Dynamic Pricing in a Frictional Product Market

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Abstract

Abstract I consider a product market characterized by matching frictions. Specifically, I assume that a buyer has to sample the variety of the good produced by a particular seller in order to find out whether he likes it or not. Also, I assume that buyers are anonymous in this market. Therefore, sellers cannot price discriminate between a first-time buyer and a repeated customer, even though the two have a different reservation price. Finally, sellers are subject to idiosyncratic shocks to their production cost. As a preliminary step, I study an environment where buyers directly observe production costs. First, I find that the optimal pricing schedule is decreasing over time. Intuitively, the seller prefers to offer a lower price tomorrow than today because tomorrow’s price directs the inflow of new customers both today and tomorrow. Secondly, I find that the optimal pricing schedule is not time-consistent. Intuitively, once prospective buyers are attracted by the expectation of low future prices, the seller would have an incentive to renege on its promise. Next, I consider the case where buyers cannot observe production costs. When shocks are i.i.d., I find that the optimal pricing schedule is such that the terms of trade are independent from the realization of the productivity shock. Otherwise, the seller would always report a negative shock in order to increase its price and renege on the promises made to its customers. When shocks are mildly persistent, I find that prices are sticky. Specifically, if a seller reports a bad productivity shock, the price increases slowly over time. Intuitively, price stickiness is the most efficient way to screen a seller that actually has a high cost of production from a seller that just says so. Finally, I find that—when shocks are sufficiently persistent—prices are fully flexible.

Keywords: Directed Search; Costumer Base; Time-Inconsistency; Sticky Prices

1 Introduction

TBA

2 The Environment

The market for good $x$ is populated by a continuum of buyers and a finite number of sellers. Buyers’ lifetime utility is defined by the mathematical expectation over the sum of periodical utilities discounted at rate $\beta$. In turn, the periodical utility function is defined over units of consumption of good $x$ and dollars $y$. Specifically, the periodical utility maps $\{0, 1\} \times \mathbb{R}$ into $\mathbb{R}$ and is given by

$$u(x, y) = y + xu.$$

(1)

Buyers are endowed with zero units of good $x$ and $y_e$ dollars. Since the utility function (1) is quasi-linear in dollars, $y_e$ can be normalized to zero without loss in generality.

Sellers operate the technology to produce good $x$. In particular, a seller can produce one unit of good $x$ at cost $c$. At any point in time, this seller can either be a low or a high cost firm, i.e. $c$ belongs to $\{c_l, c_h\}$ with $0 < c_l < c_h < u$. The seller’s production function is persistent over time. Specifically, the probability that the seller’s cost remains unchanged from one date to the next is equal to $\alpha$, where $\alpha$ is a number greater or equal than $\frac{1}{2}$. The seller maximizes the expected sum of profits discounted at rate $\beta$.

The market for good $x$ is characterized by search frictions. Specifically, when a buyer decides to visit a particular seller, he reaches the destination with probability $\lambda$ and gets lost on the way with probability $1 - \lambda$. If the buyer reaches the destination, he is offered the good and can choose whether to buy it or not. On the next day, the buyer is exogenously forced to leave the seller’s location with probability $\sigma$. With probability $1 - \sigma$, the buyer has the choice of remaining on the seller’s location or leaving and looking for a different trading partner.

Because of congestion effects, the probability $\lambda$ that a buyer finds his way to the seller’s location is a decreasing function of the measure $q$ of other buyers that are searching the very same seller on the very same date. Because of positive externalities
created by repeat costumers, the probability $\lambda$ that a buyer finds a seller is a decreasing function of the measure $n$ of buyers that have visited the firm in the previous period. Moreover, for any $n$, we assume that the probability $\lambda(q,n)$ converges to one as $q$ approaches zero.

In every period, events unfold in the following order. First, each seller announces its terms-of-trade. After having perfectly observed all of the posted prices, each buyer decides whether to spend the day in the market for good $x$ or in some other market where they expect to receive utility $k$, where $k$ belongs to the open interval $(0, u-c_h)$. In the marketplace for good $x$, unmatched buyers decide which seller to visit while matched buyers decide whether to remain on their current location or to search elsewhere. We assume that buyers are anonymous and therefore sellers cannot offer different terms-of-trade to first-time buyers and repeat costumers. Obviously, this restriction on the terms-of-trade is binding because prospective first-time buyers have a different expected utility from searching a seller than repeat costumers. This concludes the description of the physical environment.

### 3 Pricing with Observable Costs

In this section, we study the optimal pricing policy under the assumption that sellers can commit to a state-contingent sequence of prices and that buyers can directly observe the seller’s cost. For the sake of analytical tractability, from now on we are going to assume that the probability of a successful match $\lambda(q,n)$ is homogeneous of degree zero in the costumer base $n$—i.e. $\lambda(q,n) = \lambda(q/n,1)$.

Denote with $K$ the lifetime utility for a buyer that is not matched with any seller at the beginning of period $t$. Because more buyers can enter the marketplace for good $x$ at cost $k$, in equilibrium $K$ has to be equal to $(1-\beta)^{-1}k$. Denote with $W_t$ the lifetime utility of a buyer that ends up on a certain seller’s location at the end of period $t$. Let $n_t$ be the measure of such seller’s costumer base at the beginning of period $t$. Because search is directed, the equilibrium measure $q_t$ of buyers looking for the seller’s location is such that

$$\lambda(q_t/n_t,1)W_t + (1-\lambda(q_t/n_t,1))\beta K = K$$

(2)

if $W_t \geq K$, and $q_t$ is equal to zero if $W$ is smaller than $K$. More generally, equation (2) implicitly defines the measure of buyers looking for the seller as a function $q(W,n)$ of the expected value $W$ from reaching the destination and the seller’s costumer base $n$. From
the properties of the function $\lambda(q, n)$, it follows that $q(W, n)$ is increasing in $W$ and homogeneous of degree one in $n$. Finally, notice that—because of search frictions—not every single buyer looking for the seller becomes a first-time costumer. In particular, the inflow of first-time buyers is equal to $n_t \lambda(q(W_t, 1), 1) q(W_t, 1) < q(W_t, n_t)$.

Unlike a prospective buyer, a repeated costumer does not have to worry about finding his way to the seller’s location. Therefore, if $W_t$ is greater than $K$, all repeated costumers remain on the seller’s premises unless they are exogenously displaced. On the other hand, if $W_t$ is smaller than $K$, all repeated buyers leave the seller and bring their business elsewhere.

By combining the expression for the inflow of first time buyers and for the outflow of repeated costumers, we obtain a law of motion for the firm’s costumer base

$$n_{t+1} = n_t (1 - \sigma + \eta(W_t)), \quad (3)$$

where $\eta(W)$ is given by $\sigma - 1$ if $W < K$ and by $\lambda(q(W, 1), 1) q(W, 1)$ otherwise. In the remainder of the paper, we are going to assume that $\eta(W)$ is a differentiable and concave function of $W$ for all $W > K$. An illustrative example of the law of motion for the costumer base is plotted in figure 1 below.

Now we are in the position to formulate the pricing problem for a seller that enters the market at date 0 with a costumer base of measure $n_0$ and a production cost $c_0$. The seller maximizes expected discounted profits taking as given the law of motion (3) for the costumer base. The seller chooses state-contingent prices—a mapping $p$ from the history $h^t$ into non-negative real numbers. In our context, history is perfectly summarized by the realization of cost shocks up to date $t$, i.e. $h^t = \{c_0, c_1, \ldots c_t\}$. Therefore, the optimal pricing policy solves the following sequence problem

$$\max_{t=0}^\infty \beta^t \left\{ \sum_{h^t} \Pr (h^t|h_0) n(h^t) [p(h^t) - c(h^t)] \right\}, \quad \text{s.t.}$$

$$n(h^t) = n(h^{t-1}) (1 - \sigma + \eta(W(h^t))),$$

$$W(h^t) = u - p(h^t) + \beta \left\{ (1 - \sigma) \left[ \sum_{h^{t+1}} \Pr (h^{t+1}|h^t) \max \{W(h^{t+1}), K\} \right] + \sigma K \right\}.$$

Notice that the optimal state-contingent prices do not depend on the size $n(h^{-1}) = n_0$ of the costumer base at date 0. Therefore, we can set $n_0 = 1$ without loss in generality. Moreover, notice that—after any history $h^t$—the profits of the firm are homogeneous of
degree one in the customer base $n(h^t)$. Finally, notice that—after any history $h^t$—the optimal state-contingent prices must be such that the continuation profits for the seller and the continuation value for the buyer are Pareto efficient.

Exploiting the properties of the sequence problem and the Markovian structure of the stochastic process for the cost of production, the appendix proves that the optimal state-contingent prices can be derived from the policy function associated to the following Bellman equation

$$
\Pi (V, c) = \max_{p, V_i'} (1 - \sigma + \eta(W)) \left\{ p - c + \beta \left[ \alpha \Pi (V_i', c) + (1 - \alpha) \Pi (V_{i-1}', c_{-i}) \right] \right\}, \text{ s.t.}
$$

$$
V \leq W = u - p + \beta \left\{ (1 - \sigma) \left[ \alpha V_i' + (1 - \alpha)V_{i-1}' \right] + \sigma K \right\}.
$$

In the recursive problem above, the seller chooses today’s price $p$ and tomorrow’s continuation values $\{V_i', V_{i-1}'\}$ in order to maximize expected profits. The seller’s choice set is restricted to the pairs $\{p, V'(c')\}$ that endow the buyer with a lifetime utility non-smaller than $V$. Denote with $p(V, c)$ and $V'(c'; V, c)$ the policy functions associated with the recursive problem. Then, the optimal state-contingent prices $p^*(h^t)$ are such that: (a) $p^*(h^0)$ is equal to $p(0, h_0)$, (b) $p^*(h^t)$ is equal to $p(V(h^t), h_t)$, (c) $V(h^t)$ is given by $V'(h_t; V(h^t-1), h_{t-1})$.

The maximization problem in the Bellman equation can be expressed as a two-stage problem. In the second stage, the seller chooses the continuation values $\{V'(c')\}$ taking as given the delivered value $W$. In the first stage, the seller chooses $W$ subject to the promise-keeping constraint $V \leq W$. Since $W$ does not enter in the second-stage problem, the following Lemma applies.

**Lemma 1:** The policy function $V'(c|s)$ solves the unconstrained maximization problem

$$
\pi(c) = \max \alpha \left[ (1 - \sigma) V'(c) + \Pi (V'(c), c) \right] + (1 - \alpha) \left[ (1 - \sigma) V'(c_{-i}) + \Pi (V'(c_{-i}), c_{-i}) \right].
$$

Moreover, $V'(c|s)$ is state-independent: $V'(c|s) = V'(c|s')$ for all $s$ and $s'$.

**Proof:** From the definition of $W$ in the Bellman equation, we can express the seller’s price policy as

$$
p(s) = u - W(s) + \beta \left\{ (1 - \sigma) \left[ \alpha V'(c) + (1 - \alpha) V'(c_{-i}) \right] + \sigma K \right\}.
$$
Substituting (5) into the seller’s problem, we obtain

\[
\max (1 - \sigma + \eta(W)) \left\{ u - c_i - W + \beta \left\{ \sigma K + \alpha [(1 - \sigma) V'(c_i) + \Pi (V'(c_i), c_i)] + (1 - \alpha) [(1 - \sigma) V'(c_i) + \Pi (V'(c_i), c_i)] \right\} \right\} 
\]

Because \(1 - \sigma + \eta(W)\) is non-negative, the constrained maximization problem (6) can be broken down into two stages. The continuation values \(V'(c)\) solve the second-stage problem

\[
\pi(c_i) = \max \alpha \left[ (1 - \sigma) V'(c_i) + \Pi (V'(c_i), c_i) \right] + (1 - \alpha) \left[ (1 - \sigma) V'(c_{i-1}) + \Pi (V'(c_{i-1}), c_{i-1}) \right] 
\]

The delivered value \(W\) maximizes the first-stage problem

\[
\max_{V \leq W} (1 - \sigma + \eta(W)) \{ u - c_i - W + \beta \{ \sigma K + \pi(c_i) \} \}. \tag{8} 
\]

First, notice that—since the promised value \(V\) does not enter the second-stage problem—the policy function \(V'(c|s)\) does not depend on \(V\). Secondly, notice that—since the objective function in the second-stage problem is the sum of one function of \(V'(c_i)\) only and one function of \(V'(c_{i-1})\) only—the policy \(V'(c|s)\) does not depend on \(c_i\).

In light of the previous lemma, we can express the seller’s value function as

\[
\Pi (V, c_i) = \max_{V \leq W} (1 - \sigma + \eta(W)) \{ u - c_i - W + \beta \{ \sigma K + \pi(c_i) \} \} \tag{9} 
\]

Consider the objective function in (9). For \(W\) smaller than \(K\), repeated costumers leave the seller’s premises and no first-time buyers show up. The seller makes no profits. For \(W\) equal to \(K\), the seller fails to attract any first-time costumers but it maintains a fraction \(1 - \sigma\) of its costumer base. Since \(u - c_i\) is greater than \(k\), the seller makes strictly positive profits. For all \(W\) greater than \(K\), repeated costumers continue visiting the seller and a positive measure \(\eta(W)\) of new buyers flows in. In this region, seller’s profits are a differentiable and strictly concave function of \(W\). It is useful to denote with \(W^*(c_i)\) the buyers’ lifetime utility that maximizes profits, i.e.

\[
\eta'(W^*(c_i)) \{ u - c_i - W^*(c_i) + \beta [\sigma K + \pi(c_i)] \} - (1 - \sigma + \eta(W^*(c_i))) = 0. \tag{10} 
\]

In problem (9), the seller is constrained to deliver at least as much utility to its buyers as it has promised them. For \(V\) smaller than \(W^*(c_i)\), the promise-keeping constraint is not binding and the seller offers the lifetime utility \(W^*(c_i)\). In this region,
the seller’s value function \( \Pi(V, c_i) \) takes the constant value \( \Pi(W^*(c_i), c_i) \). For \( V \) greater than \( W^*(c_i) \), the promise-keeping constraint is binding and the seller offers exactly \( V \) to its buyers. In this region, the seller’s value function \( \Pi(V, c_i) \) is strictly decreasing and strictly concave in \( V \). More specifically, the first and second derivatives of the value function are

\[
\Pi'(V, c_i) = \eta'(V) \{ u - c_i - V + \beta [\sigma K + \pi(c_i)] - (1 - \sigma + \eta(V)) \},
\]

\[
\Pi''(V, c_i) = \eta''(V) \{ u - c_i - V + \beta [\sigma K + \pi(c_i)] \} - 2\eta'(V).
\] (11)

Having characterized the solution to the first stage problem (9), we know how much lifetime utility \( W_t \) the seller offers to its buyers as a function of the promised-value \( V_t \) and the production cost \( c_t \). But what is the cheapest way to deliver \( W_t \)? The answer to this question is given by the solution to the second stage problem (4).

Consider an arbitrary price schedule that gives buyers expected utility \( W_t \). If the seller was to increase its date-\( t \) price \( p_t \) by \( (1 - \sigma) \beta \) dollars and lower its date-\( (t+1) \) price by one dollar, it would still deliver \( W_t \) to its buyers. Therefore, the change to the price schedule is feasible. The increase in the date-\( t \) price reduces the inflow of first-time buyers by \( (1 - \sigma) \beta \eta'(W_t) \). The decrease in the date-\( (t+1) \) price increases the date-\( t \) inflow of new costumers by \( (1 - \sigma) \beta \eta'(W_t) \) and the date-\( (t+1) \) inflow by \( \eta'(W_{t+1}) \). The change in profits caused by the proposed deviation is equal to zero if and only if

\[
\eta'(V'(c)) \{ u - c - V'(c) + \beta [\sigma K + \pi(c)] \} - (1 - \sigma + \eta(V'(c))) = -(1 - \sigma).
\] (12)

Equation (12) is the necessary and sufficient condition for the optimality of the continuation values.

Now, we are in the position to characterize the dynamics of prices. At date 0, the seller has no prior commitment to its buyers and faces a cost of production \( c_0 \). In light of our discussion of the first-stage problem, we know that the seller will offer the lifetime utility \( W^*(c_0) \) to its buyers. At date 1, the seller will have committed to provide the contingent values \( \{V'(c)\} \). From the analysis of the second-stage problem, we know that the promised values \( V'(c) \) satisfy equation (11). Since the left hand side of (11) is strictly decreasing in \( V \), it follows that the optimal continuation value \( V'(c) \) is greater than \( W^*(c) \). Therefore, at date 1, the seller will deliver exactly \( V'(c_1) \). In light of Lemma 1, it follows that \( W(h_t^T) = V'(h_t^T) \) at all dates \( t = 2, 3, \ldots \). From the path of \( W(h_t^T) \), we can recover the equilibrium price dynamics.
Proposition 2: There exists a unique optimal state-contingent price schedule \( p^*(h^t) \) such that: (i) given the same production cost, prices are decreasing over time: \( p^*(h_0) < p^*(h^t) \) for all \( h_0 \) and \( h^t \) such that \( t \geq 1 \) and \( h_0 = h_t \); (ii) given the same date, prices are decreasing with cost: \( p^*(h^t) < p^*(h'^t) \) for all \( h^t \) and \( h'^t \) such that \( h_t < h'^t \); (iii) prices are history independent from date-1 on: \( p^*(h^t) = p^*(h'^t) \) for all \( h^t \) and \( h'^t \) such that \( h^t = h'^t \) and \( t, t^t \geq 1 \).

The optimal price schedule \( p^*(h^t) \) is time-inconsistent—i.e. if in state \( h^1 \) the seller was allowed to reset prices, it would want to reduce the lifetime utility offered to its buyers from \( V'(h^1) \) to \( W^*(h^1) \). Time-inconsistency is due to the timing of costs and benefits associated with promising a certain price \( p(h^1) \). On the one hand, the cost of promising \( p(h^1) \) is fully borne at date 1 when trade takes place. On the other hand, the benefit of promising \( p(h^1) \)—i.e. the inflow of first-time buyers—takes place in part at date 0 and in part at date 1. As a result, when date-1 arrives, the seller would have an incentive to revise its price upwards.

Exercises: (a) no repeat purchases: suppose that \( \sigma = 1 \). Then the contract is time-consistent; (b) no repeat purchases and no directed search. Then Diamond’s paradox.

4 Pricing with Unobservable Costs

4.1 Very Persistent Shocks: Flexible Prices

A seller enters the market and commits to the optimal state-contingent price schedule \( p^*(h^t) \). At the beginning of each period, buyers observe the seller’s production cost and read off the schedule \( p^* \) the current and future terms-of-trade. Unexpectedly, at date \( t \), the buyers wake up to find out that they are unable to observe the current production cost \( c_t \). Does the seller have an incentive to truthfully report \( c_t \) to its buyers?

First consider a seller that has a high cost of production at date \( t \). If the seller announces to the public that \( c_t \) is equal to \( c_h \), its expected profits are given by \( \Pi(V'(c_h), c_h) \). If the seller announces a low cost of production, then the date-\( t \) price is lower by \( (1 - \beta (1 - \sigma) (2\alpha - 1)) (V'(c_t) - V'(c_h)) \) dollars. From date-\( t + 1 \), prices do not depend upon the seller’s report. On the other hand, if the seller announces \( c_t \), buyers expect lower prices both today and in the future and their lifetime expected utility increases by \( V'(c_t) - V'(c_h) \) dollars. Therefore, at date-\( t \) the inflow of first-time buyers increases by approximately \( \eta' (V'(c_t) - V'(c_h)) \) units. Under the assumption
that function $\eta(W)$ is linear in $W$, the high-cost seller has an incentive to truthfully report its type if and only if

$$(1 - \sigma) - (1 - \sigma + \eta(V'(c_h))) \beta (1 - \sigma) (2\alpha - 1) + \eta' (1 - \beta (1 - \sigma) (2\alpha - 1)) \left[V'(c_l) - V'(c_h)\right] \geq 0. \quad (13)$$

The first term represents the (first-order) fall in profits caused by an increase in the lifetime utility promised and delivered by the seller. The second term represents the (first-order) increase in profits due to the fact that not all of the promised increase in lifetime utility is actually delivered. The final term is a second order effect associated with the concavity of the seller’s profits function. Since $(1 - \sigma + \eta(V'(c_h))) \beta$ is smaller than 1, the truth-telling condition for the high-cost seller is always satisfied.

Next, consider a seller with a low cost of production at date $t$. Under the assumption that the function $\eta(W)$ is linear, the seller has an incentive to truthfully report its type if and only if

$$(1 - \sigma) - (1 - \sigma + \eta(V'(c_l))) \beta (1 - \sigma) (2\alpha - 1) - \eta' (1 - \beta (1 - \sigma) (2\alpha - 1)) \left[V'(c_l) - V'(c_h)\right] \leq 0. \quad (14)$$

The first term in (14) represents the (first-order) increase in profits associated with a reduction in the lifetime utility promised and delivered by the seller. The second term is the decrease in profits due to the fact that the fall in the utility expected by the buyers is larger than the increase in current and future prices. The final term is the sum of the higher order effects on profits. Since the sum of the first two terms is positive while the third is negative, it is not clear whether the low-cost seller wants to truthfully report its type or lie. Nevertheless, one can easily verify that the left hand side of (14) is strictly decreasing in $\alpha$, the parameter that captures the persistence of productivity shocks. Therefore, when productivity shocks are transitory, the low-cost seller has an incentive to lie. When productivity shocks are persistent, the seller has an incentive to announce its true type.

The thought experiment that we have just studied is of particular interest. Indeed, if and only if the seller has an incentive to truthfully report its type in any arbitrary state $h_t$, then the optimal state-contingent price schedule $p^*(h_t)$ can be implemented even if the cost of production is not directly observable by the buyers at any date. This remark leads to the following proposition.

**Proposition 3:** Assume that $\eta(W)$ is linear in $W$. Then there exists an $\alpha^*$ in the interval $(1/2, 1]$ such that: (i) for all $\alpha < \alpha^*$, the price schedule $p^*(h_t)$ is implementable
when the production cost is privately observed by the seller; (ii) for all $\alpha \geq \alpha^*$, the price schedule $p^*(h^t)$ is not implementable.

### 4.2 Incentive Compatibility and Renegotiation Proofness

Suppose that productivity shocks are private information of the seller and that $\alpha$ is lower than the critical threshold $\alpha^*$. From the previous subsection, we know that the optimal price schedule $p^*(h^t)$ is not feasible. In the following pages, we want to identify the set of schedules that can be implemented under asymmetric information and characterize the one that maximizes the profits of the seller.

First, we have to introduce some notation. Define a reporting strategy $\rho$ as a sequence $\{\rho(h^t)\}$ mapping histories $h^t$ into a report of the current cost realization. Define a pricing schedule $p$ as a sequence of functions $\{p_t\}$ mapping reported histories into non-negative real numbers. We say that a pricing schedule $p$ is incentive compatible if it induces the seller to truthfully report its production cost. An incentive-compatible price schedule $p$ satisfies, for all reporting strategies $\rho$,

$$
\sum_{t=0}^{\infty} \beta^t \left\{ \sum_{h^t} \Pr(h^t|h_0)n(h^t) \left[ p(h^t) - c(h^t) \right] \right\} \geq \sum_{t=0}^{\infty} \beta^t \left\{ \sum_{h^t} \Pr(h^t|h_0)n(\rho(h^t)) \left[ p(\rho(h^t)) - c(h^t) \right] \right\},
$$

where the seller’s customer base $n(\rho(h^t))$ is given by

$$
n(\rho(h^t)) = n(\rho(h^{t-1})) \left( 1 - \sigma + \eta \left( W(\rho(h^t)) \right) \right),
$$

$$
W(\rho(h^t)) = u - p(\rho(h^t)) + \beta \left\{ \sigma K + (1 - \sigma) \sum_{h^{t+1}} \Pr(h^{t+1}|h^t) \max\{W(h^{t+1}), K\} \right\}.
$$

As noticed in Fernandez and Phelan (2000), when costs are unobservable and correlated over time, the optimal incentive-compatible schedule may be ex-post Pareto inefficient. Intuitively, ex-post inefficiencies can be used to reduce the expected continuation profits for a seller that has misreported its type in the past. Since both the buyer and the seller would agree to renegotiate an ex-post inefficient price schedule, we restrict attention to schedules that are Pareto efficient after any history $h^T$. Formally, we say that a price schedule $p$ is renegotiation proof if—after any history $h^T$—there is no other price schedule that gives higher lifetime utility to both the seller and the buyer. A renegotiation-proof price schedule $p$ satisfies, for all incentive compatible schedules
\[ p' \text{ and all histories } h^t, \]
\[ \sum_{t=0}^{\infty} \beta^{t-\tau} \left\{ \sum_{h^t} \text{Pr}(h^t|h^\tau)n(h^t) \left[ p(h^t) - c(h^t) \right] \right\} \geq \]
\[ \sum_{t=0}^{\infty} \beta^{t-\tau} \left\{ \sum_{h^t} \text{Pr}(h^t|h^\tau)n(h^t) \left[ p'(h^t) - c(h^t) \right] \right\}, \]
whenever \( W(h^t|p') \geq W(h^t|p). \)

Given the date-0 cost \( c_0 \), the seller’s problem is defined as choosing a pricing schedule \( p \) to maximize expected profits
\[ \max_p \sum_{t=0}^{\infty} \beta^t \left[ \sum_{h^t} \text{Pr}(h^t|h_0)n(h^t) \left[ p(h^t) - c(h^t) \right] \right] \]
subject to the incentive compatibility (15) and the renegotiation proofness (16) constraints.

In the appendix, we show that the optimal pricing schedule can be recovered from the policy functions associated to the following Bellman equation
\[ \Pi_j(V, c_i) = (1 - \sigma + \eta(W)) \{ c_i - c_j + \beta (2\alpha - 1) [\Pi_j(V'(c_j), c_j) - \Pi_i(V'(c_i), c_i)] \} + \]
\[ \max (1 - \sigma + \eta(W)) \{ p - c_i + \beta [\alpha \Pi_i(V'(c_i), c_i) + (1 - \alpha) \Pi_{i-1}(V'(c_{i-1}), c_{i-1})] \}, \text{ s.t.} \]
\[ V \leq W = u - p + \beta \{(1 - \sigma) [\alpha V'(c_i) + (1 - \alpha) V'(c_{i-1})] + \sigma K \}, \]
\[ \Pi_x(V'(c_x), c_x) \geq \Pi_x(V'(c_{-x}), c_{-x}) \text{ for } x = l, h. \]

In the recursive problem above, a state \( s \) is given by the actual cost of production \( c_j \), the reported cost \( c_i \) and the lifetime utility \( V \) promised by the seller to its buyers. The current price \( p \) and the continuation value \( \{V'(c)\} \) are chosen to maximize the expected profits of a seller with production cost \( c_i \). The choice set is restricted to the pairs \( \{p, V'(c)\} \) such that (a) tomorrow the seller has no incentive to misreport its type, (b) today’s buyers expect a lifetime utility non-smaller than \( V \). Denote with \( \{p(s), W(s), V'(c|s)\} \) the policy functions associated with the Bellman equation. Then, the optimal price schedule \( p(h^t) \) is such that: (a) \( p(h_0) \) is equal to \( p(0, h_0) \), (b) \( p(h^t) \) is equal to \( p(V(h^t), h_t) \), (c) \( V(h^t) \) is given by \( V'(h_t; V(h^{t-1}), h_{t-1}) \).

Also under asymmetric information, the maximization problem associated with the Bellman equation above can be solved in two stages. In the second-stage problem, the
continuation values \( \{ V'(c_i) \} \) are chosen to maximize a weighted average of buyer’s and seller’s lifetime utilities subject to the incentive-compatibility constraints. Formally, the second-stage problem is

\[
\pi(c_i) = \max \alpha [(1 - \sigma) V'(c_i) + \Pi_i (V'(c_i), c_i)] + (1 - \alpha) [(1 - \sigma) V'(c_{-i}) + \Pi_{-i} (V'(c_{-i}), c_{-i})],
\]

\[
\Pi_x (V'(c_x), c_x) \geq \Pi_x (V'(c_{-x}), c_{-x}) \text{ for } x = l, h.
\]

(17)

Notice that—since the current promised value does not enter (17)—the optimal continuation values are independent from \( V \). On the other hand—since \( V_0(c_i) \) and \( V_0(c_{-i}) \) enter simultaneously in the incentive-compatibility constraints—the optimal continuation values do depend on the current cost of production \( c_i \).

In the first-stage problem, the lifetime utility \( W \) is chosen to maximize the expected profits of a seller with production cost \( c_i \) subject to the promise-keeping constraint. Formally, the first-stage problem is given by

\[
\max_{V \leq W} \Psi(W; c_i) = (1 - \sigma + \eta(W)) \{ u - W - c_i + \beta (\sigma K + \pi(c_i)) \}.
\]

(18)

The function \( \Psi(W; c_i) \) is discontinuous at \( K \). For all \( W \) strictly smaller than \( K \), \( \Psi(W; c_i) \) is equal to zero. At \( W \) equal to \( K \), \( \Psi(W; c_i) \) is strictly positive. For all \( W \) greater than \( K \), the function is strictly concave. The global maximum of \( \Psi(W; c_i) \) is \( W^*(c_i) \). Therefore, if \( V \) is smaller than \( W^*(c_i) \), the promise-keeping constraint does not bind and the seller offers \( W^*(c_i) \) to its buyers. If \( V \) is greater than \( W^*(c_i) \), the promise-keeping constraint binds and the seller delivers precisely \( V \) to its buyers.

From the characterization of the first-stage problem, we can recover the qualitative properties of the value function \( \Pi_i(V, c_i) \). First, consider the case of a seller that has truthfully reported its cost type, i.e. \( j = i \). In this case, the seller’s expected profits \( \Pi_i(V, c_i) \) are given by the solution to the first-stage problem (18). Therefore, \( \Pi_i(V, c_i) \) is constant at \( \Pi_i(W^*(c_i), c_i) \) for \( V \) smaller than \( W^*(c_i) \) and \( \Pi_i(V, c_i) \) is decreasing and concave for \( V \) greater than \( W^*(c_i) \). Secondly, consider the case of a seller than has misreported its type, i.e. \( j = -i \). In this case, the seller’s expected profits are given by

\[
\Pi_{-i}(V, c_i) = \Pi_i(V, c_i) + (1 - \sigma + \eta(W(V, c_i))) \cdot \{ c_i - c_{-i} + \beta (2\alpha - 1) [\Pi_{-i} (V'(c_{-i}|c_i), c_{-i}) - \Pi_i (V'(c_{i}|c_i), c_i)] \}.
\]

(19)
The value function is increasing in $V$ over the (possibly empty) interval $[W^*(c_i), W^*(c_{-i})]$ and strictly decreasing for $V$ greater than $W^*(c_{-i})$. The value function $\Pi_{-i}(V, c_i)$ is strictly concave.

### 4.3 IID Shocks: Rigid Prices

It is useful to begin the analysis of the optimal incentive-compatible price schedule with the simple, yet informative, case of i.i.d productivity shocks.

When $\alpha$ is equal to 1/2, the current realization of the production cost has no effect on the probability distribution of costs in the future. As a result, both the objective function and the constraint set of the second-stage problem become independent from the cost reported by the seller today, i.e. $V'(c|c_i) = V(c)$ and $\pi(c_i) = \pi$ for all $c_i$. In turn, the latter observation implies that—from the seller’s perspective—the sole difference between truthfully reporting its cost and lying is the price he can extract from its customers in the current period. Therefore, for $\alpha$ equal to 1/2, the incentive compatibility constraint reduces to

$$
\Psi\left(W'(c_x); c_x\right) \geq \Psi\left(W'(c_{-x}); c_{-x}\right), \text{ for } x = l, h,
$$

where $W'(c)$ is equal to $W(V'(c), c)$.

When the seller has a high cost of production, profits are strictly decreasing in the lifetime utility delivered to the buyers for all $W \geq W^*(c_h)$. The renegotiation-proofness assumption implies that continuation value $W'(c_h)$ delivered by a high-cost seller is greater than $W^*(c_h)$. Similarly, $W'(c_l)$ is greater than $W^*(c_l)$. Since $W^*(c_l) > W^*(c_h)$, it follows that the incentive compatibility constraint for a high-cost seller is satisfied if and only if $W'(c_l)$ is greater or equal than $W'(c_h)$. When the seller has a low production cost, the incentive compatibility constraint can be satisfied in two different ways. First, the seller is induced to report its true type if $W'(c_h)$ is greater or equal than $W'(c_l)$. Secondly, the seller has an incentive to announce a low cost of production if $W'(c_h)$ lies on the increasing portion of the profit function $\Psi(W; c_l)$ and is so small that the benefit from reducing the current price is outweighed by the cost of lowering the inflow of first-time customers. In the latter case, the higher is $W'(c_l)$ the lower $W'(c_h)$ has to be. Combining the incentive compatibility constraints for the high and the low cost seller, we obtain the feasible set for the continuation values $V'(c_l)$ and $V'(c_h)$. The feasible set is illustrated in Figure 2.
Observing the shape of the feasible set, it follows that there are two candidate solutions to the second-stage problem (17). According to one candidate solution, the high and the low cost sellers are required to deliver the same lifetime utility to their buyers, i.e. $V'(c) = V'$ for $c = c_l, c_h$. The common promised value $V'$ is chosen to satisfy the optimality condition (12) in expected terms

$$\frac{1}{2} \Psi'(V'; c_l) + \frac{1}{2} \Psi'(V'; c_h) = -(1 - \sigma). \quad (21)$$

According to the alternative solution, the high-cost buyer offers strictly less utility to its buyers than the low-cost seller. The promised values are chosen to equate the marginal profit among seller’s types and to make the low-cost seller indifferent between announcing $c_l$ and $c_h$. Formally, $V'(c_l)$ and $V'(c_h)$ are such that

$$\Psi'(V'(c_l); c_l) = \Psi'(V'(c_h); c_h) = \Psi'(V'(c_l); c_l). \quad (22)$$

Notice that—when $c_h$ gets arbitrarily close to $c_l$—the solution to the first order condition (21) is equal to the optimal continuation values under perfect information. On the other hand, the solution to the first order condition (22) converges to $W^*(c)$. Therefore, for $c_h - c_l$ sufficiently small, it is optimal to make the continuation utility offered by the seller to its buyers independent from the cost of production.

From the properties of the continuation values, we can recover the price dynamics.

**Proposition 4:** Let $\alpha$ be equal to $1/2$. Denote with $p(h^t)$ the optimal state-contingent price schedule subject to the incentive compatibility and renegotiation proofness constraints. (i) At date 0, the seller demands the price $p_0(c_0)$, where $p_0(c_l) < p_0(c_h)$. (ii) At date $t \geq 1$, the seller demands the price $p_1(c_l)$, where $p_1(c_l) \leq p_1(c_h)$ and $p_1(c) < p_0(c)$. (iii) There exists a $\Delta^*$ such that, for all $c_h - c_l$ smaller than $\Delta^*$, the seller cannot change its price in response to a productivity shock: $p_1(c_l) = p_1(c_h)$.

### 4.4 Moderately Persistent Shocks: Sticky Prices

TBA

### 5 Conclusions

TBA
References


A Appendix