Job Market Signaling and Employer Learning∗

PRELIMINARY VERSION

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Abstract

This paper proposes a job-market signaling model where workers use education to signal their type, as in Spence (1973), but firms are able to learn the ability of their employees over time. If learning is sufficiently efficient, we find that the intuitive criterion does not select the 'Riley' separating equilibrium. In general, for separating equilibrium outcomes, the education gap between low and high types increases with the speed of learning. Introducing search frictions into the model, we show that asymmetric information leads to inefficient separations so that signaling yields employment and welfare gains.

1 Introduction

Spence’s (1973) signaling model ranks among the most influential contribution to labor economics. It has provided the seminal impulse for ground-breaking lines of research in game theory and econometrics. On the one hand, the existence of multiple equilibria has been a motivation for the vast literature on refinement concepts. On the other hand, the difficulty to identify the effects of signaling empirically has stimulated econometricians to devise ingenious tests about unobservable characteristics. In spite of their common roots, these two strands of literature have followed diverging paths. Whereas the theoretical literature focuses on the signaling stage, econometric research places the emphasis on the learning process through which employers elicit the ability of their employees. This paper proposes to reconcile these two views by embedding the signaling model of Spence (1973) into a dynamic framework with Bayesian learning on the side of firms.

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As in Spence (1973), workers of different abilities can acquire education before entering the labor market. For simplicity, we focus on the signaling content of education and thus consider that the productivity of a given worker is uncorrelated with his educational achievement. Whereas workers know their average productivity with certainty, firms are a priori unable to determine the actual ability of job applicants. This is why high types have an incentive to acquire education in order to signal their capacity. They may prefer, however, to save on education costs by choosing to reveal their ability while being on-the-job. This alternative strategy is particularly attractive if there is a strong correlation between the worker’s characteristic and his observable performance. Standard signaling models do not take into account this countervailing incentive because they generally assume that all the information available to employers is collected prior to the entry into the labor market. Instead, we combine the asymmetric information problem with a signal extraction process whereby workers’ types are gradually revealed over time. Our model therefore proposes a synthesis of the signaling and job matching literatures. We find that the two dimensions interact with each other: the equilibria of the signaling game are affected by the learning process and the labor market outcomes are influenced by the signaling stage.

In order to make the model suitable for aggregation, we adopt Spence’s stylized assumption according to which workers’ abilities are either high or low. Although this simplification is not crucial and can easily be relaxed in the original Spence model,\(^1\) it plays a key role in our enlarged set-up as it allows us to characterize the cross-sectional distribution of beliefs in closed-form. Our analysis also bears similarities to Moscarini’s (2005) matching model, since we assume that jobs’ outputs are randomly drawn from an exogenous Gaussian distribution. As Moscarini (2005), we model the learning process in continuous time. Our problem differs from the standard job matching model of Jovanovic (1979) because uncertainty is not anymore match-specific but instead worker-specific. One contribution of the paper is to show that the matching model with asymmetric information remains tractable. An appropriate change of variable enables us to analytically characterize pooling equilibria and to derive new findings about the robustness and properties of separating equilibria.

First of all, we show that, in stark contrast with Spence’s model, the intuitive criterion proposed by Cho and Kreps (1987) does not necessarily rule out all but one separating equilibrium. When learning is sufficiently efficient, workers of the high type prefer working until their ability is recognized instead of paying the education costs in order to signal their type upfront. This differs from models without learning where all pooling equilibria fail the intuitive criterion. The technical reason is that low and high types have different asset values even when pooling is the equilibrium outcome. The gap increases with the speed of learning as high and low types become respectively more and less

\(^1\)This relaxation, though, leads to well-known difficulties in the application of the Intuitive Criterion.
optimistic about their future prospects. Thus the incentive of low ability workers to send misleading signals increases with the speed of employer learning.

Accordingly, the separating level of education is higher in occupations where employers can better infer the ability of their employees. Surprisingly enough, Riley (1979b) reached the opposite conclusion in his seminal paper about educational screening and employer learning. We argue that the divergence arises due to the differences in the equilibrium concepts. Whereas Riley characterizes the informational equilibrium of the economy, we instead focus on the Perfect Bayesian (Nash) equilibrium that is consistent with the intuitive criterion. As discussed by Riley (1979b) himself in his companion paper, informational equilibria are not non-cooperative Nash equilibria. Most importantly, informational equilibria are such that all asymmetric information is purged from the learning process. Our results therefore suggest that the vast empirical literature based on Riley’s (1979b) model has tested the implications of signaling for the relationship between education and symmetric rather than asymmetric uncertainty.

After having characterized the pooling and separating equilibria in competitive labor markets, we bridge the gap with modern theories by introducing search frictions. The existence of frictions imply that workers and firms have to share a quasi-rent. As commonly assumed in the search literature, we consider wages set by Nash bargaining. Furthermore, we restrict our attention to Markov strategies, so that prospective employers do not base their offers on the labor market history of job applicants. Accordingly, unemployed workers are homogenous from the firm’s point of view. Under these two premises, we show that pooling equilibria exhibit socially inefficient job destructions. Given that high ability workers cannot credibly signal their type, they prefer to leave their jobs when their employers entertain low belief about their capacity because of an unlucky history of idiosyncratic shocks. Furthermore, it is in the interest of low types to mimic the behavior of high types in order to avoid detection. Given that low types are more likely to breach the destruction threshold, endogenous job destruction induces a negative selection effect which lowers the average ability among job seekers. Pooling equilibria are therefore characterized by a higher rate of job destruction and a smaller average productivity of new jobs. These two effects obviously concur to increase the equilibrium rate of unemployment. Hence, job market signaling generates positive social returns when search frictions are taken into account.

The paper proceeds as follows. Section 2 characterizes the pooling equilibrium. It describes the sig-

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2The related literature is too large to be comprehensively referenced. As arbitrary entries, see the seminal paper by Farber and Gibbons (1996) or Altonji and Pierret (2001), Kaymak (2006) and Lange (2006) for more recent contributions.

3On this issue, a recent working paper by Haberlmaz (2006) proposes a partial equilibrium model which also underlies the ambiguity of the relationship between the value of job market signaling and the speed of employer learning.
nal extraction problem and proposes closed-form solutions for the workers’ asset values. In Section 3, we analyze the conditions under which the intuitive criterion bites and how the separating level of education depends on the learning efficiency. Search frictions are introduced in Section 5. It contains a derivation of the wage schedule and an analysis of the job separation game. Section 6 closes the model so as to endogenize the equilibrium rate of unemployment. Section 7 concludes. The proofs of Propositions and Lemmata are relegated to the Appendix.

2 The pooling equilibrium

2.1 Signal Extraction

We restrict our attention to the case where there are only two types of workers. For simplicity, we also assume that the expected productivity of a given worker remains constant through time$^4$ and is equal to $\mu_h$ for the high type or to $\mu_l < \mu_h$ for the low type. Output realizations are random draws from a Gaussian distribution centered on the worker’s average productivity. The variance of the shocks $\sigma$ is common knowledge and independent of the worker’s type. Information is asymmetric because the worker knows his ability with certainty, while the employer has to infer it from the observation of realized outputs. Obviously, the employer’s ability to identify the mean of the output distribution is hindered by the noise due to the idiosyncratic shocks.

In order to characterize the optimal filtering problem, we notice that the cumulative output $X_t$ of a match of duration $t$ follows a Brownian motion with drift, so that

$$dX_t = \mu_i dt + \sigma dZ_t,$$

where $dZ_t$ is the increment of a standard Brownian motion and, depending on the worker’s ability, $i$ equals $h$ or $l$. The cumulative output $\langle X_t \rangle$ is observed by both parties. The employer uses the filtration $\mathcal{F}_t^X$ generated by the output sample path to revise his belief about the worker’s average productivity. Starting from a prior $p_0$ equals to the fraction of high ability workers in the population, the employer applies Bayes’ rule to update his belief $p_t \equiv \Pr(\mu = \mu_h \mid \mathcal{F}_t^X)$. His posterior is therefore given by

$$p(\langle X_t, t \rangle \mid p_0) = \frac{p_0 e^{-(X_t - \mu_h)^2 / 2\sigma^2 t}}{p_0 e^{-(X_t - \mu_h)^2 / 2\sigma^2 t} + (1 - p_0) e^{-(X_t - \mu_l)^2 / 2\sigma^2 t}}.$$

$^4$It would be reasonable to assume that workers accumulate general human capital. However, this would substantially complicate the aggregation procedure without adding substantial insights.
The derivations are simplified by the following change of variable $P_t \equiv p_t/(1-p_t)$. $P_t$ is the ratio of "good" to "bad" belief. Since $p_t$ is defined over $(0, 1)$, $P_t$ takes values over the positive real line. It immediately follows from (1) that

$$P_t(X_t, t|P_0) = P_0 e^{\frac{X_t(\mu_h-\mu_l)}{\sigma^2} - \frac{(\sigma_h^2-\sigma_l^2)t}{2\sigma^2}}.$$  

(2)

According to Ito’s lemma, the law of motion of the posterior belief ratio is

$$dP_t(X_t, t|P_0) = \frac{\partial P_t(X_t, t|P_0)}{\partial X_t}dX_t + \frac{\delta^2 P_t(X_t, t|P_0)}{2}dP_t + \frac{\partial P_t(X_t, t|P_0)}{\partial t}dt$$

$$= P_t(X_t, t|P_0) \left( \frac{\delta}{\sigma} \right) [dX_t - \mu_l dt],$$  

(3)

where $s \equiv (\mu_h - \mu_l)/\sigma$ is the signal/noise ratio of output. The higher $s$, the more efficient is the learning process. Reinserting into (3) the workers’ beliefs about the law of motion of $X_t$ yields the following stochastic differential equations:

(i) **High ability worker**: Substituting $dX_t = \mu_h dt + \sigma dZ_t$ yields

$$dP_t = P_t s (sdt + dZ_t),$$

(ii) **Low ability worker**: Substituting $dX_t = \mu_l dt + \sigma dZ_t$ yields

$$dP_t = P_t s dt.$$

Given that the signal/noise ratio $s$ is a positive constant, the posterior belief $P_t$ increases with time when the worker’s ability is high. To the contrary, $P_t$ follows a martingale when the worker is of the low type.\(^6\) In both cases, an increase in $\sigma$ lowers the variance of beliefs since larger idiosyncratic shocks hamper signal extraction.

### 2.2 Asset values

The workers’ output realizations are publicly observed. Employers compete a la Bertrand so that wages $w(p)$ are equal to the expected worker’s productivity: $p\mu_h + (1-p)\mu_l$. Although high and

\(^5\)Notice that the law of motion’s for employers beliefs is obtained substituting $dX_t = (p_t\mu_h + (1-p_t)\mu_l) dt + \sigma dZ_t$, where $Z_t$ is a standard Brownian motion with respect to the filtration $\{\mathcal{F}_t\}$, so that

$$dP_t = P_t s \left( \frac{P_t}{1+P_t} \right) dt + dZ_t.$$  

\(^6\)It may be surprising that the belief ratio $P_t$ does not drift downward when the worker is of the low type. But one has to remember that the belief ratio is a concave function of the posterior belief $p_t$. Accordingly, reversing the change-of-variable shows that the belief $p_t$ is a strict supermartingale when he worker’s ability is low.
low ability workers earn the same wage for a given cumulative output realization, their asset values
differ. This is because high ability workers are more optimistic about future prospects. Moreover,
workers’ expectations are also different from those of their employers. Accordingly, three different
asset values are associated to the same cumulative output realization. Using the law of motions of
beliefs characterized above, we are in a position to derive the two Hamilton-Jacobi-Bellman (HJB
henceforth) equations satisfied by the workers’ asset values. Assuming that workers are risk-neutral
and that they discount the future at rate \( r \), we obtain

\[
\begin{align*}
  rW_h(P) &= w(P) + Ps^2W'_h(P) + \frac{(Ps)^2}{2}W''_h(P) - \delta W_h(P) \\
  rW_l(P) &= w(P) + \frac{(Ps)^2}{2}W''_l(P) - \delta W_l(P)
\end{align*}
\]

where \( W_h(P) \) and \( W_l(P) \) denote the asset values for a high and low ability worker, respectively. The
parameter \( \delta \) measures the rate at which workers leave the labor market.\(^7\) The closed-form solutions
of these two ordinary differential equations are detailed in the following proposition.

**Proposition 1.** The expected lifetime income of a low-ability worker as a function of the belief ratio
\( P \) is

\[
W_l(P) = \frac{2(\mu_h - \mu_l)}{s^2(\alpha^+ - \alpha^-)} \left( P^{\alpha^-} \int_0^P \frac{1}{(1 + x)x^{\alpha^-}} \, dx + P^{\alpha^+} \int_P^{\infty} \frac{1}{(1 + x)x^{\alpha^+}} \, dx \right) + \frac{\mu_l}{r + \delta},
\]

where \( \alpha^- \) and \( \alpha^+ \) are the negative and positive roots of the quadratic equation

\[
\alpha(\alpha - 1) s^2 - r - \delta = 0.
\]

The expected lifetime income of a high-ability worker as a function of the belief ratio \( P \) is

\[
W_h(P) = \frac{2(\mu_h - \mu_l)}{s^2(\gamma^+ - \gamma^-)} \left( P^{\gamma^-} \int_0^P \frac{1}{(1 + x)x^{\gamma^-}} \, dx + P^{\gamma^+} \int_P^{\infty} \frac{1}{(1 + x)x^{\gamma^+}} \, dx \right) + \frac{\mu_l}{r + \delta},
\]

where \( \gamma^- \) and \( \gamma^+ \) are the negative and positive roots of the quadratic equation

\[
\gamma(\gamma - 1) s^2 + \gamma s^2 - r - \delta = 0.
\]

\(^7\)Assuming that \( \delta \) is an increasing function of labor market experience would improve the realism the model. Nonetheless, we leave it to further research because such a specification would greatly complicate the derivations by making the asset values non-stationary.
Wages are a linear function of beliefs since \( w(p) = p(\mu_h - \mu_l) + \mu_l \). Thus the wage distribution follows from the distribution of beliefs by a location transformation. In order to derive the ergodic density, we notice that the Kolmogorov forward equation characterizes how the cross-sectional density of beliefs \( f(p) \) evolves through time, so that

\[
\frac{df(p)}{dt} = \frac{d^2}{dp^2} \left( \frac{p^2 (1-p)^2 s^2}{2} f(p) \right) - \delta f(p).
\]

The first term on the right-hand side deducts, for any given belief, the outflows from the inflows. The second term takes into account the fact that workers leave the labor market at rate \( \delta \). The ergodic density is derived imposing the stationarity condition: \( df(p)/dt = 0 \). As shown in the Appendix, the general solution involves solving for two constants of integration. They can be pinned down by the normalization \( \int_0^1 f(p) dp = 1 \) and the continuity requirement \( f(p^-_0) = f(p^+_0) \).

**Proposition 2.** The ergodic distribution of beliefs is given by

\[
f(p) = \begin{cases} 
K_0 p^{-3/2+\sqrt{1/4+25/s^2}} (1-p)^{-3/2-\sqrt{1/4+25/s^2}}; & \text{if } p \in (0, p_0) \\
C_1 p^{-3/2-\sqrt{1/4+25/s^2}} (1-p)^{-3/2+\sqrt{1/4+25/s^2}}; & \text{if } p \in [p_0, 1) 
\end{cases}
\]
where $K_{0f}$ and $C_{1f}$ solve two linearly independent equations given in Appendix.

The ergodic distribution is piecewise and composed of two Beta functions. Depending on the values of the parameters, the wage distribution can be either U-shaped or hump-shaped. As documented in the data, the second case is the most realistic. It occurs for sufficiently high values of the ratio $\delta/s^2$. Then the right-tail of the distribution is decreasing at a slower rate than the Gaussian distribution from which underlying shocks are sampled. Hence, the model can replicate the heavy tail property of empirical wage distributions. As noticed by Moscarini (2005), another specific implication of the learning model is that the dispersion of beliefs is decreasing in the variance of shocks. This is of course different from the predictions of models without signal extraction where the uncertainty of the economic environment naturally generates more inequality. To the contrary, when firms have to filter the noise from the observations, a higher degree of uncertainty reduces the rate at which information is acquired. This is why the inertia in belief revisions is stronger when there is more noise, which in turn implies that wage dispersion is decreasing in the degree of uncertainty.

3 Employer Learning and the Intuitive Criterion

We now allow workers to signal their types by acquiring education before entering the labor market. Given that workers are also able to reveal their ability after the signaling stage, our framework encompasses the job market-signaling model of Spence (1973). As in the previous section, nature assigns a productivity $\mu \in \{\mu_l, \mu_h\}$ to the worker (sender), with $\mu_h > \mu_l > 0$. The worker then chooses an education level $e \in [0, +\infty)$. To isolate the effect of signaling, we assume that education does not increase the labor productivity of the worker. Thus its only use is to signal the worker’s ability, which is initially unobserved by the industry (receiver).

It is well known that one can find a plethora of Perfect Bayesian Equilibria (hereafter PBE) in Spence’s (1973) model. The refinement concept known as the Intuitive criterion strongly reinforces the model’s predictive power. As shown by Cho and Kreps (1987), it rules out all but a single separating equilibrium outcome in the signaling game. The purpose of this section is to show that this does not hold true when signal extraction also takes place on-the-job. More precisely, we prove that all pooling PBE also satisfy the Intuitive criterion when the signal/noise ratio exceeds a given threshold.

Let the cost function $c : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ specify the cost of acquiring education. That is, $c(e, \mu)$ is the cost that a worker with innate productivity $\mu$ has to pay in order to acquire education level

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8Spence’s (1973) model corresponds to the particular case where the signal/noise ratio $s = 0$. 

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8
The cost function is strictly increasing \( (c_e(e, \mu) > 0) \) and strictly convex \( (c_{ee}(e, \mu) > 0) \) in the level of education, which ensures the existence of an interior optimum. As commonly assumed in the literature, we also consider that total and marginal costs of education are strictly decreasing in the worker’s ability, so that \( c_\mu(e, \mu) < 0 \) and \( c_{e\mu}(e, \mu) < 0 \). The last requirement ensures that low-ability workers have steeper indifference curves than high-ability workers.

The industry sets the wage \( w \in [0, +\infty) \). In the classic signaling model, the payoffs of the worker are given by \( w - c(e, \mu) \). The payoffs of the industry are set (e.g. through a loss function) such that it will always offer a wage equal to the expected productivity; this is a proxy for either a competitive labor market or a finite number of firms engaged in Bertrand competition for the services of the worker. In a signalling PBE:

(a) The worker selects his education level so as to maximize his expected utility given the industry’s offer.

(b) The industry offers a starting wage equal to the expected productivity of the worker given the industry’s beliefs.

(c) The industry’s beliefs are derived from Bayes’ rule for any educational attainment \( e \) that is selected with a positive probability.

We depart from Spence’s model as follows. Once the worker has accepted a wage and been matched to a firm, i.e. after the signalling game has been played out, production initiates. The firm’s initial belief about the type of the worker is the equilibrium belief. Afterwards, the employer revises his prior in a similar fashion than in Section 2. Hence, the worker’s payoff is not simply \( w - c(e, \mu) \) but rather \( W_i(p_0) - c(e, \mu) \), where \( W_i(p_0) \) is the expected lifetime income of the worker \( (i = l, h) \) as a function of the firm’s prior \( p_0 \).

In a separating equilibrium, however, the ability of the worker is perfectly revealed by his education: depending on the education signal, the initial belief of the firm is either zero or one. As can be seen from the definition of \( p_t \) in equation (1), firms do not update their beliefs when \( p_0 \in \{0, 1\} \). This implies that the asset values of the high and low ability workers in the separating equilibrium are \( W^*_h = \mu_h/(r + \delta) \) and \( W^*_l = \mu_l/(r + \delta) \), respectively. Comparing these asset values with the ones derived in Section 2, it is relatively easy to verify that, for some parameter configurations, all pooling equilibria do not fail the Intuitive criterion. We now show this formally.

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9 Even though workers will only have productivity levels in \( \{\mu_l, \mu_h\} \), it is convenient to consider the cost function to be defined on all potential productivities.

10 If we model specifically Bertrand competition among multiple firms, the equilibrium concept has to be refined to include the additional condition that all firms have the same beliefs about the worker, given an education level.
First of all, we characterize the pooling equilibria that are sequentially rational. As shown by Lemma 1, sequential rationality is satisfied as long as the “participation constraint” for the low-type is fulfilled. This substantiates the claim made at the beginning of this section that the model admits a continuum of PBE.

**Lemma 1.** There is a pooling PBE where both types educate at level \(e_p\) if and only if \(e_p \in [0, \bar{c}]\), where \(\bar{c}\) is given by

\[
W_l(P_0) - c(\bar{c}, \mu_l) = W_l^* - c(0, \mu_l).
\]

In order to reduce the set of potential equilibria, we impose further restrictions on out-of-equilibrium beliefs. The Intuitive criterion requires that, after receiving an out-of-equilibrium signal, firms place zero probability on the event that the sender is of type \(i\) whenever the signal is equilibrium-dominated for type \(i\). In our set-up, a pooling equilibrium with education level \(e_p\) fails the Intuitive criterion if and only if there exists an education level \(e_S\) such that:

\[
W_s(P_0) - c(e_S, \mu_l) \leq W_l(P_0) - c(e_p, \mu_l), \quad \text{equilibrium dominance for the low type, (ED)}
\]

\[
W_h(P_0) - c(e_S, \mu_h) \geq W_h(P_0) - c(e_p, \mu_h), \quad \text{high-type’s participation constraint. (PC)}
\]

It is easily seen that these two conditions and the Intuitive criterion rule out pooling at \(e_p\). The equilibrium dominance condition (ED) implies that, even in the best-case scenario where the worker could forever masquerade as a high-type, deviating to \(e_S\) is not attractive to the low-type. Thus, the firm can infer by forward induction that any worker with an off the equilibrium signal \(e_S\) is of the high type. But the participation constraint (PC) implies in turn that credibly deviating to \(e_S\) is profitable for the high type. Thus \(e_p\) is not stable\(^{12}\) or, in other words, fails the Intuitive criterion.

Combining conditions (ED) and (PC), it is straightforward to show analytically that all pooling equilibria fail the Intuitive criterion in the model without learning. Let \(e^*(e_p)\) denote the minimum education level that does not trigger a profitable deviation for low ability workers, so that (ED) holds with equality at \(e^*(e_p)\). Analogously, let \(e^{**}(e_p)\) be such that (PC) holds with equality.\(^{13}\) These two thresholds always exists due to the continuity and strictly increasing profile with respect to \(e\) of the

\(^{11}\)Notice that we implicitly use the standard tie-breaking rule where the separating equilibrium is chosen over the pooling equilibrium in case of indifference.

\(^{12}\)We follow the terminology of Cho and Kreps (1987) by refering to equilibria that satisfy the Intuitive criterion as stable equilibria.

\(^{13}\)Note that, since \(W_l(P_0) < W_h^*\) and \(c(e, \mu) > 0\), condition (ED) implies that \(e^*(e_p) > e_p\). For the same reasons, condition (PC) implies that \(e^{**}(e_p) > e_p\).
cost function \( c(\mu, e) \). Conditions (ED) and (PC) above reduce to \( e^*(e_p) < e^S < e^{**}(e_p) \). Now assume that \( e_p \) meets the Intuitive criterion, so that condition (PC) is not satisfied at \( e^*(e_p) \). This can be true if and only if

\[
W_h(P_0) - W_l(P_0) \geq c(e^*(e_p), \mu_l) - c(e^*(e_p), \mu_h) - [c(e^*(e_p), \mu_h) - c(e_p, \mu_h)].
\] (5)

In the absence of learning, the workers’ asset values in a pooling equilibrium do not depend on their abilities, i.e. \( W_h(P_0|s = 0) = W_l(P_0|s = 0) \). Thus the left hand side of equation (5) is equal to zero, while the right-hand side is positive since \( c_{ue}(\mu, e) < 0 \). We have therefore obtained a contradiction which shows that, in the basic signaling model, one can always find a credible and profitable deviation from any pooling equilibrium for the high-type. When workers’ abilities are also revealed on-the-job, however, the premise leading to a contradiction is not anymore true. To the contrary, with learning, high-ability workers have a higher asset values even in a pooling equilibrium, so that \( W_h(P_0|s > 0) > W_l(P_0|s > 0) \), and thus equation (5) can hold true for some parameter configurations. Obviously, the larger the signal/noise ratio is, the bigger the difference between the workers’ asset values. This suggests that a pooling equilibrium is more likely to be stable when signal extraction is efficient. **Proposition 3** substantiates this intuition.

**Proposition 3.** For any arbitrary pooling level of education \( e_p \), there exists a threshold signal/noise ratio \( s^*(e_p) \) such that \( e_p \) fulfills the Intuitive criterion for any \( s \geq s^*(e_p) \).

**Figure 2** illustrates the mechanism behind **Proposition 3**. It contains the high and low types’ indifference curves when \( s = 0 \) and when \( s = s^*(e_p) \). The curves without dots correspond to the former case, that is the Spence model without learning. The level of education \( e^*(e_p|s) \) where condition (ED) holds with equality is given by the point where the low-type indifference curve crosses the horizontal line with intercept \( W^S_h \). Similarly, the level of education \( e^{**}(e_p|s) \) where condition (PC) holds with equality is given by the point where the high-type indifference curve crosses the same horizontal line. As discussed above, the pooling equilibrium \( e_p \) fails the Intuitive criterion if and only if \( e^*(e_p) < e^{**}(e_p) \). Hence, when the low-type indifference curve intersects the horizontal line before the high-type indifference curve, we can conclude that \( e_p \) is not stable.

First of all, consider the basic model without learning. Then, at the pooling level of education \( e_p \), the two types enjoy the same asset value \( W(P_0|s = 0) = W_h(P_0|s = 0) = W_l(P_0|s = 0) \). The single-crossing property therefore implies that \( e^*(e_p|s = 0) \) lies to the left of \( e^{**}(e_p|s = 0) \), as shown by the indifference curves without dots in **Figure 2**. This illustrates that any pooling equilibrium fails the Intuitive criterion when the signal/noise ratio \( s = 0 \). Now, consider what happens when \( s \) increases. Since high-ability workers reveal their actual productivity more quickly, their asset value increases and
so their indifference curve shifts up. Conversely, the indifference curve of low-ability workers shifts down when $s$ increases because their actual productivity is more rapidly detected. Obviously, these opposite adjustments shrink the gap between $e^*(e_p|s)$ and $e^{**}(e_p|s)$. The threshold signal/noise ratio $s^*(e_p)$ is identified by the point where this gap vanishes as the two indifference curves concurrently cross the horizontal line with intercept $W^S_h$. It is clear that $s^*(e_p)$ is unique (see Figure 2). Furthermore, for any given $e_p$, one can always find such a point because $\lim_{s \to \infty} W_h(P_0|s) = W^S_h$.

By checking that conditions (ED), (PC) and the low-type’s “participation constraint” given in Lemma 1 are simultaneously satisfied, one can establish whether or not $e_p$ is a stable pooling PBE. This approach, however, solely yields local results. Before concluding this section, we propose instead a global characterization of the region where the Intuitive criterion fails to select a unique separating equilibrium.

**Proposition 4.** When the marginal cost of education $c_e(e, \mu)$ is weakly log sub-modular, there exists a unique value of the signal/noise ratio, which we denote $s_p$, such that

(a) For all $s \in [0, s_p]$, the separating equilibrium is the only equilibrium satisfying the intuitive criterion.

(b) For all $s > s_p$, $\exists \varepsilon > 0$ such that the pooling equilibrium meets the intuitive criterion for any
education values $e \in [0, \varepsilon)$. As $s$ diverges to infinity, $\varepsilon$ converges to zero.

Figure 3: Graphical interpretation of Proposition 4. A pooling equilibrium with education level $e$ exists if and only if $e \leq \overline{e}(s)$. It satisfies the Intuitive Criterion if and only if $e \leq \tilde{e}(s)$. Same parameters than in Figure 1 with $c(e, \mu) = \exp(e/\mu) - 1$ and $p_0 = 0.5$.

The economics behind Proposition 4 is quite intuitive. When the signal/noise ratio is high, firms easily infer the actual type of their employees. Then the benefits derived from ex-ante signaling are not important. Symmetrically, when the signal/noise ratio is low, firms learn little from the observation of the output realizations. This leaves fewer opportunities for high ability workers to reveal their ability while being on-the-job and so raises their incentives to send a signal. In the extreme case where $s$ equals zero, all the relevant information is collected prior to the entry into the labor market. This situation corresponds to the signaling model of Spence (1973). Thus it should not be surprising that we recover, as a limit case, the well known result according to which all but one separating equilibrium fails the Intuitive criterion when there is no learning.

In order to derive Proposition 4, we have imposed slightly stronger restrictions on the cost of education: we assume that the log of the derivative of $c(e, \mu)$ with respect to $e$ has weakly increasing differences in $(e, \mu)$ or, in other words, that $c_e(e, \mu)$ is weakly log sub-modular. Although the weak log sub-modularity of $c_e(e, \mu)$ might seem quite restrictive, it is actually satisfied by the distributions
commonly used to illustrate the single crossing property, such as power and exponential functions. The requirement is sufficient but not necessary to ensure that the single crossing property is satisfied. It is more stringent since it implies that, not only total educational costs, but also marginal educational costs diverge. If that property is not satisfied, there are cases where an increase in the level of education restores the stability of the pooling equilibrium.\textsuperscript{14} Hence, when $c_e(\mu, e)$ is not weakly log sub-modular, equilibrium stability does not generally divide the $(s, e)$ space into two non-overlapping regions, as done in Figure 3 by the $\tilde{e}(s)$ locus.\textsuperscript{15}

4 Educational Achievement and the Speed of Learning

The previous section focused on the implications of employer learning for the strategic analysis of the signaling game. We now reverse the perspective by focusing instead on how signal extraction influences observable outcomes. This connects our analysis to the long-lasting debate on the empirical content of job market signaling. As illustrated by the extensive literature on this topic, one of the main drawbacks of signaling theory is the difficulty to substantiate it empirically. The main reason is that job market signaling and human capital theories share most of their predictions. An influential attempt at disentangling the two approaches was proposed by Riley (1979b). Riley exploits the idea that the importance of screening is likely to differ across occupations. Thus, conditional on being able to make that distinction empirically, one can test the implications of the signaling model for the relationship between the speed of learning and educational achievement.

More precisely, Riley’s (1979b) model predicts that average years of schooling should be lower in occupations where employers can better infer the ability of their employees. This prediction, however, is not supported by the data. Using a two step estimation technique, Murphy and Topel (1990) and more recently Farber and Lange (2006) document that workers with more schooling tend to work in industries and occupations that reward more unobserved characteristics. This finding is widely interpreted as evidence against the importance of signaling in labor markets. We show in this section that this interpretation is unfair to signaling theory as it does not take into account the effect of learning about asymmetric information.

The intuitive criterion pins down the education gap between low and high types in the separating equilibrium, i.e. $e^*(0)$. This is because each type minimizes its education costs so that worker with a low ability choose the minimum education requirement (normalized to zero in our set-up), whereas

\textsuperscript{14}See Claim 2 in the Proof of Proposition 4.
\textsuperscript{15}Notice that when the cost function is log-linear, e.g. $c(e, \mu) = e^2/\mu$, the $\tilde{e}(s)$ locus is vertical. See the proof of Proposition 4 for a discussion of this point.
high types select the lowest education level which allows them to credibly signal their ability. The following proposition characterizes how the signal/noise ratio affects this education gap.

**Proposition 5.** The separating level of education \( e^s(0; s) \) is an increasing function of the signal/noise ratio \( s \), for all \( s \leq s_p \).

Thus we reach a conclusion opposite to that of Riley (1979b) since the separating level of education is higher when employers learn more rapidly. At first sight, this might seem counterintuitive because the payoff from deviating and masquerading as an high type is larger when learning is difficult. This suggests that educational attainment should increase with the degree of uncertainty. In the separating equilibrium, however, such deviations are not profitable. The separating education gap is high enough to ensure that low types would never benefit from being treated as high types, even if they could fool their employer forever. In other words, employers do not need to learn in the separating equilibrium because they believe with certainty that educated workers are of the high type and their beliefs are actually correct in equilibrium. Hence, the separating level of education is such that off-the-equilibrium deviations are not profitable for both types. Since the earnings of low-type in the pooling equilibrium is a decreasing function of the speed of learning, they have more incentive to deviate when signal extraction is fast and so the separating level of education increases.

## 5 Search frictions

In this section, we add a layer of complexity to the model by introducing search frictions. This allows us to bridge the gap with modern theories of the labor market and to discuss the interactions between signaling and unemployment. As before, we first focus on the pooling equilibrium. Section 5.1 shows that the surplus-splitting rule lead to an affine wage function and derives the asset values. Section 5.2 analyzes the job destruction game and proposes a refinement which allows to select a unique equilibrium.

### 5.1 Wage Bargaining and asset values

The asset value of being unemployed depends on the worker’s ability and is equal to

\[
rU_i = b + \lambda (W_i(P_0) - U_i) - \delta U_i \quad \text{for } i = h, l,
\]

where \( b \) is the flow value of non-market activity and \( \lambda \) is the job-finding rate.\(^{16}\) As commonly assumed in the literature, we consider that wages are set by *surplus-splitting* so that bargaining yields a pro-

\[^{16}\text{Notice that because } \delta \text{ is the rate of exit, the value of being unemployed contains the additional term } -\delta U.\]

15
portional share allocation of the match’s surplus. Our problem is not completely standard, however, since the worker knows his ability with certainty and thus do not have the same expectation than his employer. Given that the worker cannot credibly signal his type, bargaining is based on the estimated worker’s surplus, so that

$$\beta J(p) = (1 - \beta)E_p [W(p) - U] = (1 - \beta) (p (W_h(p) - U_h) + (1 - p) (W_l(p) - U_l))$$

where $\beta$ denotes the exogenous bargaining power parameter and $J(p)$ is the asset value for the firm of a job with posterior belief $p$. Although the problem is not standard, the negotiated wage has the same functional form than when uncertainty is purely match-specific.

**Proposition 6.** The bargained wage is an affine function of the posterior belief $p$ given by

$$w(p) = p\zeta + (1 - \beta)b + \beta\mu_l + \lambda(1 - \beta)(W_l(p_0) - U_l)$$

where $\zeta \equiv \beta (\mu_h - \mu_l) + \lambda(1 - \beta)(W_h(p_0) - W_l(p_0) + U_l - U_h)$. The wage paid to the worker weights with the bargaining share his estimated opportunity cost of employment, $b + p(W_h(p_0) - U_h) + (1 - p)(W_l(p_0) - U_l)$, and expected output, $\mu(p) = p\mu_h + (1 - p)\mu_l$. When compared to the case where information is symmetric, the Nash-bargaining solution is complicated by the fact that high and low types have different outside options. Hence, the surplus evolves not only due to variations in the job output but also in the threat point of the worker. This additional source of fluctuation is captured by the term $\lambda(1 - \beta)(W_h(p_0) - W_l(p_0) + U_l - U_h)$ in the definition of $\zeta$. Although it substantially complicates the analytical expression, asymmetric information does not entail other technical difficulties since it preserve the linearity between $w$ and $p$. Accordingly, reinserting the wage function (6) into the workers’ HJB equations yields similar solutions to those in Section 2.2.

**Proposition 7.** The expected lifetime income of the workers with low-ability as a function of the belief ratio $P$ and separation belief $\bar{P}$ is

$$W_l(P) = \varphi_l(P) + [U_l - \varphi_l(P)] \left(\frac{P}{\bar{P}}\right)^{\alpha^-}$$

The function $\varphi_l(P)$ is given by

$$\varphi_l(P) = \frac{2\zeta}{s^2(\alpha^+ - \alpha^-)} \left( P^{\alpha^-} \int_P^P \frac{1}{(1 + x)x^{\alpha^-}} dx + P^{\alpha^+} \int_P^{\infty} \frac{1}{(1 + x)x^{\alpha^+}} dx \right) + \frac{(1 - \beta)b + \beta\mu_l + \lambda(1 - \beta)(W_l(p_0) - U_l)}{r + \delta}$$

17 The linearity of the wage as a function of $p$ does not hold when worker and match uncertainty are combined.

18 Only when $p = p_0$ does the wage, $w(p_0) = \beta\mu + (1 - \beta)b + \beta J(p_0)$, have the same solution than in standard search-matching models.
where \( \alpha^- \) and \( \alpha^+ \) are defined in Proposition 1.

The expected lifetime income of the workers with high-ability as a function of the belief ratio \( P \) and separation belief \( P \) is

\[
W_h(P) = \varphi_h(P) + [U_h - \varphi_h(P)] \left( \frac{P}{P} \right)^{\gamma^-}
\]

The function \( \varphi_h(P) \) is given by

\[
\varphi_h(P) = \frac{2\zeta}{s^2(\gamma^+ - \gamma^-)} \left( P^{\gamma^-} \int_P^P \frac{1}{(1 + x)x^{\gamma^-}} dx + P^{\gamma^+} \int_P^\infty \frac{1}{(1 + x)x^{\gamma^+}} dx \right) + \frac{(1 - \beta)b + \beta \mu_l + \lambda(1 - \beta)(W_l(P_0) - U_l)}{r + \delta}
\]

where \( \gamma^- \) and \( \gamma^+ \) are defined in Proposition 1.

The asset value of the firm is

\[
J(P) = \frac{\overline{\mu}(P) - w(P)}{r + \delta} - \left( \frac{\overline{\mu}(P) - w(P)}{r + \delta} \right) \left( \frac{P}{P} \right)^{\alpha^-} \left( \frac{1 + P}{1 + P} \right)
\]

where \( \overline{\mu}(P) \) is the expected output of the job.

5.2 Job destruction game

5.2.1 The game

Consider a worker-firm match that has been going on up to time \( t \). The firm currently entertains the belief \( p_t \) that the worker is of the good type (high-ability). This belief comes from continuous-time updating based on the observed productivity of the worker. For the instantaneous game taking place at \( t \), though, and from the firm’s point of view, this belief can be taken as an objective prior.

This allows us to describe the job-separation decision as an extensive-form game as follows. Nature chooses the type of the worker, \( i = h, l \), with probabilities \( p_t, 1 - p_t \). Then the worker decides whether to Separate or Not. Denote by \( D_i(p_t) \in \{S, N\} \) the decision of the worker of type \( i \).

If the worker separates, the game ends with payoffs \( U_i \) for the worker and 0 for the firm. These payoffs come from the fact that then the worker goes to the unemployment pool and the firm has to look for a new worker. If the worker does not separate, the firm still does not know the worker’s type, but might update beliefs from the observed worker’s decision. The beliefs of the firm at the information set are denoted \( (q_t, 1 - q_t) \). The firm has to take a single decision, namely to separate or not, and so \( F(q_t) \in \{S, N\} \). If it separates, again the payoffs are \((U_i, 0)\) If it does not separate, then the worker and the firm renegotiate the wage. In game-theoretic terms, we just assume that there is a further step of the game where bargaining takes place and the payoffs are such that the firm will offer a wage...
as in the dynamic model but based on the updated beliefs $q_t$, i.e. $w(q_t)$. This results in payoffs equal to the asset values $(W_h(q_t), J(q_t))$. These values come from Nash Bargaining and satisfy

$$
\beta J(q_t) = (1 - \beta) [q_t (W_h(q_t) - U_h) + (1 - q_t) (W_l(q_t) - U_l)]
$$

(7)

See Figure 4 for a qualitative representation of the job destruction game.\textsuperscript{19}

We concentrate on pure strategies. At each point in time, the tuple $[D_h(p_t), D_l(p_t), F(q_t), q_t]$ has to constitute a Perfect Bayesian Equilibrium of the instantaneous game. This requires the following:

\textsuperscript{19}As in Spence's (1973), strictly speaking, the reduced form cannot be taken literally as a standard extensive form game, because the payoffs depend on the beliefs. This is, however, inconsequential for the analysis. Our aim is merely to directly incorporate the result of Nash bargaining in the job separation game as an equilibrium property. It is possible to complete the game specification by rewriting the last stage as follows. If the worker does not separate, the firm (better interpreted as an industry in this case) can choose to either separate or make an (instantaneous) wage offer. The payoffs in case of a wage offer are given by an appropriate function such that the expected payoff is maximized at the corresponding asset values for the workers. Asset values themselves are interpreted as utilities for the wage offered by the firm. The payoffs of separation for the firm are renormalized so that the firm prefers separation only if $J(q_t) < 0$. In other words, the asset payoffs $W_i(q_t)$ are already equilibrium payoffs obtained through sequential rationality given the beliefs $q_t$. 

Figure 4: Reduced form of the job destruction game. Payoffs after the Nash Bargaining Stage are already equilibrium (expected) payoffs and hence depend on the beliefs.
(i) The high-ability worker optimizes given the strategy of the firm.

(ii) The low-ability worker optimizes given the strategy of the firm.

(iii) If any of the two types of workers decides not to separate, \( q_t \) is updated from \( p_t \) using Bayes’ Rule. That is: \( q_t = p_t \) if \( D_h(p_t) = D_l(p_t) = N \); \( q_t = 1 \) if \( D_h(p_t) = N, D_l(p_t) = S \); and \( q_t = 0 \) if \( D_l(p_t) = N, D_h(p_t) = S \). If both types of workers decide to separate, the PBE places no requirement on \( q_t \).

(iv) The firm decides optimally given the belief \( q_t \).

In order to single out behavior consistent with our overall model, we concentrate on the following refinement, which amounts to a specific tie-breaking rule for workers. The PBE \([D(p_t), F(p_t), q_t]\) is an optimal stopping PBE if, in case a worker type is indifferent between separating or not, the decision is taken as follows

\[
D_i(p_t) = \begin{cases} 
S & \text{if } W_i(p_t) \leq U_i \\
N & \text{if } W_i(p_t) > U_i 
\end{cases}
\]  

(8)

The rationale for this refinement is as follows. If (and only if) a worker is indifferent given the decision of the firm, which is based on the updated beliefs \( q_t \), it still considers the possibility that by not separating the firm might offer a wage based on the original, unupdated beliefs—a “market value”. If even under this belief the worker is still indifferent, \( W_i(p_t) = U_i \), we assume that the worker separates. This particular modeling decision is made to allow for a feedback of the optimal stopping problem into the equilibrium. \( W_i(p_t) = U_i \) means that the firm has hit the separation threshold. Since \( p_t \) follows a random walk, the sample path is not of bounded variation, hence for any small time interval \([t, t + \varepsilon]\) with probability one the beliefs \( p_{t'} \) will be such that \( W_i(p_{t'}) < U_i \) for some \( t' \in [t, t + \varepsilon] \). In other words, whenever the worker is indifferent under the unupdated beliefs, he will strictly prefer to separate “an instant later”, hence it anticipates this decision. Note that if \( q_t = p_t \), our refinement amounts to the assumption that the worker separates whenever he is indifferent. We review now all possible pure-strategy profiles.

- \( D(p_t) = (N, S) \) implies \( q_t = 1 \), i.e. the firm knows that the worker who has not separated is of the high ability type and, since \( J(1) > 0 \), decides not to separate and subsequently offers wage \( w(1) \). The low-ability worker receives a payoff of \( U_l \) from separating, but would obtain payoff \( W_l(1) \) from not separating. Hence this profile can never be part of a PBE provided \( U_l < W_l(1) \) (and \( J(1) > 0 \)).
• **Job Continuation:** \( D(p_t) = (N, N) \) implies \( q_t = p_t \), i.e. no information is revealed. Suppose \( F(p_t) = N \). This will be a PBE provided the following conditions hold: (i) \( W_h(p_t) \geq U_h \); (ii) \( W_l(p_t) \geq U_l \); and (v) \( J(p_t) \geq 0 \). But the latter is implied by the previous two by (7). This is the PBE where the match goes on. But, since workers separate whenever indifferent, this is only an optimal stopping PBE provided (i) \( W_h(p_t) > U_h \) and (ii) \( W_l(p_t) > U_l \).

• Consider \( D(p_t) = (N, N) \) (thus \( q_t = p_t \)) and \( F(p_t) = S \). That is, both types are fired. Workers are indifferent and thus (since \( q_t = p_t \)) should separate, a contradiction.

• **Fire the low type:** \( D(p_t) = (S, N) \) implies \( q_t = 0 \), thus the firm separates, provided \( J(0) < 0 \). Given this decision, both types of worker are actually indifferent. But, due to the tie-breaking assumption, this can only be an optimal stopping PBE if \( W_h(p_t) \leq U_h \) and \( W_l(p_t) > U_l \).

• **Job Separation:** \( D(p_t) = (S, S) \) places no requirement on \( q_t \). A PBE can obtain with a \( q_t \) such that \( F(q_t) = S \) only if \( q_t \) is low enough so that \( J(q_t) \leq 0 \). Again, workers are then indifferent, thus this profile is an optimal stopping PBE if and only if \( W_h(p_t) \leq U_h \) and \( W_l(p_t) \leq U_l \).

• **Leave the firm:** Again, \( D(p_t) = (S, S) \) which places no requirement on \( q_t \). A PBE can obtain with a \( q_t \) such that \( F(q_t) = N \) only if \( J(q_t) \geq 0 \), \( W_l(q_t) \leq U_l \), and \( W_h(q_t) \leq U_h \). By (7), this can only occur if \( J(q_t) = W_h(q_t) - U_h = W_l(q_t) - U_l = 0 \), i.e. \( q_t = p^* \). However, workers are indifferent, thus this profile is an optimal stopping PBE if and only if \( W_h(p_t) \leq U_h \) and \( W_l(p_t) \leq U_l \).

In summary: (i) If \( W_h(p_t) > U_h \) and \( W_l(p_t) > U_l \), there is a unique optimal-stopping PBE where the job goes on; (ii) If \( W_h(p_t) \leq U_h \) and \( W_l(p_t) > U_l \), there is a unique optimal-stopping PBE where the job is destroyed. The high-types leave voluntarily, the firm deduces that a worker who stays is of the low type, and fires him; (iii) If \( W_h(p_t) \leq U_h \) and \( W_l(p_t) \leq U_l \), there is an equilibrium where the job is destroyed by mutual agreement; (iv) If \( W_h(p_t) > U_h \) and \( W_l(p_t) \leq U_l \), there exists no optimal-stopping PBE.

### 5.2.2 Optimal stopping problem

It has been shown in Section 5.2.1 that the job destruction game has a unique optimal stopping-PBE when employees follow the rule (8) and \( W_l(p) > U_l \) whenever \( W_h(p) > U_h \). Rule (8) can be translated into an optimal stopping problem by considering the values of the separation beliefs

\[
P^* = \arg \max_{P \in (0, +\infty)} W_l(P | P) \quad \text{for all } P > P^*.
\]
By continuity of the asset values, the requirement on the sign of the surpluses is satisfied when \( P^l < P^h \). The optimal stopping problem is well defined if and only if the optimal separation beliefs are homogenous of degree zero with respect to the current belief \( P \). This property is easily established by differentiation of the asset values\(^{20} \) given in Proposition 7. It is also worth noticing that the optimal separation beliefs can indifferently be determined by using the smooth-pasting condition \( \partial W_l (P|P) / \partial P|_{P=P_l} = 0 \). This provides us with an alternative criterion to assess the requirement that \( P^l < P^h \): if \( \partial W_l (P|P) / \partial P|_{P=P_h} > 0 \), we can infer that low-ability workers would prefer staying rather than separating.

6 Signaling and Unemployment

6.1 The equilibrium

The purpose of this section is to show that signaling is not wasteful but on the contrary enhances welfare when there are search frictions. If education reveals workers’ abilities, the economy is composed of two disconnected labor markets. Then there are no endogenous job destructions and the equilibrium rate of unemployment is lower.

In order to close the model, we propose two additional equilibrium conditions. First of all, we have to derive the job-finding rate \( \lambda \) from primitive parameters. An additional complication is due to the fact that the share \( p_0 \) of high types in the unemployment pool is also endogenous. It differs from the share \( \chi_h \) of high types in the whole population because workers with a low ability are more likely to breach the reservation threshold. Endogenous separation therefore generates a negative selection effect which reduces the average ability of job seekers. Although firms ignore the actual ability of a given job applicant, they are aware of this selection effect. Moreover, in a rational expectation equilibrium, firms are able to infer how selection distorts the distribution of types among job seekers.

Thus we need two equilibrium conditions in order to pin down \( \lambda \) and \( p_0 \). The first one is standard and known in the search literature as the Free-entry condition. If the aggregate matching function \( m(u, v) \) is homogenous of degree one, the job finding rate only depends on the vacancy-unemployment ratio \( \theta \) since \( \lambda \equiv m(u, v)/u = q(\theta)\theta \). Obviously the job finding rate is increasing in the v-u ratio, so that \( q'(\theta)\theta + q(\theta) > 0 \). By homogeneity, vacancies are filled at rate \( q(\theta) \). Hence, there are no arbitrage opportunities when

\[
c_v = q(\theta(\lambda))J(p_0, \lambda),
\]

\(^{20}\) Notice that, when differentiating the asset values, \( W_l (P_0) \) and \( W_h (P_0) \) are treated as constants because they refer to the worker’s outside option and so do not depend on the choice of the reservation belief for the current job relationship.
where \( c_v \) is the flow cost of vacancy posting. As in standard search-matching models, the *Free-entry condition* enables us to determine the equilibrium job-finding rate \( \lambda \) for a given value of \( p_0 \). But the value of \( p_0 \) is in turn a function of \( \lambda \) because the job finding rate raises the size of the flows and consequently intensifies the selection effect. To quantify this mechanism, we use the ergodic distribution of beliefs. It can be derived in a similar fashion than in Section 2.3. The sole difference is that the endogenous threshold \( p \) truncates the support of the cross-sectional distribution.

**Proposition 8.** The ergodic distribution of beliefs is given by

\[
g(p | p_0, p) = \begin{cases} 
  C_0 p^{-1-\eta}(1-p)^{\eta-2} & \text{if } p \in (p, p_0) \\
  C_1 p^{-1-\eta}(1-p)^{\eta-2} & \text{if } p \in [p_0, 1)
\end{cases}
\]

where

\[
\eta = \frac{1}{2} + \sqrt{\frac{1}{4} + 2 \delta^2},
\]

while \( C_0 \) and \( C_1 \) solve the normalization, \( \int_0^1 g(p) \, dp = 1 \), and continuity, \( \lim_{p \to p_0^-} g(p) = \lim_{p \to p_0^+} g(p) \), requirements.

Workers enter the labor market through the unemployment pool. The mass of entrants is \( \delta \) and nature endows a share \( \chi_h \) of them with an high ability. The second source of flows into the unemployment pool is due to workers leaving their jobs. A worker-firm match separates when the belief breaches the threshold \( p \), which occurs at rate \( (s p (1-p)^{1/2}) \) \( g'(p^+) \). Given that firms’ beliefs are on average correct in a rational expectation equilibrium, the workers that quit are of the high type with probability \( p \). Concurrently, workers leave the unemployment pool because they either find a job or exit the labor market. Thus the flows of high types in and out of the unemployment pool are equal when

\[
\frac{\delta \chi_h}{\text{Entry}} + d(p_0, \lambda) p(p_0, \lambda) (1 - u(p_0, \lambda)) = (\lambda + \delta) p_0 u(p_0, \lambda),
\]

where the job destruction rate \( d(p_0, \lambda) \) reads

\[
d(p_0, \lambda) = \left( \frac{(p(p_0, \lambda) (1-p(p_0, \lambda)) s)^2}{2} \frac{\partial g(p | p_0, p(p_0, \lambda))}{\partial p} \bigg|_{p=p(p_0, \lambda)} \right)
\]

\[
= \frac{s^2}{2} p^{-\eta}(1-p)^{\eta-1}(2\eta - 1).
\]

The equilibrium rate of unemployment balances the aggregate flows in and out of employment, so that

\[\text{Job separation} = \text{Job creation and exit}\]

\[d(p_0, \lambda) = \frac{(p(p_0, \lambda) (1-p(p_0, \lambda)) s)^2}{2} \frac{\partial g(p | p_0, p(p_0, \lambda))}{\partial p} \bigg|_{p=p(p_0, \lambda)} = \frac{s^2}{2} p^{-\eta}(1-p)^{\eta-1}(2\eta - 1) .\]

\[\text{Job separation} = \text{Job creation and exit}\]

\[d(p_0, \lambda) = \frac{(p(p_0, \lambda) (1-p(p_0, \lambda)) s)^2}{2} \frac{\partial g(p | p_0, p(p_0, \lambda))}{\partial p} \bigg|_{p=p(p_0, \lambda)} = \frac{s^2}{2} p^{-\eta}(1-p)^{\eta-1}(2\eta - 1) .\]

\[\text{Job separation} = \text{Job creation and exit}\]

\[d(p_0, \lambda) = \frac{(p(p_0, \lambda) (1-p(p_0, \lambda)) s)^2}{2} \frac{\partial g(p | p_0, p(p_0, \lambda))}{\partial p} \bigg|_{p=p(p_0, \lambda)} = \frac{s^2}{2} p^{-\eta}(1-p)^{\eta-1}(2\eta - 1) .\]
\[ u = \frac{\delta + d(p_0, \lambda)}{\lambda + \delta + d(p_0, \lambda)}. \]  

(12)

Reinserting equation (12) into (11) yields

\[ \delta(\chi_h - p_0) = d(p_0, \lambda)(1 - u(p_0, \lambda))(p_0 - \bar{p}(p_0, \lambda)). \]  

(13)

Given that \( p_0 > \bar{p} \), the share \( p_0 \) of high type in the unemployment pool is smaller than the share \( \chi_h \) in the whole population. The left hand side therefore measures the increase in job seekers’ ability due to the higher average ability of entrants. The right hand side measures the opposite decrease in ability due to the inflow of workers leaving their job. Quite intuitively, condition (13) requires that the two effects balance out, so that the average ability of job seekers remains constant through time. Equation (13) implicitly depends on the smooth-pasting condition and on equation (10). It is highly non-linear and so may not define a one-to-one mapping between \( \lambda \) and \( p_0 \). However, differentiating (13) shows that it is strictly increasing in \( p_0 \) when \( \frac{\partial p(p_0, \lambda)}{\partial p_0} < 1 \) and \( \frac{\partial d(p_0, \lambda)}{\partial p_0} > 0 \).

Figure 5: Asset Values as a function of P.
A stationary equilibrium for the pooling economy is defined as a vector of scalars \( \{ \lambda^*, p_0^* \} \) satisfying equations (9) and (13).

### 6.2 A Numerical Example

We now illustrate the analysis of the model with search frictions through a numerical example. We set both interest rate \( r \) and exogenous rate of exit \( \delta \) to 0.1. For the illustrative purpose of this section, we use \( s = \sqrt{r + \delta} \) so that the roots \( \alpha \) and \( \gamma \) are integers. This greatly simplifies the solution of the model since then the indefinite integrals in Proposition 7 have analytical solutions. We consider a Cobb-Douglas matching function. Hence \( q(\theta) = A\theta^{-\varepsilon} \), where \( -\varepsilon \) is the elasticity of the matching function with respect to vacancies and \( A \) is a scaling factor which captures the efficiency of the matching process. As commonly done in the literature, we set the bargaining power of the worker \( \beta = 0.5 \) and ensure that the "Hosios condition" is satisfied by setting \( \varepsilon = \beta \). We normalize the average productivity of high types \( \mu_h \) to be one and use a flow value of non-market activity \( b \) equals to 40\% of their average output. Finally, we set the productivity of the low type \( \mu_l \) to 0.6 and \( \chi_h \) to 0.5 so that high types account for half of the population.

![Figure 6: Ergodic density of beliefs.](image)

The upper panel in Figure 5 contains the surplus of the workers and of the firm as a function of the belief ratio \( P \). As expected, the surplus of the high type lies above that of the low type. The threshold \( P \) is close to 0.2. By definition, the three surpluses are equal to zero at \( P \). For higher beliefs, the
firm’s surplus lies in the middle of the interval separating the surpluses of the workers. This is because the bargaining power is symmetric. Consequently, the firm’s surplus is equal to the average worker’s surplus.

The derivatives of the asset values are reported in the lower panel of Figure 5. The slope of the high type’s asset value is effectively zero at $P$ which confirms that the smooth-pasting condition is satisfied. Conversely, the low type’s asset value has a positive slope at $P$, indicating that he would rather continue the job relationship than separate. Thus, for these particular parameter values, the model satisfies the requirement for the existence and uniqueness of the PBE discussed in Section 5.2.

Figure 6 depicts the ergodic distribution of beliefs $p$. Its mode coincides with the endogenous value of $p_0$. One can see that it is well below $\chi_h = 0.5$ since $p_0 = 0.38$. This illustrates the importance of the negative selection effect. It reduces by nearly 25% the proportion of high type in the unemployment pool. Accordingly, endogenous job destruction account for 60% of overall destruction ($d = 0.06$). Finally, the fat tail property is highlighted by the slow decline and eventual increase in the right tail as it converges to the upper bound $p = 1$.

We conclude by briefly discussing the effect of asymmetric information on unemployment and welfare. To illustrate our discussion, we report in Figure 7 the equilibrium unemployment rate and welfare gross of education costs as a function of the scale factor $A$ of the matching function. An increase in $A$ reduces the importance of search frictions so that unemployment goes to zero as $A$ diverges to infinity. We also include in Figure 7 the equilibrium outcomes in the “Spence” pooling equilibrium (without learning) and separating equilibrium. To begin with, let us focus on these last two. The upper panel shows that unemployment is actually higher in the separating equilibrium. The employment gains for the high types due to the revelation of their ability are more than offset by the employment losses for the low type. Nevertheless, as shown in the lower panel, separating is unambiguously beneficial for welfare. The two results are not contradictory because the employees’ average productivity is much higher in the separating equilibrium.

Consider now the pooling equilibrium with learning. The first striking feature is that its equilibrium rate of unemployment is nearly 2% higher than in the separating equilibrium. The negative selection effect dwarfs the employment gains for low types. Thus learning enables us to recover a positive relationship between information revelation and employment that is probably more in line with intuition. From a welfare point of view, however, the effect of learning is not that detrimental as shown by the lower panel in Figure 7. Again, the reason is that average productivity is higher in the pooling equilibrium with learning because the jobs filled by workers with a low ability are more likely to be destroyed.
Figure 7: Unemployment and Welfare as a function of matching efficiency.

7 Conclusion

We have postulated an analytically tractable model where education plays a role as a job-market signal but, contrary to the classical paradigms, worker productivity is also revealed on-the-job as the result of continuous-time Bayesian updating on the part of the firms. The addition of this realistic element causes the failure of standard arguments, such as the unavoidable selection of the 'Riley' separating equilibrium via the Intuitive Criterion.

The separating equilibrium, though, still plays a key role provided firms’ learning is not too fast. We find that the education gap increases with the speed of learning, which conforms to empirical stylized facts which in turn are difficult to reconcile with alternative models of the job market.

Finally, we complete the model by incorporating job-destruction decisions as the outcome of an instantaneous extensive-form game played between workers and firms. Endogenous separation creates a negative selection effect as low-ability workers separate more frequently and hence are overrepre-
sented in the unemployment pool. We examine a numerical example to illustrate how this mechanism affects both welfare and unemployment. We argue that, in the presence of search frictions, signaling is in general welfare-enhancing.

APPENDIX

Proof of Proposition 1: We solve first the asset value of the low type. The wage does not directly depend on the worker’s type, but solely on the current belief \( P \). It is equal to the expected output \( \bar{\mu}(P) = (\mu_h - \mu_l) \left( \frac{P}{1+P} \right) + \mu_l \). Given that

\[
dP_t = P_t s dZ_t
\]

when the worker’s ability is low, the asset value solves the following Hamilton-Jacobi-Bellman equation

\[
(r + \delta) W_l(P) - \frac{(Ps)^2}{2} W''_l(P) = (\mu_h - \mu_l) \left( \frac{P}{1+P} \right) + \mu_l.
\]

This is a second order non-homogeneous ODE with non-constant coefficients. First of all, we solve the associated homogenous problem. It is given by an Euler equation so that the homogenous solution is

\[
W^H_l(P) = C_1 l P^{\alpha_-} + C_2 l P^{\alpha_+},
\]

where \( \alpha_- \) and \( \alpha_+ \) are the negative and positive roots of the following quadratic equation

\[
\alpha (\alpha - 1) \frac{s^2}{2} - r - \delta = 0.
\]

To solve for the non-homogenous equation we use the method of variations of parameters. First, we notice that the non-homogenous term is composed of a non-linear function of \( P \) plus a constant term. Thus we can assume that the particular solution is of the form

\[
W^{NH}_l(P) = \left[ y_1(P) P^{\alpha_-} + y_2(P) P^{\alpha_+} \right] + \frac{\mu_l}{r + \delta},
\]

Standard derivations yield the following system of equations

\[
\begin{pmatrix}
P^\alpha_-
\alpha_- P^{\alpha_- - 1}

P^\alpha_+
\alpha_+ P^{\alpha_+ - 1}
\end{pmatrix}
\begin{pmatrix}
y_1'(P)
y_2'(P)
\end{pmatrix}
= \begin{pmatrix}
0
- \frac{2(\mu_h - \mu_l)}{(1+P)Ps^2}
\end{pmatrix}.
\]

The Wronskian of the two linearly independent solutions is

\[
\Delta = P^\alpha_+ P^{\alpha_- - 1} - P^\alpha_- P^{\alpha_+ - 1} = \frac{P^{\alpha_+ + \alpha_-}}{P} (\alpha_+ - \alpha_-) = \alpha_+ - \alpha_-.
\]
where the last equality holds because
\[
\alpha^- = \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{r + \delta}{s^2}} \quad \text{and} \quad \alpha^+ = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{r + \delta}{s^2}}
\]
implies that \(\alpha^- + \alpha^+ = 1\). The non-homogenous terms \(y_1(P)\) and \(y_2(P)\) are therefore equal to
\[
\begin{align*}
\{ 
&y_1(P) = \frac{2(\mu_h - \mu_l)}{s^2(\alpha^+ - \alpha^-)} \int_0^P \frac{1}{(1+x)\alpha^-} dx + P^{\alpha^+} \int_P^\infty \frac{1}{(1+x)\alpha^+} dx \} \\
y_2(P) = \frac{-2(\mu_h - \mu_l)}{s^2(\alpha^+ - \alpha^-)} \int_0^1 \frac{1}{(1+x)\alpha^+} dx.
\end{align*}
\]
Thus the general form of the particular solution reads
\[
W_t^{NH}(P) = \frac{2(\mu_h - \mu_l)}{s^2(\alpha^+ - \alpha^-)} \left( P^{\alpha^-} \int_0^P \frac{1}{(1+x)\alpha^-} dx + P^{\alpha^+} \int_P^\infty \frac{1}{(1+x)\alpha^+} dx \right) + \frac{\mu_l}{r + \delta}. \tag{14}
\]
The bounds of integration and constants \(C_{1l}\) and \(C_{2l}\) of the homogenous solution are pinned down by the following boundary conditions
\[
\begin{align*}
\{ 
&W_t(P) \xrightarrow{P \to 0} \frac{\mu_l}{r + \delta} \\
&W_t(P) \xrightarrow{P \to \infty} \frac{\mu_l}{r + \delta}
\end{align*} \tag{15}
\]
Let us first consider the homogenous solution. Since \(P^{\alpha^-} \to \infty\) as \(P \uparrow 0\), the first boundary condition can be satisfied if and only if \(C_{1l}\) equals zero. Similarly, since \(P^{\alpha^+} \to \infty\) as \(P \uparrow \infty\), the second boundary condition allows us to set \(C_{2l}\) equal to zero. All that remains is to determine the integration bounds in equation (14). Consider the following function
\[
W_t(P) = \frac{2(\mu_h - \mu_l)}{s^2(\alpha^+ - \alpha^-)} \left( P^{\alpha^-} \int_0^P \frac{1}{(1+x)\alpha^-} dx + P^{\alpha^+} \int_P^\infty \frac{1}{(1+x)\alpha^+} dx \right) + \frac{\mu_l}{r + \delta}. \tag{16}
\]
Let us examine first the limit when \(P \uparrow 0\). Given that \(P^{\alpha^-} \to \infty\) and \(\int_0^P [(1+x)\alpha^-]^{-1} dx \to 0\) as \(P \uparrow 0\), we can apply l’Hospital’s rule to determine the limit. Straightforward calculations show that \(P^{\alpha^-} \int_0^P [(1+x)\alpha^-]^{-1} dx \to -P/[(1+P)\alpha^-] \to 0\) as \(P \uparrow 0\). A similar argument yields \(P^{\alpha^+} \int_P^\infty [(1+x)\alpha^+]^{-1} dx \to P/[(1+P)\alpha^+] \to 0\) as \(P \uparrow 0\). \(^{22}\) Hence, we have established that (16) satisfies the first boundary condition in (15). Now, consider the limit when \(P \uparrow \infty\). We can again use l’Hospital’s rule since \(P^{\alpha^-} \to 0\) and \(\int_0^P [(1+x)\alpha^-]^{-1} dx \to \infty\) as \(P \uparrow \infty\), so that \(P^{\alpha^-} \int_0^P [(1+x)\alpha^-]^{-1} dx \to -1/\alpha^-\) as \(P \uparrow \infty\). Similarly, we obtain \(P^{\alpha^+} \int_P^\infty [(1+x)\alpha^+]^{-1} dx \to 1/\alpha^+\) as \(P \uparrow \infty\). Hence we have
\[
\lim_{P \to \infty} W_t(P) = \frac{2(\mu_h - \mu_l)}{s^2(\alpha^+ - \alpha^-)} \left( -\frac{1}{\alpha^-} + \frac{1}{\alpha^+} \right) + \frac{\mu_l}{r + \delta} = \frac{2(\mu_h - \mu_l)}{s^2} \left( -\frac{1}{\alpha^-} + \frac{1}{\alpha^+} \right) + \frac{\mu_l}{r + \delta} = \frac{\mu_h}{r + \delta},
\]
\(^{22}\) Notice that \(\int_P^\infty [(1+x)\alpha^+]^{-1} dx < \int_P^\infty x^{-\alpha^+} dx = P^{-\alpha^+}/\alpha^+\). Thus \(\int_P^\infty [(1+x)\alpha^+]^{-1} dx\) is bounded for all \(P > 0\) and so the asset equation is well defined.
where the last equality follows from $\alpha^{-}\alpha^{+} = -2(r + \delta)/s^2$. Hence we have established that (16) also satisfies the second boundary condition in (15), which completes the derivation of $W_l(P)$.

The asset value of the high type is derived in a similar fashion. Given that

$$dP_t = P_t s (sdt + dZ_t)$$

when the ability of the worker is high, the asset value solves

$$(r + \delta) W_h (P) - P s^2 W'_h (P) - \frac{(Ps)^2}{2} W''_h (P) = w(P).$$

The homogenous solution is

$$W^H_h (P) = C_1 P^{\gamma^-} + C_2 P^{\gamma^+},$$

where $\gamma^-$ and $\gamma^+$ are the negative and positive roots of the following quadratic equation

$$\gamma (\gamma - 1) \frac{s^2}{2} + \gamma s^2 - r - \delta = 0.$$  

The non-homogenous solution is of the form

$$W^{NH}_h (P) = z_1 (P) P^{\gamma^-} + z_2 (P) P^{\gamma^+} + \frac{\mu_l}{r + \delta},$$

where the functions $z_1 (P)$ and $z_2 (P)$ satisfy

$$\begin{pmatrix} P^{\gamma^-} & P^{\gamma^+} \\ \gamma^- P^{\alpha^- - 1} & \gamma^+ P^{\gamma^+ - 1} \end{pmatrix} \begin{pmatrix} z'_1 (P) \\ z'_2 (P) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2(\mu_h - \mu_l)}{(1 + P) Ps^2} \end{pmatrix}.$$  

Following the same steps as before yields the solution in Proposition 1.

**Proof of Proposition 2:** The dynamics of the transition density of beliefs is captured by the Kolmogorov forward equation given in equation (4). By definition, the ergodic density satisfies the stationarity condition $df(p)/dt = 0$. The general solution of $f(p)$ reads

$$f(p) = C_{j(p)} p^{-1 - \eta}(1 - p)^{\eta - 2} + K_{j(p)} p^{\eta - 2}(1 - p)^{1 - \eta},$$

where $j(p) = 1_{\{p \geq p_0\}}$ and

$$\eta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{s^2} \delta}.$$  

The value of $C_{0f}$ can be set to zero since the integral of $f(p)$ from 0 to 1 should sum up to one, whereas $\int_{p_0}^p x^{-1 - \eta} dx \rightarrow \infty$ as $p \uparrow 0$. Similarly $K_{1f} = 0$ since $\int_{p_0}^p (1 - x)^{-1 - \eta} dx \rightarrow \infty$ as $p \uparrow 1$. The values of the two remaining constants are pinned down by the following requirements: (i) $\int_0^1 f(p) dp = 1$, 

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(ii) $\lim_{p \rightarrow p_0} f (p) = \lim_{p \rightarrow p_0^+} f (p)$. Condition (i) can be expressed analytically by integration of the ergodic density and use of the change of variable $P = p / (1 - p)$ to obtain

$$
\int_0^{p_0} f (p) \, dp = K_0 f \int_0^{p_0} p^{\eta - 2} (1 - p)^{-1 - \eta} \, dp
= K_0 f \int_0^{p_0} P^{\eta - 2} (1 + P) \, dP = K_0 f \left( \frac{\left( \frac{p_0}{1 - p_0} \right)^{\eta - 1}}{\eta - 1} + \frac{\left( \frac{p_0}{1 - p_0} \right)^{\eta}}{\eta} \right),
$$

while

$$
\int_{p_0}^{1} f (p) \, dp = C_1 f \int_{p_0}^{1} p^{-1 - \eta} (1 - p)^{\eta - 2} \, dp
= C_1 f \left( \frac{\left( \frac{p_0}{1 - p_0} \right)^{-\eta}}{\eta} + \frac{\left( \frac{p_0}{1 - p_0} \right)^{1 - \eta}}{\eta - 1} \right).
$$

Thus conditions (i) and (ii) are satisfied when

$$
\begin{pmatrix}
\left( \frac{p_0}{1 - p_0} \right)^{\eta - 1} + \left( \frac{p_0}{1 - p_0} \right)^{\eta} \left( \frac{p_0}{1 - p_0} \right)^{-\eta} + \left( \frac{p_0}{1 - p_0} \right)^{1 - \eta} \\
\left( \frac{p_0}{1 - p_0} \right)^{\eta - 2} (1 - p_0)^{-1 - \eta}
\end{pmatrix}
\begin{pmatrix}
K_0 f \\
C_1 f
\end{pmatrix} = \begin{pmatrix}
1 \\
0
\end{pmatrix}.
$$

**Proof of Lemma 1:** In a pooling equilibrium, no information is revealed by the education level $e_p$. Hence, firms’ initial beliefs are equal to the proportion of high-ability worker in the population, that is $p_0$. Define

$$L(e) = W_l (P_0) - c(e, \mu_l) - W_l^a + c(0, \mu_l).$$

By definition, $\bar{e}$ is such that $L(\bar{e}) = 0$. In a pooling equilibrium, the low type’s “participation constraint” is satisfied when

$$W_l (P_0) - c(e_p, \mu_l) \geq W_l^a - c(0, \mu_l),$$

i.e. $L(e_p) \geq 0$. Actually, the condition $L(e_p) \geq 0$ is necessary and sufficient for the existence of a pooling equilibrium at level $e_p$. For, if $L(e_p) \geq 0$, we can specify weakly consistent beliefs, e.g. $P(e_p) = P_0$ and $P(e) = 0$ for $e \neq e_p$. It follows that $W_l^a - c(0, \mu_l) > W_l^a - c(e, \mu_l)$ for all $e \neq 0, e_p$; and thus $e_p$ is sequentially rational for the low type. Since $W_h (P_0) > W_l (P_0)$, we have that $W_h (P_0) - c(e_p, \mu_l) > W_l^a - c(0, \mu_l)$. Thus the “participation constraint” for the high type is also satisfied, which in turn implies sequential rationality for the high type with the given equilibrium beliefs.

All that remains is to notice that, since

$$L'(e) = -c_e(e, \mu_l) < 0,$$

30
\( L \) is a strictly decreasing function, and hence \( L(e_p) \geq 0 \) if and only if \( e_p \in [0, \bar{e}] \).

**Proof of Proposition 3:** Define for \( i = h, l \)

\[
I_i(e, e_p) = W_h^s - c(e, \mu_i) - W_i(P_0) + c(e_p, \mu_i).
\]  

(18)

As explained in Section 3, the pooling equilibrium \( e_p \) fails the Intuitive criterion if and only if there exists an education level \( e^S \) such that \( I_l(e^S, e_p) \leq 0 \) and \( I_h(e^S, e_p) \geq 0 \). Let \( e^*(e_p) \) denote the minimum education level that does not trigger a profitable deviation for the low type, so that \( I_l(e^*(e_p), e_p) = 0 \). Since \( I_h(e, e_p) \) is strictly decreasing in \( e \), the pooling equilibrium is stable when \( I_h(e^*(e_p), e_p) < 0 \). To show that one can always find a signal/noise ratio above which this requirement is fulfilled, we need the following claim.

**Claim 1:** The asset values of high-ability and low-ability workers are respectively increasing and decreasing functions of the signal-noise ratio \( s \) for all \( p \in (0, 1) \).

We first establish this claim for low-ability workers. By definition

\[
W_l(P_t) = E_{P_t} \left[ \int_t^{\infty} e^{-r(t+\delta)(\tau-t)} w(P_\tau) d\tau \right]
\]

(19)

The second equality follows from Fubini’s theorem. The final inequality holds because \( w(\cdot) \) is a concave function and \( P_t \) is a martingale, so that \( E_{P_t} [w(P_\tau)] < w(P_t) \) for all \( \tau > t \). Notice that when \( P_t \) is equal to zero or one, its stochastic component vanishes which provides us with the boundary conditions described in equation (15).

Consider two different values of the signal/noise ratio such that \( \bar{s} > s \). We have for all \( P \in (0, 1) \)

\[
W_l''(P|s = \bar{s}) = \frac{2}{(P\bar{s})^2} [(r+\delta)W_l(P|s = s) - w(P)] < \frac{2}{(P\bar{s})^2} [(r+\delta)W_l(P|s = s) - w(P)],
\]

where the last inequality follows from equation (19). Hence

\[
W_l(P|s = \bar{s}) \leq W_l(P|s = s) \Rightarrow W_l''(P|s = \bar{s}) < W_l''(P|s = s) .
\]

Obviously, the requirement on the the second-order derivative is consistent with the boundary condition when \( P \to \infty \) in equation (15), if and only if \( W_l'(P|s = \bar{s}) > W_l'(P|s = s) \). However, this cannot be true for values of \( P \) in the neighborhood of 0, since \( W_l(0|s = \bar{s}) = W_l(0|s = s) \). Thus \( W_l(P|s = \bar{s}) \) can only intersect \( W_l(P|s = s) \) from below, which leads to a contradiction. The property for the high-ability workers can be established along the same lines. This completes the proof of the claim.
Recalling the definition of $I_h(e, e_p)$, we have

$$\lim_{s \to \infty} I_h(e^*(e_p), e_p) = \lim_{s \to \infty} (W_h^s - W_h(P_0)) + \lim_{s \to \infty} (c(e_p, \mu) - c(e^*(e_p), \mu)).$$

We first show that the second term on the right-hand side is negative. Given that $W_i(P_0|s = 0) < W_h^s$ and $c_e(e, \mu) > 0$, the relation $I_l(e^*(e_p), e_p) = 0$ requires that $e^*(e_p|s = 0) > e_p$. Claim 1 implies that $e^*(e_p|s) > e^*(e_p|s = 0)$ for all $s > 0$, which in turn yields $\lim_{s \to \infty} (c(e_p, \mu) - c(e^*(e_p), \mu)) < 0$. But Claim 1 also implies that $\lim_{s \to \infty} (W_h^s - W_h(P_0)) = 0$. Hence, one can always find a sufficiently high signal/noise ratio to ensure that $I_h(e^*(e_p), e_p) < 0$.

Finally define

$$s^*(e_p) = \begin{cases} 0 & \text{if } I_h(e^*(e_p), e_p|s = 0) \leq 0 \\ s & \text{otherwise.} \end{cases}$$

The proposition is established noticing that, since $I_h(e, e_p|s)$ is decreasing in $e$ and $e^*(e_p|s)$ is increasing in $s$, $I_h(e^*(e_p), e_p|s) < 0$ if and only if $s > s^*(e_p)$.

**Proof of Proposition 4:** For ease of notation, we define $I_i(e) = I_i(e^*(e), e)$, for $i = h, l$. Recall that a pooling equilibrium with education level $e_p$ fulfills the Intuitive criterion if and only if $I_h(e_p) < 0$. In order to characterize the region where this requirement is fulfilled, we prove the following claim.

**Claim 2:** When the marginal cost of education $c_e(e, \mu)$ is weakly log sub-modular, if the pooling equilibrium fails the Intuitive criterion for a given level of education, this is also true for any education level above it.

To prove this claim, we differentiate the equality defining $e^*(e_p)$ to obtain

$$\frac{de^*(e_p)}{de_p} = \frac{c_e(e_p, \mu)}{c_e(e^*(e_p), \mu)}.$$

hence $de^*(e_p)/de_p \in (0, 1)$. Differentiating $I_h(e)$ with respect to $e$ and replacing the solution above yields

$$\frac{dI_h(e_p)}{de_p} = -c_e(e^*(e_p), \mu) \left(\frac{de^*(e_p)}{de_p}\right) + c_e(e_p, \mu_h)$$

$$= c_e(e_p, \mu_h)c_e(e^*(e_p), \mu) - c_e(e_p, \mu)c_e(e^*(e_p), \mu).$$

When $c_e(e, \mu)$ is weakly log sub-modular, the last expression is superior or equal to zero. Thus $dI_h(e_p)/de_p \geq 0$ and so, if $I_h(e_p) < 0$ for a given $e_p$, this is also true for any education level above it, as stated in Claim 2.

Let $\tilde{e}(s)$ denote the lowest education level such that $I(\tilde{e}(s)) = 0$, if it exists. According to Claim 2, the region where the pooling equilibrium fulfills the Intuitive criterion should be below $\tilde{e}(s)$. When
the signal/noise ratio \( s = 0 \), \( I(0) > 0 \) and so all pooling equilibria fail the intuitive criterion. To characterize how \( \bar{c} \) changes with \( s \), consider the derivative of \( I_h(e_p|s) \) with respect to \( s \). Claim 2 implies that

\[
\frac{\partial I_h(e_p|s)}{\partial s} = -\frac{\partial W_h(P_0|s)}{\partial s} - c_\epsilon (e^*(e_p), \mu_h) \frac{\partial e^*(e_p)}{\partial s} < 0.
\]

Differentiating the equality \( I(\bar{c}(s)) = 0 \) with respect to \( s \) therefore yields

\[
\frac{d\bar{c}(s)}{ds} = -\frac{\partial I_h(\bar{c}(s))/\partial s}{\partial I_h(\bar{c}(s))/\partial e} > 0.
\]

Hence the function \( \bar{c}(s) \) is strictly increasing. Notice that when the cost function is log-linear, e.g. \( c(e, \mu) = e^2/\mu \), \( dI_h(e)/de = 0 \) for all \( e \). Then \( d\bar{c}(s)/ds = +\infty \), which implies that the \( \bar{c}(s) \) locus is vertical. In the limit

\[
\lim_{s \to \infty} I_h(e|s) = -c(e^*(e), \mu_h) + c(e, \mu_h),
\]

since \( W_h(P_0|s) \) converges to \( W^S_h \) as \( s \) goes to infinity. It follows that \( \lim_{s \to \infty} e^*(\bar{c}(s)) = \bar{c}(s) \). From the definition of \( e^*(\cdot) \) and the convexity of the educational costs, this can be true if and only if \( \bar{c}(s) \) diverges to infinity.

All that remains is to recall that, as shown in Lemma 1, \( e_p \) is a PBE if and only if \( e_p \leq \bar{e} \). But \( \bar{e} \) is strictly decreasing in \( s \) because \( \partial W_l(P_0|s)/\partial s < 0 \). Furthermore, since \( W_l(P_0|s) \) converges to \( W^S_l \) as \( s \) goes to infinity, it must be the case that \( \lim_{s \to \infty} \bar{e}(s) = 0 \). We can therefore conclude that \( \bar{e}(s) \) and \( \bar{c}(s) \) eventually intersect for sufficiently high values of the signal/noise ratio.

**Proof of Proposition 5:** In the separating equilibrium, low-ability worker choose the minimum level of education, i.e. zero. Given that \( c(0, \mu_l) \) can also be normalized to be zero without loss of generality

\[
e^*(0|s) = \{ e \in \mathbb{R}^+ : c(e, \mu_l) = W^S_h - W_l(P_0|s) \}.
\]

Since both \( W^S_h \) and \( c(e, \mu_l) \) are independent of the signal/noise ratio \( s \), the proposition immediately follows from Claim 1 in the proof of Proposition 3.

**Proof of Proposition 6:** In order to ease the algebra, we reverse the change of variable from \( P_l \) to \( p_l \) and obtain

(i) **Firm’s expectation:** \( dp_l = p_l (1 - p_l) s dZ_l \);

(ii) **High ability worker’s expectation:** \( dp_l = p_l (1 - p_l) s (1 - p_l) dt + dZ_l \);

(iii) **Low ability worker’s expectation:** \( dp_l = p_l (1 - p_l) s (-sp_l dt + dZ_l) \).

Thus the Hamilton-Jacobi-Bellman equations satisfied by the workers’ asset values are

\[
\begin{align*}
 rW_h(p) &= w(p) + \Sigma (p) (1 - p) s W'_h(p) + \left( \frac{\Sigma(p)^2}{2} \right) W''_h(p) - \delta W_h(p) \\
 rW_l(p) &= w(p) - \Sigma (p) ps W'_l(p) + \left( \frac{\Sigma(p)^2}{2} \right) W''_l(p) - \delta W_l(p)
\end{align*}
\]
where \( \Sigma(p) = sp(1-p) \). Reinserting their expressions into the expected surplus of the worker yields

\[
\begin{align*}
& r [p(W_h(p) - U_h) + (1-p)(W_i(p) - U_i)] \\
= & \quad w(p) - b + p \left[ \Sigma(p)(1-p)sW'_h(p) + \left( \frac{\Sigma(p)^2}{2} \right) W''_h(p) \right] \\
& + (1-p) \left[ -\Sigma(p)psW'_i(p) + \left( \frac{\Sigma(p)^2}{2} \right) W''_i(p) \right] \\
& - \delta (W_h(p) - U_h) - \lambda (W_h(p_0) - U_h) \\
& - \delta (W_i(p) - U_i) - \lambda (W_i(p_0) - U_i)
\end{align*}
\]

Since the firm’s asset value satisfies

\[
rJ(p) = \overline{\pi}(p) - w(p) + \left( \frac{\Sigma(p)^2}{2} \right) J''(p) - \beta J(p)
\]

the surplus-splitting condition is equivalent to

\[
0 = r ((1-\beta)E_p[W(p) - U] - \beta J(p)) \\
= w(p) - (1-\beta)b - \beta \overline{\pi}(p) - \lambda (1-\beta) (p(W_h(p_0) - U_h) + (1-p)(W_i(p_0) - U_i)) \\
+ (1-\beta) \left[ p \left( \Sigma(p)(1-p)sW'_h(p) + \left( \frac{\Sigma(p)^2}{2} \right) W''_h(p) \right) \right] \\
+ (1-p) \left[ -\Sigma(p)psW'_i(p) + \left( \frac{\Sigma(p)^2}{2} \right) W''_i(p) \right] \right] - \beta \left( \frac{\Sigma(p)^2}{2} \right) J''(p)
\]

Differentiating twice the bargaining rule yields

\[
\beta J''(p) = (1-\beta) (pW''_h(p) + 2W'_h(p) + (1-p)W''_i(p) - 2W'_i(p))
\]

Reinserting this expression into the surplus-splitting condition yields

\[
w(p) = (1-\beta)b + \beta \overline{\pi}(p) + \lambda (1-\beta) (p(W_h(p_0) - U_h) + (1-p)(W_i(p_0) - U_i)) \\
- \left[ p \left( \Sigma(p)(1-p)sW'_h(p) + \left( \frac{\Sigma(p)^2}{2} \right) W''_h(p) \right) \right] \\
+ (1-p) \left[ -\Sigma(p)psW'_i(p) + \left( \frac{\Sigma(p)^2}{2} \right) W''_i(p) \right] \right] \\
- \left( \frac{\Sigma(p)^2}{2} \right) \left( \frac{pW''_h(p) + 2W'_h(p)}{(1-p)W''_i(p) - 2W'_i(p)} \right)
\]

\[
= (1-\beta)b + \beta \overline{\pi}(p) + \lambda (1-\beta) (p(W_h(p_0) - U_h) + (1-p)(W_i(p_0) - U_i)) \\
- (1-\beta) \left( W'_h(p) \left( \Sigma(p)(1-p)s - \Sigma(p)^2 \right) + W'_i(p) \left( -\Sigma(p)p(1-p)s + \Sigma(p)^2 \right) \right) \\
= (1-\beta)b + \beta \overline{\pi}(p) + \lambda (1-\beta) (W_h(p_0) - U_h) + (1-p)(W_i(p_0) - U_i) \\
= p(\beta \mu_h - \mu_i) + \lambda (1-\beta) (W_h(p_0) - W_i(p_0) + U_i - U_h) \\
+ (1-\beta)b + \beta \mu_i + \lambda (1-\beta)(1-p)(W_i(p_0) - U_i)
\]

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which establishes the proposition.

**Proof of Proposition 7:** Since the proof is essentially similar to that of Proposition 1, the exposition can be brief. The HJB for the low-type is

\[(r + \delta) W_l(P) = \left( \frac{P}{1 + P} \right) \zeta + (1 - \beta)b + \beta \mu_l + \lambda(1 - \beta) (W_l(P_0) - U_l) + \frac{(Ps)^2}{2} W_l''(P) .\]

This is a second order non-homogenous ODE with non-constant coefficients. The homogenous solution is of the following form

\[W^H_l(P) = K_1 l P^{\alpha^-} + K_2 l P^{\alpha^+},\]

where \(\alpha^-\) and \(\alpha^+\) are defined in Proposition 1. The non-homogenous term is composed of a non-linear function of \(P\) plus a constant term. Thus we can assume that the particular solution is of the form

\[W^{NH}_l(P) = \begin{cases} v_1(P) P^{\alpha^-} + v_2(P) P^{\alpha^+} \end{cases} + \frac{(1 - \beta)b + \beta \mu_l + \lambda(1 - \beta) (W_l(P_0) - U_l)}{r + \delta} .\]

Usual derivations yield

\[
\begin{aligned}
  v_1(P) &= \frac{2 \zeta}{s^2 (\alpha^+ - \alpha^-)} \int \frac{1}{(1 + x) x^{\alpha^-}} dx \\
v_2(P) &= \frac{-2 \zeta}{s^2 (\alpha^+ - \alpha^-)} \int \frac{1}{(1 + x) x^{\alpha^+}} dx .
\end{aligned}
\]

As before, the bounds of integrations and constants of the homogenous solution are pinned down using boundary conditions

\[
\begin{cases}
  W_l(P) \xrightarrow{P \to P^+} U_l \\
  W_l(P) \xrightarrow{P \to \infty} \frac{\beta \mu_h + (1 - \beta)b + \lambda(1 - \beta) (W_h(P_0) - U_h)}{r + \delta} .
\end{cases}
\tag{20}
\]

The first condition ensures that the asset value at the separation threshold is equal to the worker’s outside option. The second condition ensures that, as \(P \uparrow \infty\), the worker’s lifetime income converges to the one of a high-ability worker in the absence of uncertainty. The second boundary condition obviously requires that \(K_{2l} = 0\), since \(P^{\alpha^+} \to \infty\) as \(P \uparrow \infty\). This implies that the non-homogenous solution must converge to the upper limit in (20). Consider the following function

\[
W^{NH}_l(P) = \frac{2 \zeta}{s^2 (\alpha^+ - \alpha^-)} \left( P^{\alpha^-} \int_{P}^{P^+} \frac{1}{(1 + x) x^{\alpha^-}} dx + P^{\alpha^+} \int_{P}^{\infty} \frac{1}{(1 + x) x^{\alpha^+}} dx \right) + \frac{(1 - \beta)b + \beta \mu_l + \lambda(1 - \beta) (W_l(P_0) - U_l)}{r + \delta} .
\tag{21}
\]

Using l’Hospital’s rule in a similar fashion than in the proof of Proposition 1 yields

\[
\lim_{P \to \infty} W^{NH}_l(P) = \frac{\zeta + (1 - \beta)b + \beta \mu_l + \lambda(1 - \beta) (W_l(P_0) - U_l)}{r + \delta} = \frac{\beta \mu_h + (1 - \beta)b + \lambda(1 - \beta) (W_h(P_0) - U_h)}{r + \delta} .
\]

Hence, we only have to ensure that the boundary condition when \( P \uparrow P^+ \) is satisfied. It is easily seen that this holds true when

\[
K_{1l} = (U_l - W_l^{NH}(P)) P^{-\alpha^{-}},
\]

which yields the expression in Proposition 7.

The Asset Value of the high-ability type is derived in a similar fashion. Given that

\[
dP_t = P_t s (s dt + dZ_t),
\]

when the ability of the worker is high, the asset value solves

\[
(r + \delta) W_h(P) = w(P) + Ps^2 W'_h(P) + \frac{(Ps)^2}{2} W''_h(P)
\]

Thus the homogenous solution is

\[
W^H_h(P) = K_{1h} P^{\gamma^-} + K_{2h} P^{\gamma^+}
\]

where \( \gamma^- \) and \( \gamma^+ \) are defined in Proposition 1. The non-homogenous solution is of the form

\[
W^{NH}_h(P) = k_1(P) P^{\gamma^-} + k_2(P) P^{\gamma^+} + \frac{(1 - \beta) b + \beta \mu + \lambda(1 - \beta) (W_l(P) - U_l)}{r + \delta}
\]

where the functions \( z_1(P) \) and \( z_2(P) \) satisfy

\[
\begin{pmatrix}
P^{\gamma^-} \\
\gamma^- P^{\alpha^- - 1}
\end{pmatrix}
\begin{pmatrix}
k_1'(P) \\
k_2'(P)
\end{pmatrix}
= \begin{pmatrix}
0 \\
-\frac{2c}{(1 + p) s^2}
\end{pmatrix}
\]

Following the same steps than for the low-ability type yields the solution given in Proposition 7. Finally, the asset value of the firm is given by the expression in Moscarini (2005), Proposition 2.

Proof of Proposition 8: As shown in the proof Proposition 2, the general solution of \( g(p) \) is given by

\[
g(p) = C_j(p) p^{1-\eta} (1 - p)^{\eta-2} + K_{1g} p^{\eta-2} (1 - p)^{-1-\eta},
\]

where \( j(p) = 1_{\{p \geq p_0\}} \). \( K_{1g} \) can be set to zero since \( \int_{p_0}^p (1 - x)^{-1-\eta} dx \to \infty \) as \( p \uparrow 1 \). We have to determine the values of three constants instead of two in Proposition 2. The additional requirement is that worker should spend no time at the separation threshold, so that \( g(p^+) = 0 \). This is true if and only if \( K_{0g} = -((1 - p)/2) p^{2\eta-1} C_{0g} \). Thus, the general expression of the ergodic distribution is as proposed in the main text. The values of the two remaining constants are pinned down as before by: (i) \( \int_0^1 g(p) dp = 1 \), and (ii) \( \lim_{p \to p_0^-} g(p) = \lim_{p \to p_0^+} g(p) \). Following the same steps than in the proof of
Proposition 2, we obtain

\[ \int_{p_0}^{p} g(p) \, dp = C_{0g} \left( \frac{p}{1-p} \right)^{2\eta-1} \left( \frac{p^0}{1-p^0} \right)^{\eta-1} + C_{0g} \left( \frac{p}{1-p} \right)^{\eta-1} \left( \frac{p^0}{1-p^0} \right)^{\eta-1} \left( \frac{p}{1-p} \right)^{\eta} \left( \frac{p^0}{1-p^0} \right)^{\eta} \]

whereas \( \int_{p_0}^{1} g(p) \, dp \) is given by equation (17). Finally, notice that the endogenous rate of job separation can also be derived in closed-form since

\[ p^2 (1-p)^2 g'(p^+) = C_{0g} p^{-\eta} (1-p)^{\eta-1} (2\eta-1) . \] 

(22)

References


