Consumption Commitments, Borrowing Constraints and Preferences for Risk

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Abstract

This paper characterizes the solution to the consumer’s dynamic decision problem in the presence of consumption commitments (goods that involve transaction costs and are infrequently adjusted) and borrowing constraints. The findings in recent theoretical literature have suggested that consumption commitments amplify risk aversion in static models (Postlewaite et. el. 2006) and in a dynamic model with specific assumptions on the interest rate and the time discount factor (Chetty and Szeidl 2007).

This paper illustrates that the opposite result might obtain in a more general dynamic environment. We show that, if the price of a risk-free bond is below the time preference rate (as it is usually predicted by the general equilibrium incomplete market models), the consumers who start saving in order to increase consumption of the commitment good in the future become risk lovers. We also argue that such behavior is likely to arise due to the presence of borrowing constraints; therefore suggesting that in the economies with consumption commitments borrowing constraints can make uninsured risk desirable (in contrast, it is known that in standard models borrowing constraints increase the cost of uninsured risk).

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1 Introduction

Consumption commitments arise when some goods involve substantial transaction costs and thus are infrequently adjusted. Typical examples of such goods are housing, land, vehicles, etc. Consumption commitments have attracted attention in recent economic literature since it has been documented that they constitute a large share in households’ life-time expenses and that the adjustment costs can be substantial\footnote{Warren and Tyagi (2003) find that a typical American family earmarks 75\% of their income on “fixed expenses” such as mortgage, car payments, child care, etc. Chetty and Szeidl (2007) use CES data and estimate that around 50\% of households’ life-time wealth is spent on “commitment goods”.}. Various effects of commitment goods have been discussed; special attention being paid to their impact on consumers’ risk preferences (e.g. Chetty and Szeidl 2007, Shore and Sinai 2005, Postlewaite et. el. 2006, etc.). It has been argued that the presence of consumption commitments increases risk aversion of individuals.

Intuitively, if consumption of some of the goods remains unchanged in response to (relatively small) permanent changes in current wealth or income, consumption of the remaining (flexible) goods would vary “too much”. Such extra volatility of these goods raises the welfare cost of risk compared to the environment in which all the goods could be adjusted flexibly. Thus it has been suggested that consumption commitments magnify risk aversion. So far, this result had been rigorously established in static models (Postlewaite et. el. 2006) and in a dynamic model with specific assumptions on the interest rate and the time discount factor (Chetty and Szeidl 2007).

This paper argues that in a more general environment the effect of consumption commitments on consumers’ risk preferences might be very different; the consumers with particular wealth levels might even become risk lovers. As in the previous studies, we emphasize that consumers’ attitudes towards risk are determined not by the properties of their instantaneous utility function – but by the shape of their indirect life-time utility function arising from the solution of a dynamic optimization problem with commitments. In our model (as well as in the previous literature), the indirect life-time utility is a function of consumers’ wealth. We show that for the consumers with particular wealth levels this function may be concave, linear or even convex, depending on the relative values of the interest rate and the time discount factor.
To briefly outline the driving forces of our results, consider a case when the gross return to risk-free savings is smaller than the reverse of the time discount factor (\( \beta(1 + r) < 1 \)), as it is usually predicted by the general equilibrium macro models with uninsured risk (e.g. Aiyagari 1994). Suppose also that the agent has committed to little housing expenses in the beginning of his life, but is currently accumulating wealth in order to move to a bigger house in the future. As long as this agent stays in a small house, his consumption of a flexible good has the standard properties obtained in a neoclassical growth model. In particular, the current marginal utility from consuming the flexible good is equalized with the discounted future marginal utility times the gross rate of return \((u'(c_t) = \beta(1 + r) u'(c_{t+1}))\); thus the flexible good’s consumption declines over time. At the same time, by the envelope condition, the consumption level of the flexible good determines the slope of the value function (indirect life-time utility function): as consumption declines, the value function gets steeper. Since consumption declines simultaneously with an increase in wealth (recall that the consumer makes savings in order to move to a bigger house), the value function (as a function of wealth) gets steeper as wealth rises, thereby generating demand for wealth lotteries.

Why would some consumers decide to commit to a low level of housing consumption in the beginning of their life and then start accumulating wealth in order to move to a more expensive housing in the future? One possibility could be the presence of the borrowing constraints. If buying a small house requires lower adjustment cost than purchasing a big house, if consumers have little wealth in the beginning of their life and cannot borrow against their future (possibly large) income, they would have no other choice but to move into a small house and start accumulating wealth in order to pay bigger adjustment cost and eventually switch to a bigger house. While they are living in a small house, their value function has the properties described in the previous paragraph and thus these consumers could actually become risk lovers. Therefore, the presence of commitment goods has a somewhat surprising impact on the role of borrowing constraints. It is commonly believed that borrowing constraints make risk more costly (because people have to give up their consumption when they are hit with the temporary negative shocks). This paper points out to a situation in which borrowing constraints have a reverse effect: they generate an interval of wealth levels, within which consumers are actually willing to pay for taking a lottery; though outside of this interval borrowing constraints have

3
a standard effect by raising consumers’ risk aversion.

Previous literature has discussed a number of situations in which risk loving behavior might arise endogenously. Most of them are various version of discrete choice models which generate kink in the indirect utility function. For example, Vereshchagina and Hopenhayn (2006) model a discrete occupational choice model and argue that entrepreneurs might be willing to take excess risk because their value function has a kink at a wealth level at which agents are indifferent between being workers and being entrepreneurs. Albuquerque and Hopenhayn (2004) show that risk taking might be the feature of the optimal borrowing contract between the bank and a risk-neutral firm; such risk taking is used to eliminate the kink occurring at the intersection of the firm’s endogenous value and its exogenous liquidation cost. Athreya (2002) argues that poor consumers might have additional incentives to make risky investment and are more likely to default because their value has a kink at a point at which the value of default intersects the value of no-default. Even Chetty and Szeidl (2006) acknowledge that consumption commitments can generate demand for moderate-stake risks due to local convexity of the value function at a point at which the consumer is indifferent between staying in an house and adjusting his housing consumption to a new level.

In contrast to all these papers, we describe how the local convexity of the indirect utility function arises not due to a presence of a kink, but as an outcome of optimal saving behavior of consumers either with little patience or facing low interest rate (so that \( \beta(1 + r) \) is sufficiently low). The resulting value function is convex not at one single point at which the consumer is indifferent between two discrete options; instead, it is convex within an interval of wealth levels. Therefore this project brings up attention to a (hopefully) novel relationship between agents’ patience, saving opportunities, borrowing constraints and risk preferences.

The paper is organized as follows. Section 2.1 lays out a basic model. Section 2.2 characterizes its solution in the absence of borrowing constraints. Sections 2.3 and 2.4 describe possible effects of commitment goods on consumers’ risk attitudes. Section 2.5 discusses the role of the borrowing constraints. Section 2.6 studies analytically and numerically the example with CRRA utility function. Section 3 makes final remarks and briefly outlines the shortcomings of the current version of the paper which we are trying to improve now.
2 The Model

2.1 Wealth dynamics and indirect utility function

We will illustrate our main results in a discrete-time infinite horizon model. Assume that a consumer receives income \( y \geq 0 \) in every period and can spend it on food and housing. The consumer’s life-time utility is time-additive with geometric discounting:

\[
\sum_{t=0}^{+\infty} \beta^t (u(c_t) + v(h_t)),
\]

where \( \beta \in (0, 1) \) is the time discount factor, \( u(c_t) \) is the utility derived from the consumption of food and \( v(h_t) \) is the utility derived from housing in period \( t \). Both utility functions, \( u(c) \) and \( v(h) \), are defined on \( (0, +\infty) \) and are strictly increasing, strictly concave, bounded from above and satisfy Inada conditions.

We will assume that food consumption is flexible and can be adjusted at no cost. In contrast, housing will play the role of the commitment good, which can be adjusted only at a cost \( \eta(h_t - h_{t-1}, h_t) \). Assume also that \( \eta(h_{t-1}, h_t) \) is bounded away from zero, i.e. \( \lim_{h_t \to h_{t-1}} \eta(h_{t-1}, h_t) = 0 \). It is well known that under such assumption the consumer’s choice of the corresponding consumption good would be endogenously discrete. Thus, without loss of generality, we will assume that housing expenses can be either \( h^1 > 0 \) or \( h^2 > h^1 \). Denote the corresponding utility levels by \( v(h^1) = v^1 \) and \( v(h^2) = v^2 \) and the corresponding adjustment costs by \( \eta(h^1, h^2) = \eta^1 \) and \( \eta(h^2, h^1) = \eta^2 \).

Finally, assume that the consumers can save in a risk-free bond which offers interest rate \( r \). Thus the consumer’s decision problem can be written as

\[
\max_{c_t, h_t, a_t} \sum_{t=0}^{+\infty} \beta^t (u(c_t) + v(h_t)) \]

s.t. \( c_t + h_t + \frac{a_{t+1}}{1+r} + \eta^1 I_{h_{t}>h_{t-1}} + \eta^2 I_{h_{t}<h_{t-1}} \leq a_t + y, \quad t \geq 0 \) (1)
\[
a_t \geq a, \quad t \geq 0, \quad \lim_{t \to +\infty} \frac{a_t}{(1+r)^t} \geq 0,
\]
\[
a_0 \text{ and } h_{-1} \text{ are given},
\]

Separability of the utility function simplifies the analysis (in particular, it allows to establish single crossing of particular value functions in one step); however it is not crucial for obtaining the main result.
where $a_t$ is the net wealth of the consumer in the beginning of period $t$ and $I_A$ is the indicator function which is equal to 1 if $A$ is true and to 0 otherwise. In other words, the consumer maximizes his life-time utility subject to the period-by-period budget constraint, the borrowing constraint and the no-Ponzi game condition.

We also assume that $h_{-1} = h^1$ and $\eta^2$ is sufficiently large so that the consumer would never want to switch from high to low level of housing.\(^3\) The former assumption is crucial for obtaining the main results of the paper (as we will show later, only the consumers who are planning to switch from low to high level of housing might become risk lovers).\(^4\) The latter assumption is not essential, and is adopted here to simplify the analysis; it implies that the life-time utility of the consumer who has already switched to high housing consumption can be easily determined.

Under these assumptions, the solution to (1) is equivalent to the solution of the problem, in which the agent can choose the food consumption and wealth profiles $\{c_t, a_t\}_{t=0}^{+\infty}$ as well as the moment of switching from low to high housing consumption:

$$\max_{\{c_t, a_t\}, T \in \{0, 1, 2, \ldots, +\infty\}} \sum_{t=0}^{T-1} \beta^t (u(c_t) + v^1) + \sum_{t=T}^{+\infty} \beta^t (u(c_t) + v^2)$$

s.t. $c_t + h^1 + \frac{a_{t+1}}{1+r} \leq a_t + y$, $0 \leq t < T$,

$c_t + h^2 + \eta^1 + \frac{a_{t+1}}{1+r} \leq a_t + y$, $t = T$,

$c_t + h^2 + \frac{a_{t+1}}{1+r} \leq a_t + y$, $t > T$,

$a_t \geq a$, $t \geq 0$, $\lim_{t \to +\infty} \frac{a_t}{(1+r)^t} \geq 0$,

$a_0$ is given.

(2)

Note that the agent might decide to switch to $h^2$ right away by setting $T = 0$ (in which case the first budget constraint is irrelevant) or to remain in a small house $h^1$ forever by setting $T = +\infty$.

\(^3\)Since $u(c)$ is bounded from above, such $\eta^2$ exists.

\(^4\)Such assumption could be justified by interpreting $\eta^1$ as a cost of moving into $h^2$ and assuming that it is costless to move into $h^1$. 
2.2 The solution without borrowing constraints

In the absence of the borrowing constraints the consumer’s problem (2) has a simple characterization. In this case the only restriction on the consumer’s asset level is the no-Ponzi game condition; thus the constraints in the problem (2) can be replaced with the life-time budget constraint equalizing the present values of the life-time expenses and wealth:

\[
V(a_0) = \max_{\{c_t, T \in \{0, 1, 2, ..., +\infty\}} \sum_{t=0}^{T-1} \beta^t (u(c_t) + v^1) + \sum_{t=T}^{+\infty} \beta^t (u(c_t) + v^2) \\
\text{s.t.} \sum_{t=0}^{T-1} c_t + h^1 \frac{1 + r}{(1 + r)^t} + \sum_{t=T}^{+\infty} c_t + h^2 \frac{1 + r}{(1 + r)^t} + \eta^1 \frac{1 + r}{(1 + r)^t} \leq a_0 + 1 + \frac{r}{r} y. 
\]

Since \( u(c) \) is defined on \((0, +\infty)\), \( V(a_0) \) is defined for \( a_0 \in (1 + \frac{r}{r} (h^1 - y), +\infty) \).

Alternatively, (3) can be represented as a two-stage maximization problem, in which the consumer solves for the optimal consumption profile given that the switch from \( h^1 \) to \( h^2 \) occurs in period \( T \), and then chooses the most favorable \( T \):

\[
V(a_0) = \max\{V_\infty(a_0), V_0(a_0), V_1(a_0), V_2(a_0), \ldots\}, 
\]

Here \( V_\infty(a_0) \) is the life-time utility of the consumer whose housing consumption remains at \( h_t = h^1 \) forever:

\[
V_\infty(a_0) = \max_{\{c_t\}} \sum_{t=0}^{+\infty} \beta^t (u(c_t) + v^1) \\
\text{s.t.} \sum_{t=0}^{+\infty} c_t + h^1 \frac{1 + r}{(1 + r)^t} \leq a_0 + 1 + \frac{r}{r} y. 
\]

The value \( V_\infty(a_0) \) is defined for \( a_0 > \hat{a}_\infty = \frac{1 + r}{r} (h^1 - y) \). Similarly, \( V_T(a_0) \ (T \geq 0) \) is the life-time utility of the consumer who switches from \( h^1 \) to \( h^2 \) in period \( T \):

\[
V_T(a_0) = \max_{\{c_t\}} \sum_{t=0}^{T-1} \beta^t (u(c_t) + v^1) + \sum_{t=T}^{+\infty} \beta^t (u(c_t) + v^2) \\
\text{s.t.} \sum_{t=0}^{T-1} c_t + h^1 \frac{1 + r}{(1 + r)^t} + \sum_{t=T}^{+\infty} c_t + h^2 \frac{1 + r}{(1 + r)^t} + \eta^1 \frac{1 + r}{(1 + r)^t} \leq a_0 + 1 + \frac{r}{r} y. 
\]
Note that the difference between the problems (3) and (6) is that switching time $T$ is the choice variable in (3), but is a parameter in (6). Let us denote the domain of $V_T(a_0)$ by $(\hat{a}_T, +\infty)$. It can be easily verified that $\hat{a}_0 = \frac{1+r}{r}(h^2 - y) + \eta^1$ and $\hat{a}_T = \frac{1}{(1+r)^T} \hat{a}_0 + (1 - \frac{1}{(1+r)^T}) \hat{a}_{\infty}$. Note that $\hat{a}_0 > \hat{a}_{\infty}$ and thus the sequence $(\hat{a}_T)_{T=0}^{+\infty}$ is monotone, $\hat{a}_T > \hat{a}_{T+1}$ for all $T \geq 0$, and converges to $\hat{a}_{\infty}$.

Due to separability of instantaneous utility function in food and housing consumption, each of the value functions $\{V_T(a), T = \infty, 0, 1, \ldots\}$ is an affine transformation of the value $W(w)$ of the consumer with life-time wealth $w$ who derives utility only from food consumption:

$$W(w) = \max_{\{c_t\}} \sum_{t=0}^{+\infty} \beta^t u(c_t)$$

s.t.

$$\sum_{t=0}^{+\infty} \frac{c_t}{(1+r)^t} \leq w.$$  

(7)

In particular,

$$V_T(a) = W(a - \hat{a}_T) + \hat{v}_T, \quad T \in \{\infty, 0, 1, \ldots\},$$

(8)

where $\hat{v}_{\infty} = \frac{\eta^1}{1-\beta}$, $\hat{v}_0 = \frac{\eta^2}{1-\beta} > \hat{v}_{\infty}$ and $\hat{v}_T = (1 - \beta^T) \hat{v}_{\infty} + \beta^T \hat{v}_0$. Obviously, the sequence $(\hat{v}_T)_{T=0}^{+\infty}$ is monotone, $\hat{v}_T > \hat{v}_{T+1}$ for all $T \geq 0$, and converges to $\hat{v}_{\infty}$.

The monotonicity of $(\hat{a}_T)_{T=0}^{+\infty}$ and $(\hat{v}_T)_{T=0}^{+\infty}$ clearly illustrates the tradeoff associated with the choice of the switching period: moving into a new house in later periods lowers the life-time utility derived from housing but raises the amount of resources that can be spent on food consumption. The latter increase in life-time wealth occurs due to two reasons: cutting the housing cost and postponing the payment of the switching cost (thereby lowering its value from $\eta^1$ to $\frac{\eta^1}{(1+r)^T}$).

Correspondingly, the motive for postponing the moment of switching from $h_1$ to $h_2$ would disappear if (i) the consumer could freely choose the level of housing after the adjustment cost is paid and (ii) the interest rate $r$ were equal to zero. It would be easy to incorporate endogenous choice of $h_2$ into our environment, however we of course cannot set $r = 0$ in our model because then the consumer’s life-time wealth $a_0 + \frac{1+r}{r} y$ would be unbounded. It might be possible to prove that with the flexible choice of $h_2$ and sufficiently low interest rate the switch, if it ever occurs, happens
only in period zero.\(^5\) However, by imposing additional restrictions on \(r\) we would loose some generality of our results. Nevertheless, the partial case in which the switch either occurs in period zero or never happens is of special interest because in this situation the presence of consumption commitments magnifies risk aversion for all consumers, independently of the initial wealth level. Such prediction would also be similar to the results obtained in Chetty and Szeidl (2006) who analyze the finite horizon economy with \(r = 0\). Thus throughout the rest of the paper we will devote special attention to the case in which \(V(a) = \max\{V_0(a), V_\infty(a)\}\).

Since \(W(w)\) is strictly increasing, strictly concave and bounded from above\(^6\) and the sequences \(\{\hat{a}_T\}_{T=0}^{\infty}\) and \(\{\hat{\eta}_T\}_{T=0}^{\infty}\) are monotone, any two functions \(V_i(a), V_j(a) \in \{V_T(a), T = \infty, 0, 1, \ldots\}\) have exactly one intersection and \(V_i(a) > V_j(a)\) for large enough \(a\) if \(i < j\). This implies that consumers with sufficiently high initial wealth choose to switch to high housing consumption in period 0, while the poorest consumers will always consume low housing services. Potentially, the consumers with the intermediate wealth levels might decide to wait for some time before they switch from \(h^1\) to \(h^2\). In order to describe their attitudes towards risk, we need to characterize the shape of their value function. Such characterization is significantly simplified once we notice that the indirect utility functions \(\{V_T(a), T = \infty, 0, 1, \ldots\}\) are related to each other recursively in the following way:

\[
V_\infty(a) = \max_{c, a'} \{u(a + y - h^1 - \frac{a'}{1 + r}) + v^1 + \beta V_\infty(a')\} \\
V_{T+1}(a) = \max_{c, a'} \{u(a + y - h^1 - \frac{a'}{1 + r}) + v^1 + \beta V_T(a')\}, \quad T \geq 0
\]

\(^5\)Consider a consumer who switches from \(h^1\) to some (endogenously chosen) \(h^2\) in period 1. His total expenses on housing (including switching cost) amount to \(h^1 + \frac{h^2}{r} + \frac{\eta}{1 + r}\) and his lifetime utility from housing is equal to \((1 - \beta)\frac{v(h^1)}{1 + \beta} + \beta \frac{v(h^2)}{1 + \beta}\). Alternatively, if this consumer decides to switch to a new housing level in period 0, he would be able to afford housing consumption \(\hat{h}^2 = (h^1 + \frac{h^2}{r} + \frac{\eta}{1 + r} - \eta) \frac{r}{1 + r}\) for the rest of his life, which would generate life-time housing value \(v(\hat{h}^2)\). Since \(v(h)\) is strictly concave, the consumer would be strictly better off to switch in period 0 if \((1 - \beta)h^1 + \beta h^2 \leq \hat{h}^2\), which is equivalent to \(r(1 - \frac{1}{1 + r}) \eta \leq (h^2 - h^1)(1 - \beta(1 + r))\). This inequality is satisfied if \(\beta(1 + r) < 1\), \(h^2 > h^1\) and \(r\) is sufficiently close to zero. However, as the choice of \(h^2\) is endogenous, it still remains to show that \(h^2\) is bounded away from \(h^1\) when \(r\) is arbitrary close to zero; whether it is true or not would depend on the properties of the instantaneous utility function \(u(c) + v(h)\).

\(^6\)The standard dynamic programming tools can be applied to characterize the properties of \(W(w)\).
\[ V_0(a) = \max_{c,a' > \frac{c}{r_r(h^2-y)}} \{ u(a + y - h^2 - \frac{a'}{1 + r} - \eta^1) + v^2 + \beta V^2(a') \}, \]  

where \( V^2(a) \) is the value of the consumer with high housing consumption

\[ V^2(a) = \max_{c,a' > \frac{c}{r_r(h^2-y)}} \{ u(a + y - h^2 - \frac{a'}{1 + r}) + v^2 + \beta V^2(a') \} \]

Such recursive representation helps to fully characterize the solution to the consumer’s decision problem (3) and describe the properties of the value function \( V(a) \) over its domain \((\hat{a}_\infty, +\infty)\).

**PROPOSITION 1 (Optimal choice without borrowing constraints)**

There exists \( \hat{a}_\infty \leq a_\infty^* < a_T^* < a_{T-1}^* < ... < a_1^* \) such that

(i) the consumer solving maximization problem (3) chooses \( h_t = h^1 \) for all \( t \geq 0 \)

if \( a_0 \in (\hat{a}_\infty, a_\infty^*) \), switches to \( h^2 \) in period \( t \) if \( a_0 \in [a_{t+1}^*, a_t^*) \), \( 1 \leq t \leq T \) (where \( a_{T+1}^* = a_\infty^* \)) and switches to \( h^2 \) right away if \( a_0 \in [a_T^*, +\infty) \);

(ii) the consumer’s optimal savings \( a'(a) \) obeys the following rules:

- \( a'(a) \in (\hat{a}_\infty, a_\infty^*) \) if \( a \in (\hat{a}_\infty, a_\infty^*) \);
- \( a'(a) \in (a_t^*, a_{t-1}^*) \) if \( a \in [a_{t+1}^*, a_t^*) \), \( 2 \leq t \leq T \);
- \( a'(a) \in [a_T^*, +\infty) \) if \( a \in [a_T^*, +\infty) \).

Figure 1 illustrates the properties of the consumer’s optimal savings rule and the indirect utility function summarized in Proposition 1. Consistently with the above intuition, the consumers with sufficiently high initial wealth \( (a_0 \in [a_T^*, +\infty) \) switch to high housing consumption right away, while the consumers with \( a_0 \in (\hat{a}_\infty, a_T^*) \) choose to consume low housing services in the beginning of their life. Those with 
\( a_0 \in (\hat{a}_\infty, a_\infty^*) \) never adjust their housing consumption; their wealth remains within 
\( (\hat{a}_\infty, a_\infty^*) \) for all \( t \geq 0 \). The consumers with intermediate initial wealth \( a_0 \in [a_\infty^*, a_T^*) \) eventually switch to high housing consumption; (ii) of Proposition 1 says that their wealth rises over time as long as their housing consumption remains low \( (a_{t+1} > a_t \) if \( h_{t+1} = h_t = h^1 \), for all \( t \geq 0 \)). Note also that the consumer’s indirect utility function is strictly concave to the left of \( a_\infty^* \) and to the right of \( a_T^* \). Thus if the consumer’s initial wealth falls into \((\hat{a}_\infty, a_\infty^*) \) or \((a_T^*, +\infty) \), the consumer is risk averse with respect to the small fair lotteries in his initial wealth level \( a_0 \).
Figure 1: Indirect utility function $V(a) = \max \{ V_\infty(a), V_0(a), V_1(a), V_2(a), \ldots \}$ (highlighted in yellow) and the optimal savings policy $a'(a)$ when $h_0 = h^1$.

2.3 When do consumption commitments magnify risk aversion?

As a partial case, it might happen that $V(a) = \max \{ V_\infty(a), V_0(a) \}$. The previous literature analyzing the effects of consumption commitments has studied the models in which this particular case would be the only solution to the agents’ problem, either because the static models were analyzed (Postlewaite et. al. 2006) or because there was no motive for postponing the moment of adjusting the level of commitment good (Chetty and Szeidl 2007). Figure 2 illustrates why this solution suggests that consumption commitments are likely to magnify risk aversion. If the adjustment cost where such that $\lim_{h'\to h} \eta^n(h, h') = 0$ (superindex $n$ stands for ‘no commitment’), the consumers would adjust their housing consumption continuously in response to the changes in the wealth level. Under some assumptions on $\eta^n(h, h')$, the consumers’ indirect life-time utility $V^n(a)$ would be strictly concave. To make the cases with and without commitment comparable, let us assume that $h^1$ is the optimal choice of housing at some $a^1 > 0$, $h^2$ is the optimal choice of housing at $a^2 > 0$ and $\eta^n(h, h')$ is such that $\eta^n(h^1, h^2) = \eta^1$. Then $V^n(a)$ is tangent to $V(a) = \max \{ V_\infty(a), V_0(a) \}$ at $a = a^1$ and $a = a^2$ as it is shown in Figure 2.

Since consumer’s attitudes towards risk are characterized by the curvature of the
Figure 2: The effect of consumption commitments on the agents’ indirect utility when $V(a) = \max \{V_\infty(a), V_0(a)\}$. $V^n(a)$ is the consumers’ indirect utility in the absence of consumption commitments ($\eta^n(h, h') \to 0$ as $h' \to h$); $h^1$ is the optimal choice of housing at $a^1$, $h^2$ is the optimal choice of housing at $a^2$ and $\eta^n(h, h')$ is such that $\eta^n(h^1, h^2) = \eta^1$.

value functions $V(a)$ and $V^n(a)$, it is obvious that housing commitments increase risk aversion of the consumers with $a_0 = a^1$ and $a_0 = a^2$. Under additional assumptions on $\eta^n(h, h')$ and the relative curvature of $u(c)$ and $v(h)$, it can be shown that the curvature of $V(a)$ exceeds the curvature of $V^n(a)$ for all $a \in (\hat{a}_\infty, a^*_\infty)$ and all $a \in (a^*_\infty, +\infty)$, thus implying that housing commitments magnify risk aversion independently of the initial wealth level. Intuitively, in the presence of housing commitments small changes in initial wealth should be absorbed only by changes in food consumption, which might generate substantial welfare losses if $u(c)$ has high risk aversion coefficient. Thus the existing theoretical studies came to the conclusion that consumption commitments make consumers more risk averse.\(^7\)

In contrast to this conclusion, we will argue in the remaining of the paper that in a dynamic environment the presence of bounded from zero adjustment costs (leading to endogenous consumption commitments) might create incentives for local risk taking if $\beta(1 + r) < 1$.

\(^7\)Some papers, e.g. Adam and Szeidl 2006, compare the model with commitments to the environment without adjustment costs; they also conclude that commitments magnify risk aversion, even though the result cannot be illustrated graphically using Figure 2 – the agent’s value in the absence of adjustment costs should exceed $V(a)$ for all $a \neq a^1$.\(^{12}\)
2.4 When do consumption commitments create incentives for risk taking?

Let us look more closely at the shape of $V(a)$ in the interval $(a^*_a, a^*_1)$ (consider the case when $V(a) > \max\{V_\infty(a), V_0(a)\}$ for some $a > a_\infty$). By the first statement of Proposition 1,

$$V'(a) = V'_{t+1}(a) = u'(c(a)) \text{ for all } a \in [a^*_t, a^*_1], 0 \leq t \leq T;$$

where $c(a)$ is the optimal food consumption of the agent with current wealth $a$. The first order conditions to (10), which hold with equality since consumers’ savings are interior (by (ii) of Proposition 1), imply that

$$u'(c(a)) = \beta(1 + r)V'(a') \text{ for all } a \in [a^*_t, a^*_1], 0 \leq t \leq T.$$  \hfill(14)

Recalling that $a'(a) \in [a^*_t, a^*_{t-1})$ by (ii) of Proposition 1, we can apply (i) of Proposition 1 once again and then, combining (14) with (13), we obtain that

$$V'(a) = \beta(1 + r)V'(a') \text{ for all } a \in [a^*_t, a^*_1], 0 \leq t \leq T.$$  \hfill(15)

This implies that if $\beta(1 + r) < 1$ then the consumer’s value function gets steeper while his wealth rises over time.

Figure 3 illustrates the dynamics of the consumer’s value and wealth starting from some $a_0 \in (a^*_\infty, a^*_T)$. Relationship (15) implies that

$$V'(a_0) < V'(a_1) < ... < V'(a_T) \quad \text{and} \quad a_0 < a_1 < ... < a_T.$$  

This creates some sort of convexity in the indirect utility function within the interval $(a^*_\infty, a^*_T)$. In particular, at time periods $t = 1...T-1$ the consumer would like to take a lottery randomizing between $a_{t-1}$ and $a_{t+1}$ with the expected payoff $a_t$. In general, this consumer would be better off if he is able to randomize over any combination of the wealth levels $\{a_0, a_1, ..., a_T\}$ experienced during the first $T$ periods of his life if the lottery’s expected payoff is equal to his current wealth level.

Note also that, due to time discreetness, the consumer would not necessarily be willing to take any fair lottery with payoffs from $(a^*_\infty, a^*_T)$; it is important that
Figure 3: The effect of consumption commitments on the agents’ indirect utility, $\beta(1 + r) < 1$

Figure 4: The shape of the indirect utility function for $\beta(1 + r) < 1$ in a continuous time model.
the outcomes of the lottery belong to the set \( \{ a_0, a_1, ..., a_T \} \). This is because the consumer’s indirect life-time utility is strictly concave within each interval \( (a^*_t, a^*_t) \), \( 0 \leq t \leq T \). However, all above arguments do not depend on the length of the time period. Thus we can always rewrite the model by shortening the time period and, thereby, allowing consumers to make adjustments more frequently (doing this would only make the model more realistic). Under such modification, the intervals within which \( V(a) \) is strictly concave would shrink. Intuition suggests that as the length of the period converges to zero (i.e. the model becomes a continuous time model), the consumer’s indirect utility function would be strictly convex in the interval \( (a^*_\infty, a^*_T) \) as it is shown in Figure 4. However, this result should still be established rigorously (either by rewriting the model in continuous time or by using some limiting argument).

Figure 5 and 6 illustrate how the shape of \( V(a) \) changes as \( \beta(1 + r) \) increases. Since Proposition 1 holds for any value of \( \beta(1 + r) \), relationship (15) implies that \( V'(a_0) = V'(a_1) = ... = V'(a_T) \) if \( \beta(1 + r) = 1 \) and \( V'(a_0) > V'(a_1) > ... > V'(a_T) \) if \( \beta(1 + r) > 1 \). By (ii) of Proposition 1, \( a_0 < a_1 < ... < a_T \) independently of the value of \( \beta(1 + r) \) (of course, the actual values of \( a_0, a_1, ..., a_T \) as well as \( T \) are different for different \( \beta(1 + r) \)). This suggests that as the time period shrinks, those consumer who save in order to switch to higher housing consumption become risk-neutral when \( \beta(1 + r) = 1 \) and risk-averse when \( \beta(1 + r) > 1 \). Note that even in the latter case the fact that \( V(a) \) is concave within \( (a^*_\infty, a^*_T) \) does not imply that these consumers are more risk averse than what they would be if there were no consumption commitments (in order to say something about the relative curvature of \( V(a) \) and \( V^n(a) \) additional assumptions would have to be imposed on \( \eta^n(h, h') \)).

In short, we have illustrated that if in the presence of consumption commitments some consumers decide to accumulate wealth for some time prior to switching to a higher level commitment good, these consumers’ risk aversion is not necessarily magnified by the presence of consumption commitments. We have shown that risk preferences of such consumers are determined by the value of \( \beta(1 + r) \): it turns out that for the values of \( \beta(1 + r) \) consistent with the typical predictions of the general equilibrium macroeconomic models (\( \beta(1 + r) = 1 \) in complete market economies and

---

8When we make the time period shorter, we need to adjust the model’s parameters correspondingly. For example, if we split each period into \( n \) equal periods, we need to set \( \beta = \beta^{1/n} \), \( \tilde{r} = (1 + r)^{1/n} - 1 \) and \( \tilde{y} = \frac{1 + r}{1 + \tilde{r}} y \).
Figure 5: The effect of consumption commitments on the agents’ indirect utility, $\beta(1 + r) = 1$.

Figure 6: The effect of consumption commitments on the agents’ indirect utility, $\beta(1 + r) > 1$. 

16
\( \beta (1 + r) < 1 \) in the models with uninsured risk, the consumers of this type become risk-neutral or even risk-lovers because they have committed to low housing value in the beginning of their life.

### 2.5 The effects of the borrowing constraints

In this section we relax one of the assumptions of our model and analyze the effects of borrowing constraints on consumers’ risk preferences. First of all, notice that in the absence of borrowing constraints the consumers’ indirect utility specified in (3) can be redefined as a function of the total life-time wealth \( w_0 = a_0 + \frac{1+r}{r} y \). Doing this would not have any impact on the shape of the value function illustrated on Figure 1; in order to plot it against life-time wealth \( w \), we simply need to shift the graph to the left by \( \frac{1+r}{r} y \). Correspondingly, the threshold levels of life-time wealth determining the consumers’ behavior can be found as \( w^*_\infty = a^*_\infty + \frac{1+r}{r} y \), \( w^*_t = a^*_t + \frac{1+r}{r} y \), ..., \( w^*_1 = a^*_1 + \frac{1+r}{r} y \).

To understand why borrowing constraints might have substantial impact on the consumer’s preferences for risk, consider the following example. Suppose that \( V(a) = \max \{ V_\infty(a), V_0(a) \} \), i.e. in the economy without borrowing constraints the consumer chooses to switch to high housing consumption right away if his period-0 life-time wealth \( a_0 + \frac{1+r}{r} y \) exceeds \( w^*_\infty = a^*_\infty + \frac{1+r}{r} y \) and decides to remain in a small house forever if \( a_0 + \frac{1+r}{r} y < w^*_\infty \). Note that only the total value of \( a_0 + \frac{1+r}{r} y \) is important for the consumer’s decision, and it does not matter how much of it is accounted by the initial wealth \( a_0 \) and how much by the present value of life-time income \( \frac{1+r}{r} y \).

Assume now that the consumer is allowed to save but cannot borrow. Then in the first period of his life the consumer can spend at most \( a_0 + y \) on food and housing consumption (including switching cost \( \eta_1 \)). Thus the consumer cannot switch to high housing consumption in the first period if \( a_0 + y < \eta_1 \). In particular, this would happen if a substantial part of the consumer’s life-time wealth is accounted by the future income stream \( (\frac{1+r}{r} y) > (w_0 - \eta_1) \) implies that \( a_0 + y < \eta_1 \). Thus it might happen that some of the (risk-averse) consumers with \( a_0 + \frac{1+r}{r} y > w^*_\infty \) would have to postpone adjusting their housing consumption and gradually accumulate sufficient funds in order to eventually pay the switching cost \( \eta_1 \). As it was discussed in the previous section, risk preferences of these consumers are determined by the value of
\( \beta(1 + r) \). That is why borrowing constraints may create incentives for risk taking in the presence of (bounded from zero) adjustment costs.

More formally, when borrowing constraints are imposed, we cannot simplify characterization of the solution to the optimization problem (2) by getting rid of period-by-period budget constraints. However, we still can rewrite (2) recursively similar to the way we have done it in (9)-(12). The indirect life-time utility \( \tilde{V}(a) \) (tilde indicates the presence of borrowing constraints) of the agent with current wealth \( a \) can be expressed as:

\[
\tilde{V}(a) = \max \{ \tilde{V}_\infty(a), \tilde{V}_0(a), \tilde{V}_1(a), \tilde{V}_2(a), \ldots \},
\]

where

\[
\tilde{V}_\infty(a) = \max_{c, a' > a} \left\{ u(a + y - h^1 - \frac{a'}{1 + r}) + v^1 + \beta \tilde{V}_\infty(a') \right\}
\]

(17)

\[
\tilde{V}_{T+1}(a) = \max_{c, a' > a} \left\{ u(a + y - h^1 - \frac{a'}{1 + r}) + v^1 + \beta \tilde{V}_{T}(a') \right\}, \quad T \geq 0
\]

(18)

\[
\tilde{V}_0(a) = \max_{c, a' > a} \left\{ u(a + y - h^2 - \frac{a'}{1 + r} - \eta^1) + v^2 + \beta \tilde{V}_2(a') \right\},
\]

(19)

\[
\tilde{V}_2(a) = \max_{c, a' > a} \left\{ u(a + y - h^2 - \frac{a'}{1 + r}) + v^2 + \beta \tilde{V}_2(a') \right\}.
\]

(20)

The only difference between the sets of recursive equations (9)-(12) and (17)-(20) is in the constraints imposed on the next-period wealth levels \( a' \). Without loss of generality, we will assume that the minimal wealth requirement \( a \) exceeds \( \tilde{a}_0 = \frac{1 + r}{r}(h^2 - y) + \eta^1 \) (e.g. if \( \tilde{a} = 0 \) then the agents’ life-time income should be sufficient to cover all the expenses associated with switching to higher housing consumption). Thus \( a' > \tilde{a} \) is the only constraint appearing in (17)-(20).

Due to the existence of this recursive representation, we can use the same tools as in the proof of Proposition 1 to characterize the properties of the consumers' optimal choice. The only novel part is that we need to establish single-crossing of \( \tilde{V}_i(a) \) and \( \tilde{V}_j(a) \) for all \( i, j \in \{ \infty, 0, 1, 2, \ldots \} \) (these functions are not affine transformations of \( W(a) \) defined in (7) any more) – but this is just a matter of mere technicality. Proposition 2 below states that in the presence of the borrowing constraints the agents’ optimal policy has exactly the same structure as in the model without borrowing constraints, only the threshold levels of wealth and the maximum number of
PROPOSITION 2 (Optimal choice with borrowing constraints)
There exist \( \bar{a} \leq \bar{a}^*_\infty < \bar{a}^*_T < \bar{a}^*_{T-1} < \ldots < \bar{a}^*_1 \) such that

(i) the consumer solving maximization problem (2) chooses \( h_t = h^1 \) for all \( t \geq 0 \)
if \( a_0 \in (\bar{a}, \bar{a}^*_\infty) \), switches to \( h^2 \) in period \( t \) if \( a_0 \in [\bar{a}^*_{t+1}, \bar{a}^*_t) \), \( 1 \leq t \leq T \) (where \( \bar{a}^*_{T+1} = \bar{a}^*_\infty \)) and switches to \( h^2 \) right away if \( a_0 \in [\bar{a}^*_1, +\infty) \);

(ii) the consumer’s optimal savings \( a'(a) \) obeys the following rules:
- \( a'(a) \in (\bar{a}, \bar{a}^*_\infty) \) if \( a \in (\bar{a}, \bar{a}^*_{\infty}) \);
- \( a'(a) \in (\bar{a}^*_t, \bar{a}^*_{t-1}) \) if \( a \in [\bar{a}^*_{t+1}, \bar{a}^*_t) \), \( 2 \leq t \leq T \);
- \( a'(a) \in [\bar{a}^*_1, +\infty) \) if \( a \in [\bar{a}^*_2, \bar{a}^*_1) \).

A preliminary numerical example in the next section illustrates that the presence of the borrowing constraints might indeed have substantial effects on consumers’ risk preferences.

2.6 An example: \( u(c) \) is CRRA

Suppose that the instantaneous utility function is given by:

\[
 u(c) + v(h) = \frac{c^{1-\sigma} - 1}{1-\sigma} + \mu \frac{h^{1-\sigma} - 1}{1-\sigma},
\]  

(21)

where \( \sigma > 1 \) and \( \mu > 0 \). The parameter \( \mu \) indicates how much consumers value housing consumption relative to food consumption. For instance, in the economy without borrowing constraints and switching costs \( \mu \) and \( \sigma \) would uniquely determine the share of the life-time wealth spent on housing consumption (which then can be expressed as \( \frac{1}{1+\mu} \)). For this utility function, optimization problem (7) has a closed-form solution and thus we can do a number of analytical comparative static exercises. For instance, we derive the conditions on the parameters of the model under which the interval \((a^*_\infty, a^*_1)\) is non-empty. Such conditions are useful to obtain because only inside this interval consumers’ risk attitudes can be different from what has been suggested in the previous literature.
**PROPOSITION 3**

Suppose that the instantaneous utility function is given by (21). Then \( V(a) > \max\{V_{\infty}(a), V_0(a)\} \) for some \( a \) if and only if \( \mu > \underline{\mu} > 0 \), where \( V(a) \) is the indirect utility function obtained by solving (3), \( V_{\infty}(a) \) and \( V_0(a) \) are defined in (9) and (11) respectively and \( \underline{\mu} \) is determined by the rest of the model’s parameters.

Intuitively, if \( \mu \) is large enough then even the agents with low life-time wealth would like to consume high level of housing. However, switching to \( h^2 \) right away might be too expensive for them, so they decide to stay in a small house for a while in order to reduce the present value of the switching cost and the life-time expenses on housing. It would probably be more intuitive to reformulate Proposition 3 by providing necessary and sufficient conditions on \( r \) rather than on \( \mu \) (to make this result consistent with the discussion in footnote 3 of section 2.2); but it appears to be more difficult to do this (though I am still trying). Numerical exercises described below seem to indicate that our earlier intuition was correct and the interval \((a_{\infty}^*, a_1^*)\) (within which \( V(a) > \max\{\max\{V_{\infty}(a), V_0(a)\}\) disappears if the interest rate \( r \) is small enough.

Proposition 3 poses a question of whether in the absence of the borrowing constraints the interval \((a_{\infty}^*, a_1^*)\) would actually be nonempty for a model with reasonable parameter values. We address this question in the following numerical exercise. We set \( h^1 = 1, \ h^2 = 2 \) and \( y = 3.5 \) (only their relative values matter). The adjustment cost is equal to 15% of life-time expenses on a small house and the preference parameters are \( \beta = 0.98 \) and \( \sigma = 1.5 \). Then we compute \( \underline{\mu} \) from Proposition 3 for different values of the interest rate \( r \). The results are reported in the first column of Table 1. As we can see, \( \mu \) is decreasing in \( r \). This also implies that for a given level of \( \mu \) the interval \((a_{\infty}^*, a_1^*)\) disappears as \( r \) declines (e.g. if \( \mu = 2 \) consumers with intermediate wealth levels wait before switching to \( h^2 \) when \( r = 0.015 \) or \( r = 0.01 \); but if the interest rate is as low as \( r = 0.05 \) then all the consumers either remain with \( h^1 \) forever or switch to \( h^2 \) right away).

To understand whether these values of \( \underline{\mu} \) are sensible, we compute in the second column of Table 1 the share of life-time wealth spent on housing by those consumers whose initial wealth falls into the interval \((a_{\infty}^*, a_1^*)\). Chetty and Saeid (2006) report that in the data this ratio is around 50%. In the model, when the interest rates are low and \( \mu \) is sufficiently high to result in \( a_{\infty}^* < a_1^* \), the consumers with initial
wealth in the interval \((a^*_\infty, a^*_1)\) spend way too much resources on housing (when \(r = 0.01\) the housing share for these consumers is at least 72%; it rises to 89% as \(r\) drops to 0.005). This suggests that studied in this paper mechanism, through which consumption commitments can reduce risk aversion, might be relevant only if the interest rate is high enough. Indeed, for \(r = 0.015\), the housing share for the consumers with \(a_0 = a^*_\infty\) gets close to 50%. In his case \(\beta(1 + r)\) is very close to 1 (see the last column of Table 1) and thus the consumers with \(a_0 \in (a^*_\infty, a^*_1)\) are almost risk neutral.

It is also important to point out that the shares reported in column 2 of Table 1 are very close to the ones that would be obtained if consumers were able to choose any housing level (not necessarily \(h^2\)) after the switching cost \(\eta^1\) is paid. This indicates that in our economy the major motive for postponing housing adjustment is driven by consumer’s willingness to allocate less than \(1 + r h^2\) but more than \(1 + r h^1\) to life-time housing expenses. Such motive should obviously disappear if we endogenize the choice of \(h^2\), and thus it is very likely that in a model with such modification the “reversal” of risk preferences described in section 2.4 would never occur (because the interval \((a^*_\infty, a^*_1)\) would disappear).

In contrast, in the model with borrowing constraints, consumers may want to postpone housing adjustment even if \(h^2\) can be chosen endogenously because they might not be able to pay the switching cost right away. This effect of the borrowing constraints is illustrated in Figures 8 and 7.
Comparison of the value functions with and without borrowing constraints, $r = 0.01$

Figure 7: The effect of borrowing constraints on the agents’ indirect utility, $r = 0.01$.

Value functions and risk premia for different interest rates

Figure 8: Risk attitudes with and without borrowing constraints, $r = 0.01$ and $r = 0.007$. 

22
We set $\mu = 1$ (so that the ratio of housing expenses is around 50%) and recompute the model with non-negative restriction on consumers’ wealth ($a \geq 0$). Figure 7 plots the consumer’s value functions for the economies with and without the borrowing constraints when $r = 0.01$. As we already know, for such combination of $r$ and $\mu$ no consumers would want to delay housing adjustment if they are able to borrow. Figure 7 also shows that for these parameter values all the consumers with $a_0 \geq 0$ switch to $h^2$ in period 0. However, when the constraint is binding and wealth level is low, the consumers cannot afford to pay the switching cost (it is 4 times bigger than their per period income). We can see that the value function in this region becomes convex and the consumers become risk lovers.

The bottom plots of Figure 8 quantify consumers’ risk attitudes. They illustrate risk premia that would make consumers indifferent between keeping his initial wealth safe and taking a lottery with 10% standard deviation. For instance, when $r = 0.01$ and borrowing is allowed, the consumers with wealth levels within $(0, a^*)$ would take such a lottery only if it pays around 1% premium. However, in the economy with borrowing constraints these consumers would be willing to pay the same amount in order to be able to invest all their initial wealth into such lottery.

Finally, also notice that outside of the interval $(0, a^*_\infty)$ the required risk premia are higher in the model with borrowing constraints. This is consistent with the standard predictions of a neoclassical growth model, in which imposing borrowing limits increases the cost of uninsured risk (e.g. Aiyagari 1994). That is why the borrowing constraints have an unusual effect only if they induce individuals to simultaneously choose increasing wealth and decreasing consumption profiles, which would be hard to derive in a model without commitments.

3 Final Remarks

The current version of the paper should be improved in several dimensions:

1. Due to imposed simplifying assumptions (exogenous $h^2$) it might happen that $V(a) > \{V_\infty(a), V_0(a)\}$ in the model without borrowing constraints. As it is discussed in sections 2.2 and 2.6, this case is likely to disappear if we allow consumers to choose $h^2$ endogenously (since one of the important motives for postponing switching to $h^2$ in our model is adjusting the level of total life-
time expenditures on housing). It should not be difficult to incorporate this decision into the model. Obviously, the main argument will still go through: some agents would be borrowing constrained and will need to save before adjusting their housing level. The technical difficulty would be to prove single crossing of $V_0(a)$ and $V_1(a)$, but we hope it will not be impossible.

2. Since the model is formulated in discrete time, the value function $V(a)$ is not strictly concave within $(a^*_\infty, a^*_1)$. As we argued in section 2.3, the intervals within which this function is strictly concave get shorter as we shorten the time period (and adjust all the parameter values correspondingly). Nevertheless, to make our results sharper, it would still be useful to either rewrite the model in continuous time or use some limiting argument.

3. We probably should be more specific about why consumers are initially committed to low housing expenses. Even though we address this question in footnote 3 and discuss it in the introduction, we probably should be modelling it in a straightforward way.

4. It would be useful to support our findings by some empirical evidence.

5. It is also worth pointing out that the mechanism described in the paper is not restricted to the models with commitment goods. It would arise in many discrete choice models (e.g. a standard occupational choice model) in an agent making discrete choice is risk averse. For example, when incorporated into an occupational choice model such mechanism would suggest that workers are more willing to get jobs with risky wage profiles prior to starting up their own businesses.
4 References


5 Appendix

Proof of Proposition 1
Let us denote by $a_t^*$ the wealth level at which $V_t(a)$ and $V_{t-1}(a)$ intersect; and by $a_t^\infty$ the wealth level at which $V_t(a)$ and $V_\infty(a)$ intersect. Since all these functions are affine transformation of a strictly concave and bounded from above function $W(a)$, all the intersection points are unique and $V_{t-1}(a) > V_t(a)$ as well as $V_t(a) > V_\infty(a)$ for sufficiently large $a$. Denote also by $a_t'(a)$ the optimal savings rule for the recursive problem (10). Then

$$V_{t+1}(a) = u(a + y - h^1 - \frac{a_{t+1}'(a)}{1+r}) + v^1 + \beta V_t(a_{t+1}'(a)), \quad t \geq 0$$

(22)

It is useful to establish the following property of the optimal saving policies $a_t'(a)$, $t \geq 1$:

**Lemma 1** The optimal saving rules $a_t'(a)$, $t \geq 1$ obtained in the set set of recursive problems (10) are such that:

(a) $a_t'(a) > a_t^*$ for all $a \geq a_t^* + 1$;

(b) $a_{t+1}'(a) < a_t^*$ for all $a \leq a_t^* + 1$.

**Proof:** Since each of $V_t(a)$, $t \geq 0$ is strictly concave, the policy functions $a_t'(a)$ are strictly increasing. Thus it is sufficient to verify that $a_t'(a_t^* + 1) > a_t^*$ and $a_{t+1}'(a_t^* + 1) < a_t^*$. We do it by contradiction. Suppose that $a_t'(a_t^* + 1) \leq a_t^*$. Then, by definition of $a_t^*$, $V_t(a_t'(a_t^* + 1)) \leq V_{t-1}(a_t'(a_t^* + 1))$. At the same time, $a_t^*(a)$ is a feasible saving rule for the consumer solving maximization problem (10) for $T = t$ (since $a_t^*(a) \geq \hat{a}_{t-1} > \hat{a}_t$). Thus

$$V_{t+1}(a_t^* + 1) = u(a_{t+1}^* + y - h^1 - \frac{a_{t+1}'(a_t^* + 1)}{1+r}) + v^1 + \beta V_t(a_{t+1}'(a_t^* + 1)) \geq$$

$$\geq u(a_{t+1}^* + y - h^1 - \frac{a_{t+1}'(a_t^* + 1)}{1+r}) + v^1 + \beta V_{t-1}(a_t'(a_t^* + 1)) = V_t(a_t^* + 1),$$

which contradicts to the definition of $a_{t+1}^*$. The first inequality in the above expression is strict because $V_{t+1}(a)$ is flatter than $V_t(a)$ at $a_{t+1}^*$, so the saving decision $a_t'(a_t^* + 1)$ is strictly suboptimal. This proves (i). The inequality $a_{t+1}'(a_t^* + 1) < a_t^*$ can
be easily verified in a similar way. □

Clearly, Lemma 1 would be useful in proving (ii) of Proposition 1. But before we can apply it, we need to establish one more result:

**Lemma 2** The sequence of value functions \( \{V_t(a)\}_{t=0}^{+\infty} \) defined by (10) has the following property: if \( V_t(a) < \max\{V_\infty(a), V_{t-1}(a)\} \) for all \( a \geq \tilde{a}_t \) then \( V_{t+1}(a) < \max\{V_\infty(a), V_{t}(a)\} \) for all \( a \geq \tilde{a}_{t+1} \).

**Proof:** Suppose that the opposite is true and there exist \( t \geq 1 \) such that \( V_t(a) < \max\{V_\infty(a), V_{t-1}(a)\} \) for all \( a \geq \tilde{a}_t \) but \( V_{t+1}(a) \geq \max\{V_\infty(a), V_{t}(a)\} \) for some \( a \). This implies that \( a^*_t \geq a^*_t > a^*_t \) (see Figure 9). Correspondingly, \( V_{t+1}(a^*_t) > V_\infty(a^*_t) \).

First, notice that \( a^*_t(a^*_t+1) > a^*_t+1 \). If the opposite is true then \( V_{t+1}(a^*_t+1) \geq V_t(a^*_t+1) \) and thus \( V_{t+2}(a^*_t(a^*_t+1)) \geq V_{t+1}(a^*_t(a^*_t+1)) \). Since \( V_{t+2}(a) \) and \( V_{t+1}(a) \) have unique intersection and \( V_{t+1}(a) > V_{t+2}(a) \) for sufficiently large \( a \), \( V_{t+2}(a) \geq V_{t+1}(a) \geq V_t(a) \) for all \( a \leq a^*_t+1 \). By induction, it follows that \( V_{t+k}(a) \geq V_{t+1}(a) \) for all \( a \leq a^*_t \) and \( k \geq 2 \), which leads to contradiction because \( \lim_{k \to +\infty} V_{t+k}(a) = V_\infty(a) \) and \( V_{t+1}(a^*_t+1) > V_\infty(a^*_t+1) \).
Second, \( a'_{t+1}(a^*_{t+1}) > a^*_{t+1} \) implies that

\[
V_{t+1}(a^*_{t+1}) = u(a^*_{t+1} + y - h^1 - \frac{a'_{t+1}(a^*_{t+1})}{1 + r}) + v^1 + \beta V_t(a'_{t+1}(a^*_{t+1})) < \\
< u(a^*_{t+1} + y - h^1 - \frac{a'_{t+1}(a^*_{t+1})}{1 + r}) + v^1 + \beta V_{t-1}(a'_{t+1}(a^*_{t+1})) \leq \\
\leq u(a^*_{t+1} + y - h^1 - \frac{a'_1(a^*_{t+1})}{1 + r}) + v^1 + \beta V_{t-1}(a'_1(a^*_{t+1})) = V_t(a^*_{t+1}),
\]

which contradicts to the definition of \( a^*_{t+1} \) \( (V_{t+1}(a^*_{t+1}) = V_t(a^*_{t+1})) \). □

In words, Lemma 2 implies that if there are no wealth levels at which an agent would want to wait for \( t + 1 \) periods, then no one would want to wait for more than \( t \) periods. Therefore, if we denote the maximum waiting time by \( T \) then \( V(a) = \max\{V_\infty(a), V_0(a), ..., V_T(a)\} \) and (i) of Proposition 1 is proven (with \( a^*_\infty = a^*_T \)).

To complete the proof of (ii) of Proposition 1, we remain to verify that \( a'(a) \in (\tilde{a}_\infty, a^*_\infty) \) if \( a \in (\tilde{a}_\infty, a^*_\infty) \) (the rest follows directly from Lemma 1). This result is trivial if \( \beta(1 + r) \leq 1 \) since then the optimal saving policy \( a'_\infty(a) \) is such that \( a'\infty(a) \leq a \) for all \( a > \tilde{a}_\infty \) (since the first order and envelope condition imply that \( V'_\infty(a) = \beta(1 + r)V'_\infty(a'_\infty(a)) \)). If \( \beta(1 + r) > 1 \) then \( a'_\infty(a) > a^*_\infty \) if \( a \) is sufficiently close to \( a^*_\infty \). But for this case it is easy to see that \( a^*_\infty = \tilde{a}_\infty \), i.e. \( V_\infty(a) < \max\{V_0(a), V_1(a), ...\} \) for all \( a > \tilde{a}_\infty \).

Since \( a'_\infty(a) > a \) if \( \beta(1 + r) > 1 \), the agent following the savings policy \( a'_\infty(a) \) would reach \( a^*_\infty \) in a finite number of periods from any initial wealth level \( a_0 < a^*_\infty \).

Denote this number of periods by \( N \). It is trivial to check that for this agent \( V_\infty(a_0) < V_N(a_0) \) since \( V_\infty(a^*_\infty) = V_0(a^*_\infty) \) and \( V_0(a) \) is steeper than \( V_\infty(a) \) at \( a^*_\infty \).

This completes the proof of Proposition 1. ■

**Proof of Propositions 2 and 3** are to be added later.