B = 0, \quad D = [\alpha + r(\alpha - \beta) / \gamma] / [\alpha / \rho + \alpha^2 + r(\alpha^2 - \beta^2) / \gamma],$$

$$N(r) = \{ \alpha / \rho - r(\alpha - \beta)^2 / \gamma - [r(\alpha - \beta) / \gamma]^2 \} / [\alpha / \rho + \alpha^2 + r(\alpha^2 - \beta^2) / \gamma],$$

$$\begin{aligned} H(r) = & \quad [(\alpha - \beta) / \gamma] [\alpha + r(\alpha - \beta) / \gamma] [2\alpha / \rho + \alpha(\alpha - \beta) + r(\alpha^2 - \beta^2) / \gamma] / \\ & \quad [\alpha / \rho + \alpha^2 + r(\alpha^2 - \beta^2) / \gamma]^2. \end{aligned}$$

Case 1 applies when this solution satisfies $\partial n / \partial B \geq 0$, which holds when $r \geq \varphi$, where $\varphi = [0.5\gamma / (\alpha - \beta)] \{[(\alpha / \rho + \alpha - \beta)^2 + 4\alpha(1 - \alpha) / \rho]^{0.5} - (\alpha / \rho + \alpha - \beta)\}$. (Here $\partial n / \partial B \geq 0$ holds when $[r(\alpha - \beta) / \gamma]^2 + (\alpha / \rho + \alpha - \beta)[r(\alpha - \beta) / \gamma] - \alpha(1 - \alpha) / \rho \geq 0$, which holds when $r \geq \varphi$.)

Otherwise, when $r < \varphi$, we have *case 2*, where $b > 0$.

In case 2, the solution, determined by the conditions $\partial n / \partial D = 0$ and $\partial n / \partial B = 0$, is:

$$B = 1 - [\alpha(1 - \alpha) / \rho + \beta r(\alpha - \beta) / \gamma] / \{ \alpha(1 - \alpha)(1 + 1 / \rho) - (\rho + 1)r(\alpha - \beta)^2 / \gamma - \rho[r(\alpha - \beta) / \gamma]^2 \},$$

$$D = [r(\alpha - \beta) / \gamma] / \{ \alpha(1 - \alpha)(1 + 1 / \rho) - (\rho + 1)r(\alpha - \beta)^2 / \gamma - \rho[r(\alpha - \beta) / \gamma]^2 \},$$

$$N(r) = \{ \alpha(1 - \alpha) / \rho - r(\alpha - \beta)^2 / \gamma - [r(\alpha - \beta) / \gamma]^2 \} / \{ \alpha(1 - \alpha)(1 + 1 / \rho) - (\rho + 1)r(\alpha - \beta)^2 / \gamma - \rho[r(\alpha - \beta) / \gamma]^2 \},$$

$$H(r) = [r(\alpha - \beta)^2 / \gamma^2][2\alpha(1 - \alpha) / \rho - r(\alpha - \beta)^2 / \gamma] / \{ \alpha(1 - \alpha)(1 + 1 / \rho) - (\rho + 1)r(\alpha - \beta)^2 / \gamma - \rho[r(\alpha - \beta) / \gamma]^2 \}^2.$$

Let θ denote the supremum of all r such that $N(r) > 0$. This can be determined from the equation $N(\theta) = 0$ using the formula for case 1, which yields

$$\theta = 0.5\gamma \{ [1 + 4\alpha / (\rho(\alpha - \beta)^2)]^{0.5} - 1 \}. \text{ (This formula makes } \theta > \varphi \text{ for case 1.)}$$

If $\gamma < (\alpha - \beta) / \{ \alpha + 0.5\beta[1 + 4\alpha / (\rho(\alpha - \beta)^2)]^{0.5} - 0.5\beta \}$ then $\gamma + (\beta / \alpha)(1 + \theta) < 1$ and so mismanagement is never worthwhile with any equilibrium $r_t \leq \theta$.

Detailed analysis of young-agent incentive problem

$$Y(r) = \text{minimum}_{(c,g,b) \geq 0} c + \rho N(r) (\alpha g + (1-\alpha)b)$$

$$\text{subject to } c^{0.5} + \rho(\alpha g^{0.5} + (1-\alpha)b^{0.5}) \geq (c+\gamma)^{0.5} + \rho(\beta g^{0.5} + (1-\beta)b^{0.5}).$$

An optimal solution must have $b=0$.

So the incentive constraint is binding when

$$g^{0.5} = [(c+\gamma)^{0.5} - c^{0.5}] / [\rho(\alpha - \beta)], \text{ and } g = c[(1+\gamma/c)^{0.5} - 1]^2 / [\rho(\alpha - \beta)]^2.$$

Let $x = (1+\gamma/c)^{0.5} \geq 1$.

$$\text{So } c = \gamma/(x^2-1), \quad g = \gamma(x-1)^2 / [(x^2-1)\rho^2(\alpha-\beta)^2].$$

Then the cost $Y(r)$ can be rewritten as a function of x

$$Y = y(x) = \gamma \{ 1 + (x-1)^2 \alpha N(r) / [\rho(\alpha - \beta)^2] \} / (x^2 - 1).$$

To minimize this over $x \geq 1$, the first-order condition $dy/dx=0$ yields

$$0 = x^2 - [2 + \rho(\alpha - \beta)^2 / (\alpha N(r))]x + 1.$$

The root with $x \geq 1$ is

$$x = [1 + 0.5\rho(\alpha - \beta)^2 / (\alpha N(r))] + \{ [1 + 0.5\rho(\alpha - \beta)^2 / (\alpha N(r))]^2 - 1 \}^{0.5}.$$

The optimal c and g can be computed from this x as above, and we get

$$Y(r) = c + N(r)\rho\alpha g = 0.5\gamma \{ [1 + 4\alpha N(r) / (\rho(\alpha - \beta)^2)]^{0.5} - 1 \}.$$

$$G(r) = H(r)\alpha g = H(r)\alpha\gamma / \{ \rho^2(\alpha - \beta)^2 [1 + 4\alpha N(r) / (\rho(\alpha - \beta)^2)]^{0.5} \}$$

$$= H(r)\alpha\gamma / \{ \rho^2(\alpha - \beta)^2 [2Y(r) / \gamma + 1] \}.$$

Appendix

Finally let us briefly consider the case when $G(r^*) < 1$.

(In our example with $\alpha=0.7$, $\beta=0.4$, and $\gamma=0.1$, we get $G(r^*) < 1$ when $\rho \geq 0.67$.)

In this case, an equilibrium from any initial condition always has $r_{t+1} = r^*$ with $J_{t+1} = I(r^*) - G(r^*)J_t$ for all $t \geq 1$, converging to $\lim_{t \rightarrow \infty} J_t = J^* = I(r^*) / [1 + G(r^*)]$.

If $V_0 \leq I(r^*) / H(r^*)$ then we start at $r_1 = r^*$ with $J_1 = I(r^*) - V_0 H(r^*)$;

otherwise we may start with $J_1 = 0$ at some $r_1 < r^*$ satisfying $I(r_1) / H(r_1) = V_0$.