Lumps and Clusters in Duopolistic Investment Games:

An Early Exercise Premium Approach*

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Abstract

This paper investigates strategic investment policies in a duopolistic continuous-time real options game. Our contribution is twofold, economic and methodological. The former is the recognition that, under fixed costs of investment and time-to-build, a firm’s exercise of its capital-replacement option leads to a significant temporary reallocation of the firm’s revenues to its competitor. The latter is the introduction of the early exercise premium representation as a valuable device for the characterization of optimal exercise policies in real options games. Assuming exogenous firm roles, we find that (i) as the leader installs its newly purchased capital, the follower’s optimal investment policy displays a markedly convex and monotonically decreasing pattern over time, which finds its justification in the temporary transfer of the leader’s consumer demand to its competitor, and (ii) once the leader has completed its investment process, the follower’s trigger boundary – i.e., the level of market demand that renders capital replacement optimal – is time-independent. Moreover, we demonstrate that the follower’s willingness to delay investment is enhanced by a longer time-to-build and a more volatile market demand, while it is weakened by a higher quality improvement upon replacement and by a higher expected growth in market demand. Finally, we study the probability that the follower mimics the leader’s decision within the leader’s time-to-build window. We conclude that, while a higher quality advancement upon investment and a higher growth rate in market demand make it more likely for the follower to exercise its investment option promptly, a higher market uncertainty and a longer time-to-build alter the probability of an investment cluster non-monotonically.

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1 Introduction

Over the recent years, the merged theory of real options and strategic firm interactions within the game-theoretical paradigm has gained significant influence among both academic researchers and management practitioners. The broad appeal of this literature lies in its recognition of three distinctive features of corporate investment decisions: (i) the initial investment cost is largely sunk, which makes the investment decision often irreversible, (ii) the future stream of profits upon investing is uncertain and it is affected by the behavior of the firm’s competitors, and (iii) the timing of investment is at the discretion of the firm’s decision-makers. The interaction of these three attributes results in an isomorphism between a firm’s real investment projects and a set of financial positions in American options, which permits the application of some of the most advanced techniques developed in contingent claims analysis.

Dixit and Pindyck (1994) and Smit and Trigeorgis (2004) provide standard textbook treatments of real investment decisions under uncertainty such as new entry, determination of the initial scale of the firm, temporary shutdown and restart, and permanent exit. The survey article by Boyer, Gravel and Lasserre (2004) attempts to collect the most notable contributions to the literature on strategic investment games, from the pioneering works of Gilbert and Harris (1984) and Fudenberg and Tirole (1985) to more recent contributions. Amram and Kulatilaka (1999) rely on case studies to show the practitioner audience how real options can help improve capital investment planning and results. Grenadier (2000) represents a core reference volume on the developments of game theory and real options when addressing competition risk.

Because firm interactions tend to play a crucial role in the adoption of new technologies, the literature has merged the option-pricing methodology with the game-theoretical paradigm to analyze what is best viewed as strategic capital replacement games. Grenadier (1996) and Huisman and Kort (1998), among many others, explore the theoretical and applied approach of game theory and real options. Grenadier (1996) develops an equilibrium model for the optimal investment timing of two symmetric firms facing a deterministic time-to-build delay. In his model, the presence of a superior technology renders the older production unit obsolete, causing its revenues to fall. Working backwards in a dynamic programming fashion, Grenadier determines the firms’ optimal exercise policies and provides a rational explanation for overbuilding in the real estate market. Huisman and Kort (1998) adapt the Stenbacka and Tombak (1994) framework to analyze technology adoption assuming a stochastic time-to-build delay as the sole source of profit uncertainty. They consider dispersed versus joint equilibria in the case of endogenous firms’ roles and they find that the profit stream belonging to the preemption equilibrium can be so low that both firms would be better off never exercising their capital-replacement option.

We consider an economy populated by two symmetric firms, each holding a unique capital-replacement option over an infinite horizon. In this respect, our paper is analogous to those of Grenadier and Huisman and Kort. We assume that until one of the two firms exercises its option, both firms operate the same technology. Upon exercising its investment

1Throughout this article, the attribute "symmetric" refers to the firms' identical physical production endowments.
option, a firm is endowed with a full stock of higher-quality productive capital. Since each firm’s investment opportunity is unique and indivisible, this economy is characterized by *lumpy investment*.\(^2\) This is not to say that multiple capital upgrades are absent in this model. Rather, firms can continuously improve existing products marginally, via "cosmetic" changes, and occasionally they commit resources to fundamental improvements in their line of business. The engineering of the firm’s improvements is delegated to a team of research and development (R&D) specialists. While the ordinary duty of the firm’s R&D desk consists of creating incremental improvements to existing processes, once over the firm’s life this team is commissioned to the development of next-generation products. The output of the R&D team generates profits for the firm. Minor ongoing improvements provide an immediate flow of revenues while fundamental improvements, those associated with the exercise of the investment option, provide a higher flow of earnings, albeit delayed by the lengthy development of the new goods.

The contribution of this paper to the existing real options literature is twofold, economic and methodological. The economic contribution is the recognition that, under fixed costs of investment and time-to-build, a firm’s exercise of its capital-replacement option leads not only to a temporary "loss of output associated with the acquisition and installation of new capital goods (Cooper et al. 1999, p. 923)" but also to a significant temporary reallocation of the firm’s revenues to its competitor. Upon exercising its investment option, the firm pays a fraction of the implicit strike price to its competitor in the form of *transferred foregone consumer demand*. This results in a competitive advantage for the second mover,\(^3\) whose opportunity cost of investment is inflated by the prospect of transitory monopoly profits. By incorporating this effect we depart from the previous literature, which has exclusively modeled the degree of "harm" to the follower’s revenues caused by the leader’s adoption of the improved technology, and we provide a new perspective on understanding competitive behavior among firms.

The combination of stochastic firm revenues with our demand transfer mechanism renders the standard partial differential equations (PDE) approach intractable, calling for a more powerful solution methodology.\(^4\) Our methodological contribution consists of demonstrating the efficacy of the *early exercise premium* (EEP) representation for the characterization of optimal exercise policies in real options games. The EEP representation expresses an American option as the combination of a European option and the right to exercise the option early. This powerful decomposition of the American option value was derived by Kim (1990), Jacka (1991) and Carr, Jarrow, and Myneni (1992), building on the seminal work of Samuelson (1965) and McKean (1965). This representation is central to the analysis of American-style derivatives in the recent book by Detemple (2006).

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\(^2\)In a recent paper documenting the relevance of microeconomic lumpiness for aggregate investment dynamics, Gourio and Kashyap (2006) characterize lumpy investment based on two empirical regularities: first, "the typical manufacturing plant is likely to have at least one year when its capital stock surges," and second, "many establishments forgo investment in some years (p.1)."

\(^3\)Second-mover advantages are the subject of the empirical work of Cottrell and Sick (2001).

\(^4\)Hoppe (2000) and Hoppe and Lehmann-Grube (2001) explore a duopoly model of new technology adoption in which second-mover advantages accrue from uncertainty and decreasing adoption costs over time. Their simpler environment still permits the use of traditional solution methods to derive optimal investment policies.
Our solution method relies on the standard Stackelberg\(^5\) mechanism for subgame perfect equilibria. Since the core of our analysis is the characterization of the follower’s optimal response to the leader’s initiation of its investment process, firm roles are exogenously predetermined.\(^6\) We find that, conditional on the leader’s having completed its capital-replacement process, the investment trigger boundary for the follower — i.e., the level of market demand that renders capital replacement optimal — is time-independent and it is proportional to the \emph{adjusted fixed cost of investment}, which we define as the actual fixed cost of investment weighted by the present value of future revenues per unit of market demand. This result enables us to conduct comparative statics on the follower’s investment initiation boundary, which demonstrate that the follower’s willingness to delay investment is enhanced by a longer time-to-build and a more volatile market demand, while it is weakened by a higher quality improvement upon replacement as well as by a higher expected growth in market demand. In addition, we show that the value of the follower’s capital-replacement option monotonically increases with current market demand, regardless of the technological enhancement achieved upon investment and it becomes more valuable as the quality of the newly installed capital rises, this latter effect being magnified by higher market demand levels. We devote special attention to the near linearity of the option’s value along the market demand dimension, which at first may appear at variance with the well-established convexity properties of plain-vanilla contingent claims. This peculiarity reflects the relative contribution of the present value of the firm’s revenues prior to investment to the overall value of the investment option.

During the interim in which the leader does not operate in the market, the follower’s optimal investment policy solves a non-linear recursive integral equation, which we derive in pseudo closed form. We obtain an atypically convex and monotonically decreasing pattern for the follower’s boundary over time, which finds its justification in the temporary reallocation of the leader’s revenues to its competitor. This, in turn, motivates the follower’s increasing willingness to initiate its capital-replacement process as the opportunity cost of investment, in terms of foregone monopoly profits, declines over time.

Finally, we solve the leader’s problem: because all the expressions relevant to the pricing of the leader’s investment option are identical to those of the follower, we find that the leader’s trigger boundary as well as the value of its option to invest coincide with those of the follower.

The competitive advantage for the second mover referred to above engenders the salient question: what is the probability that the follower mimics the leader’s decision within the leader’s time-to-build window? We define this probability as the probability of an \emph{investment cluster}. Via a numerical exercise we conclude that, while a higher quality advancement upon investment and a higher growth rate in market demand make it more likely for the follower

\(^5\)Stackelberg (1934) proposes a dynamic model of duopoly in which a dominant firm (or leader) moves first and a subordinate firm (or follower) moves second. For a classic textbook treatment of the Stackelberg game the reader is referred to Myerson (2004, p. 187) and Fudenberg and Tirole (1991, pp. 67-69). Kulatilaka and Perotti (2000) apply the Stackelberg mechanism to a two-period real options study of time-to-market capability. The authors examine the decision to invest in logistics, market profiling and distribution capabilities that allow a firm to seize market share by being able to deliver a product ahead of competitors under Cournot quantity competition.

\(^6\)Initial conditions on the market demand determining the leadership position are analyzed in details in the Appendix.
to exercise its investment option promptly, a higher market uncertainty and a longer time-
to-build alter the probability of an investment cluster non-monotonically.

The structure of this paper is as follows. In Section 2 we describe the basic model: we
adopt the stochastic process for market demand as well as the Markovian revenue equations
of Ruffino and Treussard (2006). In Section 3 we apply the Stackelberg mechanism to
derive the trigger boundary for the low-quality producer. In Section 4 we solve the leader’s
problem and we determine via comparative statics the probability that the follower mimics
the leader’s decision within the leader’s time-to-build window. In Section 5 we summarize
the results of the paper. Mathematical derivations are collected in an Appendix (Section 6)
and the Matlab code implemented to obtain the numerical results presented in the body of
this article is compiled in Section 7.

2 The Model

In this Section we review the stochastic process for market demand and the Markovian
revenue equations from Ruffino and Treussard (2006).\textsuperscript{7}

We consider an infinite-horizon economy populated by two firms. At time $t = 0$, firm
$i - i \in \{1, 2\}$ - produces $y_{i;0} = \kappa$ units of output.\textsuperscript{8} At time $t_i$, firm $i$ initiates its capital-
replacement process, which leads the new capital of quality $\lambda (\lambda > 1$ is a quality factor
common across firms) to be operational at time $t_i + \Delta$, where $\Delta$ is a strictly positive time-to-
built delay. At time $t_i + \Delta$, firm $i$ is endowed with $\kappa$, a full stock of higher-quality productive
capital. We denote $y_{i;t} = k_{i;t}$ as firm $i$’s output capacity at time $t$. The aggregate quantity
produced is given by

$$Y_t = \sum_{i=1}^{2} y_{i;t}. \tag{1}$$

The inverse demand function for this commodity is given by

$$P_t = X_t D(Y_t), \tag{2}$$

where $\frac{\partial D(Y)}{\partial Y} < 0$ and $X_t$ is an exogenous demand-shock process that obeys

$$dX_t = \alpha X_t dt + \sigma X_t dz_t. \tag{3}$$

We account for quality improvement by assuming that the adoption of the new technology
allows the firm to sell its output at a per-unit price equal to $\lambda P_t$. In this economy, the

\textsuperscript{7}Ruffino and Treussard (2006) examine strategic investment in the context of a duopolistic continuous-
time real options game and derive a set of economic and mathematical conditions under which both firms
optimally retain their investment option forever. In this article, we depart from those benchmark conditions
to describe active strategic interactions, as those observed daily in the real world.

\textsuperscript{8}Our assumption of identical initial capital stock levels across firms is made for mathematical simplicity
only. None of our results would be qualitatively altered if we departed from this condition. Optimal capacity
determination in a continuous-time real options game is studied, among others, by Wu (2006).
time-discount rate is a constant $r > \alpha > 0$ so that that the present value discount factor is

$$b_{t,v} = \frac{b_v}{b_t} = e^{-r(v-t)}.$$  

Until one of the two firms exercises its capital-replacement option, both firms operate the same technology. Since each firm’s investment opportunity is unique and indivisible, this economy is characterized by lumpy investment. We assume that firm 1 is the first firm to exercise its investment option, i.e. firm 1 is the Stackelberg leader. At time $t_1$, firm 1 suspends its production operations and pays a fixed cost $I$. The instantaneous revenues for firm 1, $R_1(t) = k_{1,t}P_t$, follow

$$R_1(t) = \begin{cases} 
  k_{1,t} \left[ X_t D\left( \sum_{i=1}^2 k_{i,t} \right) \right], & 0 \leq t < t_1 \\
  0, & t_1 \leq t < t_1 + \Delta \\
  k_{1,t} \left[ \lambda X_t D\left( \sum_{i=1}^2 k_{i,t} \right) \right], & t_1 + \Delta \leq t < t_2 \\
  k_{1,t} \left[ X_{t_1} D\left( k_{1,t} \right) \right], & t_2 \leq t < t_2 + \Delta \\
  k_{1,t} \left[ \lambda X_t D\left( \sum_{i=1}^2 k_{i,t} \right) \right], & t_2 + \Delta \leq t < \infty 
\end{cases} \quad (4)$$

On the other hand, the flow of revenues for firm 2, $R_2(t) = k_{2,t}P_t$, follows

$$R_2(t) = \begin{cases} 
  k_{2,t} \left[ X_t D\left( \sum_{i=1}^2 k_{i,t} \right) \right], & 0 \leq t < t_1 \\
  k_{2,t} \left[ X_{t_1} D\left( k_{2,t} \right) \right], & t_1 \leq t < t_1 + \Delta \\
  k_{2,t} \left[ X_t D\left( \sum_{i=1}^2 k_{i,t} \right) \right], & t_1 + \Delta \leq t < t_2 \\
  \lambda X_t D\left( \sum_{i=1}^2 k_{i,t} \right), & t_2 \leq t < t_2 + \Delta \\
  k_{2,t} \left[ \lambda X_t D\left( \sum_{i=1}^2 k_{i,t} \right) \right], & t_2 + \Delta \leq t < \infty 
\end{cases} \quad (5)$$

Equation (5) embodies our paper’s economic recognition that, under fixed costs of investment and time-to-build, a firm’s exercise of its capital-replacement option leads not only to a temporary "loss of output associated with the acquisition and installation of new capital goods (Cooper et al. 1999, p. 923)" but also to a significant temporary reallocation of the firm’s revenues to its competitor. Upon exercising its investment option, the firm pays a fraction of the implicit strike price to its competitor in the form of transferred foregone consumer demand. This effect is captured by the fact that $k_{2,t} \left[ X_{t_1} D\left( k_{2,t} \right) \right] > k_{2,t} \left[ X_t D\left( \sum_{i=1}^2 k_{i,t} \right) \right]$, under the assumption that $D(Y)$ is monotonically decreasing in $Y$. Equations (4) and (5) reflect each firm’s decision to either improve existing products marginally or to commit R&D resources to fundamental improvements. For instance, firm 2’s higher revenues during the interval $[t_1, t_1 + \Delta]$ find their justification in that the "low-hanging" improvements created

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9 As highlighted by Dixit and Pindyck (1994, p.138), $r > \alpha > 0$ is necessary to obtain non-degenerate investment policies. Accordingly, when $r = \alpha$, Ruffino and Treussard (2006) find that perpetual inaction is optimal even in the presence of positive capital depreciation, technology improvement, and harm effects to the low-technology producer.

10 The interruption of physical production over the time-to-build phase does reflect the reality of numerous industries, examples of which are the restaurant or movie theater businesses as well as the production of aircraft.

11 Our modeling assumption that adopting a new technology allows the firm to sell its output at a per-unit price equal to $\lambda P_t$ is pivotal in proving the consistency of the revenue streams in Eqs. (4) and (5) with the results of Grossman and Helpman’s model of sequential quality improvement (Ruffino and Treussard, 2006).
by its R&D team are more readily embraced by its temporarily wider market base.\textsuperscript{12}

Throughout the remainder of this article we assume a specific functional form for the market demand \( D(Y_t) \). In particular, we adopt the form \( D(Y_t) = Y_t^{-\gamma} \), where \( \gamma \) is the constant inverse elasticity parameter taking values in the interval \((0, 1)\). Our assumption of an isoelastic inverse demand function is also present in the model of Caballero and Pindyck (1996): in their calibration \( \gamma = \frac{1}{2} \), which falls within our range. The bounds we impose on the inverse demand elasticity parameter guarantee well-defined firm revenues as the aggregate quantity supplied becomes infinitesimal.

3 The Follower’s Optimal Investment Behavior

Since firm 1 is the first to invest in a new capital stock, we begin by determining the optimal policy for firm 2 in a backward induction manner. Afterwards we solve firm 1’s optimal investment problem. This solution method follows the standard Stackelberg mechanism for subgame perfect equilibria, imposing sequentiality.

Consider the problem of firm 2. Since firm 1 has already exercised its capital-replacement option, we focus on times \( t \geq t_1 \), the investment initiation time for firm 1. For any such \( t \), firm 2’s value is equal to the value of one of the following two options: (i) an option to initiate the capital-replacement process between \( t_1 \) and \( t_1 + \Delta \), whose value is denoted by \( F \), and (ii) an option to initiate the investment process beyond time \( t_1 + \Delta \), whose value is denoted by \( W \). If unexercised at time \( t_1 + \Delta \), the \( F \) option expires and is exchanged with the \( W \) option. Over the time interval \( t_1 \leq t < t_1 + \Delta \), the revenue stream for firm 2 is equal to \( k_{2,t} \left[ X_t D(k_{2,t}) \right] \) while it is equal to \( k_{2,t} \left[ \sum_{i=1}^{2} k_{i,t} \right] \) beyond \( t_1 + \Delta \) until the capital-replacement process is initiated (Eq. (5)).

3.1 Optimal Behavior Beyond \( t_1 + \Delta \)

Suppose that firm 1 has already begun to produce using its new capital stock. The strategy space for firm 2 is \( S_{t_1 + \Delta, \infty} \), the set of stopping times that take values between \( t_1 + \Delta \) and \( \infty \).

For any \( t_2 \in S_{t_1 + \Delta, \infty} \), the value of the follower’s capital-replacement option is given by

\[
W_t(EP, H, t_2) = E_t \left[ \int_{t}^{t_2} b_{t,v} dH_v + b_{t,t_2} [EP_{t_2} - I] \right]
= E_t \left[ \int_{t}^{t_2} e^{-r(v-t)} dH_v + e^{r(t_2-t)} [EP_{t_2} - I] \right]. \tag{6}
\]

In Eq. (6) \( dH_t \) is the flow of instantaneous revenues received until the time of exercise.\textsuperscript{13}

\textsuperscript{12}It is worth noting the existence of industrial sectors in which widely acknowledged followers behave consistently with our modeling assumptions. For instance, commentators and journalists have often focused on Microsoft’s strategy. On December 1, 2006, the \textit{Wall Street Journal} quotes Dave Winer’s assessment that "Microsoft isn’t an innovator, and never was. They are always playing catch-up, by design. That’s their M.O. They describe their development approach as "chasing tail lights." (...) They let others develop the markets, and have been content to catch-up."

\textsuperscript{13}Unlike in the more traditional partial differential equation (PDE) methods presented in Dixit and Pindyck (1994), we model the flow of instantaneous revenues received until the time of exercise, \( dH_t \), as
which is equal to
\[ dH_t = k_{2,t} \left[ X_t D(\Sigma_{i=1}^{2} k_{i,t}) \right] dt = 2^{-\gamma} k^{1-\gamma} X_t dt, \]
and \( EP_t \) is the value of the future flow of revenues earned beyond time \( t_2 + \Delta \) when the option to replace the firm’s capital is exercised at time \( t = t_2 \). Equation (8) below contains the closed-form expression for \( EP_t \), the derivation of which is contained in the Appendix.

\[ EP_t = E_t \left[ \int_{t+\Delta}^{\infty} b_{t,v} k_{2,v} \left[ \lambda X_v D(\Sigma_{i=1}^{2} k_{i,v}) \right] dv \right] = \lambda 2^{-\gamma} k^{1-\gamma} X_t \frac{e^{((\alpha-\gamma)\Delta)}}{r - \alpha}. \]

In what follows we rely on the early exercise premium (EEP) representation for American options to study strategic firm behavior, as that embedded in Eq. (6).\(^{14}\) The early exercise premium representation is parametric in that it expresses the American option value as a function of its unknown optimal exercise boundary. However, using the fact that immediate exercise is optimal when the boundary is reached, the EEP formula produces a recursive integral equation, which can be used to study the characteristics of the boundary. More specifically the EEP representation formula provides a decomposition of the American contingent claim into its European counterpart and the right to exercise the option early.\(^{15}\)

Accordingly, Eq. (6) can be represented as the sum of a European option and the early exercise premium, denoted \( EEP_t \). Hence, Eq. (6) becomes

\[ W_t(EP, H, t_2) = \lim_{T \to \infty} E_t \left[ \int_{t}^{T} b_{t,v} dH_v + b_{t,T} \left[ EP_T - I \right]^+ \right] + EEP_t, \]

where

\[ EEP_t = \lim_{T \to \infty} E_t \left[ \int_{t}^{T} b_{t,v} 1_{\{\tau_v = v\}} (r [EP_v - I] dv - dA_v - dH_v) \right], \]

in which \( \tau_v \) is the first time at which exercise becomes optimal. Detemple (2006, p. 43) provides intuition for the local gains from early exercise in Eq. (10): \( r [EP_v - I] dv \) is the amount of interest collected over time if one invests immediately and places the proceeds in the riskless account,\(^{16}\) \(-dA_v \) is the loss incurred by exercising early due to foregone expected appreciation in the payoffs, and \(-dH_v \) is the loss due to foregone cash flows earned prior to investment. The expected appreciation in payoffs \( dA_t \) follows from a standard application\(^{17}\) of a dividend flow on the firm’s capital-replacement option. This is in line with Merton (1998) and Detemple (2006).

\(^{14}\)For a valuable textbook exposition of the valuation of American-style derivatives, the reader is referred to Detemple (2006).

\(^{15}\)An alternative decomposition is the delayed exercise premium (DEP) representation, which emphasizes the gains from waiting to exercise. The DEP representation, developed by Carr, Jarrow, and Myneni (1992), expresses the value of the American option as the sum of the payoffs upon immediate exercise and the additional value of waiting.

\(^{16}\)The interest gains collected over time could result from a change in the firm’s ownership at the optimal exercise time for the investment option. The current owners would sell the firm’s assets worth \( EP_v - I \) and invest the proceeds in the riskless account.
of Ito’s Lemma to Eq. (8),

\[ dEP_t = EP_t [\alpha dt + \sigma dz_t] = \alpha EP_t dt + \sigma EP_t dz_t. \tag{11} \]

Armed with an explicit formula for each of the components entering Eq. (10), we obtain the closed-form formula for \( EEP_t \) as a function of the unknown optimal exercise boundary \( B \). Substituting Eqs. (7), (8) and (11) into Eq. (10) yields

\[
EEP_t = \lim_{T \to \infty} E_t \left[ \int_t^T b_{t,v} 1_{\{\tau_v = v\}} \left( r \left[ EP_v - I \right] dv - dA_v - dH_v \right) \right] = \lim_{T \to \infty} E_t \left[ \int_t^T b_{t,v} 1_{\{\tau_v = v\}} \left( r \left[ EP_v - I \right] dv - \alpha EP_v dv - 2^{-\gamma} \kappa^{1-\gamma} X_v dv \right) \right] = \lim_{T \to \infty} E_t \left[ \int_t^T b_{t,v} 1_{\{\tau_v = v\}} \left( 2^{-\gamma} \kappa^{1-\gamma} \left( \lambda e^{(\alpha - r)\Delta} - 1 \right) X_v - r I \right) \right]. \tag{12}
\]

The object of the next Proposition is the provision of a pseudo closed-form solution to Eq. (12).

**Proposition 1** The follower’s early exercise premium, \( EEP_t \), is equal to

\[
EEP_t = 2^{-\gamma} \kappa^{1-\gamma} \left( \lambda e^{(\alpha - r)\Delta} - 1 \right) X_t \left[ \int_t^T e^{(\alpha - r)(v-t)} \Phi(d_1) dv \right] - r I \left[ \int_t^T e^{(\alpha - r)(v-t)} \Phi(d_1 - \sigma \sqrt{v-t}) dv \right], \tag{13}
\]

where \( \Phi(\cdot) \) is the cumulative standard normal distribution function and

\[
d_1 = \frac{\ln \left( \frac{X_t}{I_v} \right) + (\alpha + \frac{1}{2} \sigma^2)(v-t)}{\sigma \sqrt{v-t}}.
\]

**Proof.** The crucial step in obtaining Eq. (13) is to replace the indicator function \( 1_{\{\tau_v = v\}} \) with the indicator function \( 1_{\{X_v \geq B_v\}} \), which permits to integrate over the density function of \( X_v \). A complete proof of Eq. (13) is contained in the Appendix.

In the following Proposition, we evaluate the European component of the follower’s option to invest, which, when combined with the result of Proposition 1, provides us with a pseudo closed-form formula for \( W_t(EP, H, t_2) \) in terms of the unknown boundary \( B \).

**Proposition 2** The value of the European counterpart to the American contingent claim \( W_t(EP, H, t_2) \) is

\[
\lim_{T \to \infty} E_t \left[ \int_t^T e^{-(r)(v-t)} dH_v \right] + \lim_{T \to \infty} E_t \left[ e^{-(r)(T-t)} \right] \left[ EP_T - I \right]^+ = \frac{2^{-\gamma} \kappa^{1-\gamma}}{r - \alpha} X_t, \tag{14}
\]

\footnote{See the proof of Theorem 34, p. 76 in Detemple (2006).}
which in turn determines the value of firm 2’s capital-replacement option to be

$$W_t(EP, H, t_2) = \frac{2^{-\gamma}r^{1-\gamma}}{r-\alpha}X_t$$

(15)

\[ + 2^{-\gamma}r^{1-\gamma}\left(\lambda e^{((\alpha-r)\Delta)} - 1\right)X_t \lim_{T \to \infty} \left[ \int_t^T e^{\{(a-r)(v-t)\}} \Phi(d_1) dv \right] \]

\[ - rI \lim_{T \to \infty} \left[ \int_t^T [e^{-r(v-t)}] \Phi(d_1 - \sigma\sqrt{v-t})] dv \right], \]

where \(d_1\) is as in Proposition 1.

Proof. Evaluating the second term in Eq. (14) requires that we define

$$\bar{I} = \frac{I}{\lambda 2^{-\gamma}r^{1-\gamma}e^{((\alpha-r)\Delta)}},$$

the denominator of which is the present value of future revenues upon exercise per unit of market demand, \(\frac{EP}{X_t}\), or demand multiplier. Accordingly, we refer to \(\bar{I}\) as the adjusted fixed cost of investment per unit of demand multiplier. Making use of this newly-defined parameter, we can readily demonstrate that the second term in Eq. (14) vanishes as \(T\) approaches \(\infty\). The details of this proof are contained in the Appendix.

Since immediate exercise is optimal the first time the boundary is reached, Eq. (15) produces a recursive integral equation for the trigger boundary. Theorem 1 below states the value of the trigger boundary for the follower, which is proportional to the adjusted fixed cost of investment \(\bar{I}\).

**Theorem 1** The immediate exercise boundary for the follower beyond \(t_1 + \Delta\) is given by

$$B_\infty = \bar{I} \frac{\beta}{\beta - 1},$$

in which \(\bar{I} = \frac{I}{\lambda 2^{-\gamma}r^{1-\gamma}e^{((\alpha-r)\Delta)}}\), \(\beta = \frac{\sqrt{2\pi}r^2 + \rho^2 - \rho}{\sigma^2}\) and \(\rho = \alpha - \frac{1}{2}\sigma^2\).

Proof. Recognizing that, as \(T\) approaches \(\infty\), the immediate exercise boundary becomes time-independent, i.e., \(B = B_\infty\), is crucial to our derivation. In particular, it allows us to rewrite \(d_1\) as \(\frac{(\alpha + \frac{1}{2}\sigma^2)(v-t)}{\sigma\sqrt{v-t}}\), which does not depend on the boundary. The integration over the time interval \([t, \infty)\) directly obtains by making use of the identity

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-au}}{\sqrt{u}} du = \frac{\sqrt{2}}{\sqrt{a}} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2a}},$$

resulting from the change of variable \(x = \sqrt{2a}u\). The reader can find the detailed proof of Theorem 1 in the Appendix.

Theorem 1 enables us to conduct comparative statics on the follower’s investment initiation boundary: Figure 1 and Figure 2 depict the sensitivity of \(B_\infty\) to changes in parameter
values. In Figure 1 we present the effects of jointly varying the quality-improvement factor and the time-to-build delay.

Figure 1
Comparative Statics on the Follower’s Immediate Exercise Boundary:
Quality-Improvement Factor and Time-to-Build Delay

![Graph showing the effects of quality-improvement factor and time-to-build delay on the follower's immediate exercise boundary.]

**Calibration:** The fixed cost of investment, $I$, is set equal to 100. The drift, $\alpha$, and the volatility, $\sigma$, of the demand process are equal to 0.02 and 0.05 respectively. The time-discount rate, $r$, is 0.05, the concavity parameter, $\gamma$, is 0.8 and the capital stock, $\kappa$, is equal to 1000.

Figure 1 shows that the follower’s decision to initiate its capital-replacement process is insensitive to the length of the time-to-build delay when the quality-improvement upon replacement is expected to be large (e.g., $\lambda = 1.5$). Conversely, assuming a low quality enhancement factor (e.g., $\lambda = 1.1$), the follower’s willingness to postpone investment increases as the time-to-build lag rises from six months to two years, which translates into the follower’s need for a higher level of market demand to initiate its investment process. Overall, the follower’s investment trigger boundary decreases monotonically as the quality-improvement factor becomes large. For example, for $\Delta$ equal to one year, the boundary declines from approximately 35 ($\lambda = 1.1$) to 5 ($\lambda = 1.5$). Thus we conclude that the follower’s reluctance to invest is higher the longer the time-to-build and the lower the quality-improvement factor.
In Figure 2 we consider the effects of a higher expected demand growth rate, $\alpha$, and a higher market demand volatility, $\sigma$, on the follower’s immediate exercise boundary.

**Figure 2**
*Comparative Statics on the Follower’s Immediate Exercise Boundary: Drift and Volatility of the Stochastic Demand Process*

*Calibration:* The fixed cost of investment, $I$, is set equal to 100. The quality-improvement factor, $\lambda$, is 1.2. The time-to-build delay is set at $\Delta = 1$. The time-discount rate, $r$, is 0.05, the concavity parameter, $\gamma$, is 0.8 and the capital stock, $\kappa$, is equal to 1000.

Figure 2 shows that higher demand growth rates lead to lower values for the follower’s investment trigger boundary. Indeed, if the follower expects market demand to grow at a higher rate, a lower current market demand level is sufficient to induce it to exercise its capital-replacement option. Assuming a market demand volatility $\sigma$ equal to 0.05, the boundary falls from 16 ($\alpha = 0.01$) to 11 ($\alpha = 0.0495$). Furthermore, Figure 2 reveals that higher market uncertainty increases the follower’s reluctance to invest. When market demand is expected to grow at a rate $\alpha$ equal to 0.02, the boundary rises from 13 ($\sigma = 0.01$) to 18 ($\sigma = 0.15$). Hence, we conclude that the follower’s willingness to postpone investment is reduced by a higher expected growth in market demand and it is strengthened by added market uncertainty.
Table 1 summarizes the comparative statics we conduct on the follower’s investment boundary, as depicted in Figure 1 and Figure 2.

### Table 1
Comparative Statics on the Follower’s Immediate Exercise Boundary: A Summary

<table>
<thead>
<tr>
<th>Model Parameters</th>
<th>Sensitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quality-Improvement Factor</td>
<td>$\frac{\partial B_v}{\partial \beta} &lt; 0$</td>
</tr>
<tr>
<td>Time-to-Build Delay</td>
<td>$\frac{\partial B_v}{\partial \Delta} &gt; 0$</td>
</tr>
<tr>
<td>Drift of Demand Process</td>
<td>$\frac{\partial B_v}{\partial \beta} &lt; 0$</td>
</tr>
<tr>
<td>Volatility of Demand Process</td>
<td>$\frac{\partial B_v}{\partial \sigma} &gt; 0$</td>
</tr>
</tbody>
</table>

The following Corollary provides the value of the follower’s option conditional upon the leader having reentered the market with a full stock of higher-quality productive capital.

**Corollary 1** The value of the follower’s capital-replacement option at time $t \geq t_1 + \Delta$ is given by

$$W_t(EP, H, t_2) = \begin{cases} 
2^{-\gamma} \kappa^{1-\gamma} X_t + \frac{2^{-\gamma} \kappa^{1-\gamma} (\lambda e^{((\alpha-r)\Delta)} - 1)}{r - \alpha} & \text{if } X_t < B_{\infty} \\
EP_t - I & \text{if } X_t \geq B_{\infty}
\end{cases} (16)$$

**Proof.** The value of the follower’s replacement option is obtained by replacing $B_v$ with $B_{\infty} = \tilde{I} \frac{\beta}{\beta - 1}$, as derived in Theorem 1, in Eq. (15). This allows us to apply a formula often used to price perpetual American options (e.g., Kim, 1990)

$$\int_0^\infty (r - \alpha) X_t e^{-(r - \alpha)u} \Phi \left( \frac{\ln \left( \frac{X_t}{B_{\infty}} \right) + (\alpha + \frac{1}{2} \sigma^2)u}{\sigma \sqrt{u}} \right) du$$

$$- \int_0^\infty r \tilde{I} e^{-ru} \Phi \left( \frac{\ln \left( \frac{X_t}{B_{\infty}} \right) + (\alpha - \frac{1}{2} \sigma^2)u}{\sigma \sqrt{u}} \right) du$$

$$= \frac{\tilde{I}}{\beta - 1} \left( \frac{X_t (\beta - 1)}{\tilde{I} \beta} \right)$$

which leads to the direct obtention of Eq. (16). \qed
We next analyze the behavior of the follower’s option value by conducting a set of comparative statics. Figure 3 displays the value of firm 2’s investment option as we vary the quality-improvement factor, $\lambda$, and the level of current market demand, $X_t$.

**Figure 3**

Comparative Statics on the Follower’s Investment Option: Quality-Improvement Factor and Demand Process

![Graph showing the value of the follower's investment option](image)

**Calibration:** The fixed cost of investment, $I$, is set equal to 100. The drift, $\alpha$, and the volatility, $\sigma$, of the demand process are equal to 0.02 and 0.05 respectively. The time-discount rate, $r$, is 0.05, the concavity parameter, $\gamma$, is 0.8 and the capital stock, $\kappa$, is equal to 1000. The time-to-build delay, $\Delta$, is equal to 1.

As Figure 3 reveals, the value of the follower’s capital-replacement option monotonically increases with current market demand, regardless of the technological enhancement achieved upon investment. In addition, firm 2’s option to invest becomes more valuable as the quality of the newly installed capital rises, this latter effect being magnified by higher market demand levels.

Perhaps the most striking feature of the surface displayed in Figure 3 is its near linearity along the market demand dimension, which may appear at variance with the well-established convexity properties of plain-vanilla contingent claims (e.g., Merton, 1973). However the reader should recall that, unlike the more traditional PDE treatments of real options, we model the flow of instantaneous revenues received until exercise, $dH_t$, as a dividend flow on the firm’s capital-replacement option. Therefore, the observed linearity results from the relative contribution of the present value of the firm’s revenues prior to investment, $E_t \left[ \int_t^\infty b_{t+1} dH_t \right]$, to the overall value of the investment option. Net of this component, the investment option displays an accentuated curvature, as shown in the left panel of Figure 4.
**Calibration:** The fixed cost of investment, $I$, is set equal to 100. The drift, $\alpha$, and the volatility, $\sigma$, of the demand process are equal to 0.02 and 0.05 respectively. The time-discount rate, $r$, is 0.05, the concavity parameter, $\gamma$, is 0.8 and the capital stock, $\kappa$, is equal to 1000. The quality-improvement factor $\lambda$ is equal to 1.2 and the time-to-build delay, $\Delta$, is equal to 1.

Figure 4 illustrates numerically the degree of convexity of the follower’s option as stated in Eq. (16). For values of current market demand, $X_t$, in the interval $[0, B_{\infty}]$, the value of this option is equal to

$$W_t(EP, H, t_2) = \frac{2^{-\gamma} \kappa^{1-\gamma}}{r - \alpha} X_t + \frac{2^{-\gamma} \kappa^{1-\gamma}}{r - \alpha} \left( \lambda e^{((\alpha - r)\Delta)} - 1 \right) \left[ \frac{\bar{I}}{\beta - 1} \left( \frac{X_t(\beta - 1)}{\bar{I}\beta} \right)^\beta \right],$$

which, for interpretive purposes, we rearrange as

$$W_t(EP, H, t_2) = \Pi_1 X_t + \Pi_2 X_t^\beta.$$

Based on our calibration, $\Pi_1$ numerically dominates $\Pi_2$, which delivers the approximate linearity displayed in the right panel of Figure 4.

Having completed the study of the follower’s behavior past the leader’s time-to-build phase, we turn to the question of whether firm 2 would replace its capital stock prior to $t_1 + \Delta$. Over this time interval the revenue stream for firm 2 is equal to $k_{2,t} [X_tD(k_{2,t})]$, as defined in Eq. (5).
3.2 Optimal Behavior Prior to \( t_1 + \Delta \)

Suppose that firm 1 has not begun to produce with its newly purchased stock of capital. The strategy space for firm 2 is \( S_{t_1,t_1+\Delta} \), the set of stopping times that take values between \( t_1 \) and \( t_1 + \Delta \). For any \( t_2 \in S_{t_1,t_1+\Delta} \), the value of the follower’s capital-replacement option is given by

\[
F_t(EP, H, t_2) = E_t \left[ \int_t^{t_2} b_{t,v} dH_v + b_{t,t_2} [EP_{t_2} - I] \right] = E_t \left[ \int_t^{t_1+\Delta} b_{t,v} dH_v + b_{t,t_1+\Delta} \max [EP_{t_1+\Delta} - I, W_{t_1+\Delta}(EP, H, t_2)] \right] + EEP_t
\]

in which

\[
dH_t = k_{2,t} [X_t D(k_{2,t})] dt = \kappa^{1-\gamma} X_t dt,
\]

\( EP_t \) is given by Eq. (8), and the last equality follows directly from the formula for the American option in Eq. (16).

Proposition 3 below contains the exact form for the \( F \) option’s early exercise premium, which is used to state the recursive equation for the option value.

**Proposition 3** The value of the follower’s capital-replacement option at time \( t_1 \leq t \leq t_1 + \Delta \) is given by

\[
F_t(EP, H, t_2) = \kappa^{1-\gamma} X_t \frac{1 - e^{(\alpha-r)(t_1+\Delta-t)}}{r - \alpha} + \frac{2-\gamma}{r - \alpha} b_{t_1,t_1+\Delta} X_t e^{(\alpha r)} + \frac{2-\gamma}{r - \alpha} \kappa^{1-\gamma} \left( \lambda e^{(\alpha-r)\Delta} - 1 \right) \frac{I}{\beta - 1} \left( \frac{\beta - 1}{\beta} \right)^{\beta} b_{t,t_1+\Delta}
\]

\[
\cdot \Phi \left( -\frac{\ln X_t}{B_\infty} + \frac{(\alpha + (\beta - 1/2) \sigma^2) \tau}{\sigma \sqrt{\tau}} \right) X_t^\beta e^{\frac{\beta(2\alpha + (\beta - 1/2) \sigma^2) \tau}{2}}
\]

\[+ b_{t_1,t_1+\Delta} \Phi \left( \frac{\ln X_t}{B_\infty} + \frac{(\alpha + (\beta - 1/2) \sigma^2) \tau}{\sigma \sqrt{\tau}} \right) \left[ \frac{2-\gamma}{r - \alpha} \kappa^{1-\gamma} \left( \lambda e^{(\alpha-r)\Delta} - 1 \right) - I \right]
\]

\[+ \kappa^{1-\gamma} (\lambda e^{(\alpha-r)\Delta} - 1) X_t \left[ \int_t^{t_1+\Delta} e^{(\alpha-r)(v-t)} \Phi (d_1) dv \right]
\]

\[\cdot \Phi \left( \int_t^{t_1+\Delta} e^{(\alpha-r)(v-t)} \Phi (d_1 - \sigma \sqrt{v-t}) dv \right),
\]

in which \( d_1 \) is as in Proposition 1.
Proof. In Eq. (17) the value of the follower’s option is decomposed into its European counterpart, $E_t \left[ \int_t^{t_1+\Delta} b_{t,v} dH_v + b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,t_2) \right]$, and the value of early exercise privileges, denoted as $EEP_t$. We resolve the challenge of evaluating $E_t [b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,t_2)]$ by partitioning the spectrum of values of the market demand process at time $t_1 + \Delta$. This allows us to reexpress $E_t [b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,t_2)]$ as

$$E_t [b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,t_2) | X_{t_1+\Delta} < B_\infty] \Pr (X_{t_1+\Delta} < B_\infty) + E_t [b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,t_2) | X_{t_1+\Delta} \geq B_\infty] \Pr (X_{t_1+\Delta} \geq B_\infty),$$

which we solve piecwise. The expressions for $E_t \left[ \int_t^{t_1+\Delta} b_{t,v} dH_v \right]$ and $EEP_t$ result from extending earlier derivations. The full proof of Proposition 3 is available in the Appendix.

Equation (19) provides us with the principal building block of the recursive integral equation that characterizes the follower’s investment boundary, $B_t$, for $t \in [t_1, t_1 + \Delta]$. The obtention of the integral equation requires to specify a terminal condition, which is provided in the next Lemma.

Lemma 1 The terminal condition to the recursive integral equation is given by

$$B_{(t_1+\Delta)_-} = \bar{I} \max \left( \frac{r}{r_0}, \frac{\beta}{\beta-1} \right) = \bar{I} \frac{\beta}{\beta-1} = B_\infty.$$

Proof. The key steps in the proof of Lemma 1 consist of identifying relevant lower bounds for the immediate exercise boundary $B_{(t_1+\Delta)_-}$ and determining which lower bound coincides with the true value of the boundary at time $(t_1 + \Delta)_-$. These lower bounds are $B_{(t_1+\Delta)_-} \geq \bar{I} \frac{\beta}{\beta-1}$ and $B_{(t_1+\Delta)_-} \geq \bar{I} \frac{r}{r_0}$. Following from the follower’s optimal exercise policy at time $t_1 + \Delta$ and from the $EEP_t$ integrand, respectively. We demonstrate by contradiction that the first inequality is necessarily dominant and that it must hold with equality. Finally, since $B_t = B_\infty, \forall t \geq t_1 + \Delta$ (from Eq. (17) and Theorem 1) and $B_{(t_1+\Delta)_-} = B_\infty$ (from the above demonstration), we conclude that the recursive boundary is continuous at $t_1 + \Delta$. ■

Theorem 2 supplies the precise form of this recursive equation, which the reader may recognize as the standard value-matching condition exploited in partial differential equation methods. The value-matching condition requires that the value of the net payoff upon exercise, $EP_t - I$, be equal to the value of the option, $F_t(EP,H,t_2)$. 

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Theorem 2 At each instant $t$ prior to $t_1 + \Delta$, the optimal exercise boundary for the follower satisfies the equality

$$
\left( \lambda_2^{-\gamma} \kappa^{1-\gamma} e^{(\alpha r) \Delta} \right) B_t - I = \kappa^{1-\gamma} \frac{1 - e^{((\alpha - r)(t_1 + \Delta - t))}}{r - \alpha} B_t + 2\gamma \kappa^{1-\gamma} b_{t,t_1+\Delta} e^{(\alpha r)} B_t - \\
+ \frac{2\gamma^{1-\gamma} \left( \lambda e^{((\alpha - r)\Delta)} - 1 \right)}{r - \alpha} \frac{\beta}{\beta - 1} \left( \frac{I^\beta}{I^\beta} \right) b_{t,t_1+\Delta} \\
\Phi \left( -\ln \frac{B_t}{B_\infty} + \left( \alpha + \left( \beta - \frac{1}{2} \right) \sigma^2 \right) \tau \right) \frac{\sigma}{\sqrt{\tau}} B_t^\beta e^{\left( \frac{\beta^2(2\sigma^2 + (\beta - 1)\sigma^2)}{2} \right)} \\
+ b_{t,t_1+\Delta} \Phi \left( \ln \frac{B_t}{B_\infty} + \left( \alpha + \frac{1}{2} \sigma^2 \right) \tau \right) \frac{\sigma}{\sqrt{\tau}} B_t^\beta e^{\left( \frac{\beta^2(2\sigma^2 + (\beta - 1)\sigma^2)}{2} \right)} \\
\cdot \left[ \frac{2\gamma^{1-\gamma} \left( \lambda e^{((\alpha - r)\Delta)} - 1 \right)}{r - \alpha} B_t e^{(\alpha r)} - I \right] \\
+ \kappa^{1-\gamma} \left( \lambda e^{((\alpha - r)\Delta)} - 1 \right) B_t \\
\cdot \left[ \int_t^{t_1+\Delta} e^{((\alpha - r)(v-t))} \Phi \left( \ln \frac{B_t}{B_\infty} + \left( \alpha + \frac{1}{2} \sigma^2 \right) (v - t) \right) \frac{\sigma}{\sqrt{v - t}} dv \right] \\
- r I \left[ \int_t^{t_1+\Delta} e^{-(r(v-t))} \Phi \left( \ln \frac{B_t}{B_\infty} + \left( \alpha - \frac{1}{2} \sigma^2 \right) (v - t) \right) \frac{\sigma}{\sqrt{v - t}} dv \right] ,
$$

subject to the terminal condition $B_{(t_1 + \Delta)_-} = B_\infty$.

Since Eq. (20) is not analytically tractable, we resort to numerical methods for the determination of the follower’s investment boundary. Over the past decade, financial economists and engineers have devoted much effort to the development of accurate numerical approximations for American-style derivatives prices and optimal exercise boundaries. Most notably, Broadie and Detemple’s (1996) procedure yields a lower bound for the exercise boundary, lower and upper bounds for the option value, and approximate option prices. Unlike previously developed methodologies constructed directly from the recursive equation for the optimal exercise boundary, their approach concentrates on a class of exercise policies that can be readily evaluated and chooses the best one within that class. In a noteworthy article, Kallast and Kivinukk (2003) combine a simple approximation of the recursive integral equations by quadrature formulas with the Newton-Raphson iteration procedure. This method to compute the optimal exercise boundary at each point in time leads to an efficient and accurate numerical procedure. This article implements the latter method, which is further discussed in Subsection 7.2. The Matlab function and its outer shell for the obtention of the boundary prior to $t_1 + \Delta$ produce Figure 5 below.
Immediate Exercise Boundary for the Follower prior to $t_1 + \Delta$

**Calibration:** The fixed cost of investment, $I$, is set equal to 100. The drift, $\alpha$, and the volatility, $\sigma$, of the demand process are equal to 0.02 and 0.05 respectively. The time-discount rate, $r$, is 0.05, the concavity parameter, $\gamma$, is 0.8 and the capital stock, $\kappa$, is 1000. The quality-improvement factor $\lambda$ is equal to 1.2 and the time-to-build delay, $\Delta$, is equal to 1. The investment horizon is discretized into 2000 annual time steps.

Figure 5 displays a markedly convex and monotonically decreasing\(^{18}\) pattern for the follower’s boundary over the interval $[t_1, t_1 + \Delta]$.\(^{19}\) As the time to maturity of the $F$ option, $t_1 + \Delta - t$, shortens, the slope of the boundary diminishes and the demand level separating the continuation region from the immediate exercise region converges to the terminal value $B_\infty$. This rather unusual shape finds its justification in the temporary reallocation of the leader’s revenues upon exercise to its competitor. More specifically, the leader pays a fraction of the implicit strike price to the follower in the form of transferred consumer demand, as specified in Eq. (5). As a result, the follower’s willingness to delay the initiation of its capital-replacement process decreases considerably as the opportunity cost of investment, in terms of foregone monopoly profits, declines over time.

\(^{18}\)For certain parameterizations, we observe minor local non-monotonicities near the maturity date as the result of numerical imperfections.

\(^{19}\)The reader may prefer to convert the trigger value from units of the underlying demand process, $X$, to units of expected profits. The benchmark calibration of Figure 5 yields a conversion factor of 88.75. Accordingly, to trigger the follower’s exercise at time $t_1$, expected profits should be as high as 18 times its initial fixed cost of investment, the magnitude of this number reflecting the relative contribution of the present value of the firm’s revenues prior to investment to the overall value of the investment option.
Having determined the optimal policy for firm 2, we now turn to firm 1’s problem. In the next Section we first solve the leader’s problem and we subsequently determine, via comparative statics, the probability that the follower mimics the leader’s decision within the leader’s time-to-build window.

4 Firm 1’s Optimal Investment Behavior

In the previous Section we applied the standard Stackelberg mechanism for subgame perfect equilibria to derive the trigger boundary for firm 2 conditional on firm 1’s investment policy $t_1$. We now consider firm 1’s investment decision.

The leader firm holds a perpetual American option with a strike price equal to $I$ and it receives a flow of revenues equal to

$$dH_t = k_{1,t} \left[ X_t D(\Sigma_{i=1}^2 k_{i,t}) \right] dt = \kappa X_t (2\kappa)^{-\gamma} dt = 2^{-\gamma} \kappa^{1-\gamma} X_t dt \tag{21}$$

until it initiates its capital-replacement process. We denote the strategy space for firm 1 by $S_{0,\infty}$, the set of stopping times that take values between 0 and $\infty$. For any $t_1 \in S_{0,\infty}$ the value of the leader’s capital-replacement option is

$$G_t(EP,H,t_1) = E_t \left[ \int_{t}^{t_1} b_{t,v} dH_v + b_{t,t_1}[EP_{t_1} - I] \right]$$

$$= \lim_{T \to \infty} E_t \left[ \int_{t}^{T} e^{-(r(v-t))} dH_v + e^{-(r(T-t))} [EP_T - I]^+ \right] + EEP_t. \tag{22}$$

The obtention of $EP_t$, the value of the flow of future revenues earned beyond time $t_1 + \Delta$, is the object of the following Proposition.

**Proposition 4** The value of the flow of future revenues earned beyond time $t_1 + \Delta$, when the option to replace the firm’s capital is exercised at time $t = t_1$, is equal to

$$EP_t = E_t \left[ \int_{t+\Delta}^{\infty} b_{t,v} k_{1,v} [\lambda X_v D(\Sigma_{i=1}^2 k_{i,v})] dv \right]$$

$$= E_t \left[ \int_{t+\Delta}^{\infty} \left[ 1_{\{t_2 \geq v\}} + 1_{\{v \geq t_2 + \Delta\}} + \frac{(1 - 1_{\{t_2 \geq v\}})(1 - 1_{\{v \geq t_2 + \Delta\}})}{2^{-\gamma}} \right] b_{t,v} (\lambda 2^{-\gamma} \kappa^{1-\gamma}) X_v dv \right]$$

$$= \lambda \kappa^{1-\gamma} E_t \left[ \int_{t+\Delta}^{\infty} b_{t,v} X_v dv \right] + (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{t_2 \geq v\}} b_{t,v} X_v dv \right]$$

$$+ (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{v \geq t_2 + \Delta\}} b_{t,v} X_v dv \right],$$

which simplifies to

$$EP_t = 2^{-\gamma} \lambda \kappa^{1-\gamma} X_t \frac{e^{\left(\frac{(\alpha-r)\Delta}{r-\alpha}\right)}}{r-\alpha}. \tag{23}$$
Proof. The proof of Proposition 4 relies on the application of probabilistic methods often exploited in exotic option-pricing theory (e.g., Shreve, 2004) as well as on a careful analysis of the integration regions for the transformed state variable \( Y_t = \frac{X_t}{B_t} \). The former aspect is prescribed by the isomorphism between the expectations \( E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{v \geq t_2 + \Delta\}} b_{t,v} X_v d\nu \right] \) and \( E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{v \geq t_2 + \Delta\}} b_{t,v} X_v d\nu \right] \) and the payoffs arising from knock-out and knock-in option portfolios, respectively. The latter aspect allows for significant simplifications towards the derivation of a closed-form solution for \( EP_t \): recognizing the equivalence between the indicator functions \( 1_{\{\hat{M} \geq b, (\hat{z} - \hat{z}_i) \geq k\}} \) and \( 1_{\{\hat{M} \leq b, (\hat{z} - \hat{z}_i) \geq k\}} \) permits us to express the value of the implicit knock-in option portfolio as the difference between the value of a portfolio of zero-strike-price European calls and that of their knock-out counterparts. The reader is referred to the Appendix for a complete mathematical proof of Proposition 4.

As the reader can verify, Eq. (23) is identical to the value of future revenues earned by the follower beyond \( t_2 + \Delta \), which is reported in Eq. (8). Consequently, the expected appreciation in payoffs, \( dA_v \), is common across firms and is given by Eq. (11). Furthermore, Eqs. (21) and (7), which contain the revenue flows prior to investment for the leader and for the follower respectively, are identical. Thus, substituting Eqs. (21), (23) and (11) into the EEP formula yields

\[
EEP_t = \lim_{T \to \infty} E_t \left[ \int_t^T b_{t,v} 1_{\{v = v\}} \left( r [EP_v - I] dv - dA_v - dH_v \right) \right] = \lim_{T \to \infty} E_t \left[ \int_t^T b_{t,v} 1_{\{v = v\}} \left( 2^{-\gamma} k^{1-\gamma} (\lambda e^{\{(a-r)\Delta\}} - 1) X_v - rI \right) dv \right].
\]

(24)

Since Eq. (24) is identical to Eq. (12), we appeal to Proposition 1 and we conclude that

\[
EEP_t = 2^{-\gamma} k^{1-\gamma} (\lambda e^{\{(a-r)\Delta\}} - 1) X_t \left[ \int_t^T e^{\{(a-r)\nu\}} \Phi \left( d_1 \right) d\nu \right] - rI \left[ \int_t^T e^{\{-(v-t)r\}} \Phi \left( d_1 - \sigma \sqrt{v-t} \right) d\nu \right],
\]

in which

\[
d_1 = \frac{\ln \left( \frac{X_t}{B_t} \right) + (\alpha + 1/2 \sigma^2)(v - t)}{\sigma \sqrt{v-t}}.
\]

Moreover, because all the expressions relevant to the pricing of the leader’s investment option are identical to those leading to Proposition 2, the value of the leader’s capital replacement option is given by Eq. (15).
Thus, Theorem 1 applies and yields the constant immediate exercise boundary $B_1$ equal to $ar{I} \frac{\beta}{\beta - 1}$, with $eta = \frac{\sqrt{2} \sigma^2 + \rho^2 - \rho}{\sigma^2}$, $ho = \alpha - \frac{1}{2} \sigma^2$ and $ar{I} = \frac{I}{\frac{2 \alpha - \gamma}{r - \alpha} \left( \frac{1}{(r - \alpha) \Delta} - 1 \right)}$. Accordingly, the value of the leader’s investment option is

$$G_t(EP, H, t_1) = \begin{cases} 2^{-\gamma} \kappa^{1-\gamma} X_t + \frac{2^{-\gamma} \kappa^{1-\gamma} \left( \lambda e^{\left( (\alpha - r) \Delta \right)} - 1 \right)}{r - \alpha} \left[ \frac{\bar{I} \left( X_t \left( \frac{\beta - 1}{I \beta} \right) \right)^{\beta}}{\beta - 1} \right] & \text{if } X_t < B_\infty \\ EP_t - I & \text{if } X_t \geq B_\infty \end{cases}$$

(25)
a consequence of Corollary 1.

Figure 6 below summarizes the equilibrium strategies for both the leader and the follower. It also introduces our study of the determinants of the probability that the follower emulates the leader’s decision to invest within the leader’s time-to-build window.

**Figure 6**

**Optimal Investment Strategies: A Summary of the Game’s Equilibrium**

**Calibration:** The fixed cost of investment, $I$, is set equal to 100. The drift, $\alpha$, and the volatility, $\sigma$, of the demand process are equal to 0.02 and 0.05 respectively. The time-discount rate, $r$, is 0.05, the concavity parameter, $\gamma$, is 0.8 and the capital stock, $\kappa$, is 1000. The quality-improvement factor $\lambda$ is equal to 1.2 and the time-to-build delay, $\Delta$, is equal to 1. The investment horizon is discretized into 2000 annual time steps.

Conditional on the leader having initiated its capital-replacement process, the follower’s optimal strategy is to exercise its investment option the first time that $X_t$ reaches the trigger boundary $B$. Over the interval $[t_1, t_1 + \Delta]$, the trigger boundary for firm 2 solves the recursive integral equation in Theorem 2, while beyond time $t_1 + \Delta$ it assumes the form of $B_\infty$ stated in Theorem 1. Assuming exogenous roles, the leader invests in a full stock of higher quality productive capital the first time that $X_t$ reaches the boundary $B_\infty$. This occurs at the random time $t_1$. Beyond $t_1$, the demand process continues to evolve stochastically, thereby
originating a non-degenerate equilibrium distribution of investment times for firm 2. Two sample realizations of the market demand process are depicted in Figure 6.

Next we study the probability that the follower mimics the leader’s decision to invest within the leader’s time-to-build window: this is equivalent to determining the probability that the demand process $X_t$ attains the follower’s boundary prior to $t_1 + \Delta$, which we define as the probability of an investment cluster.\(^{20}\)

**Figure 7**

Probabilities of Attaining the Trigger Boundary Prior to $t_1 + \Delta$ Varying the Quality-Improvement Factor and the Time-to-Build Delay (Years)

---

**Calibration:** The investment horizon is discretized into $D = 2000$ annual time steps. The benchmark parameters are $I = 100$, $\alpha = .02$, $\sigma = .05$, $r = .05$, $\gamma = .8$, $\kappa = 1000$. 30,000 paths are simulated.

Figure 7 shows that the probability of an investment cluster increases monotonically as the quality-improvement factor becomes large. For example, for $\Delta$ equal to one year, this probability grows from 74.9\% ($\lambda = 1.2$) to 95.2\% ($\lambda = 1.5$). Now consider the effects of lengthening the time-to-build interval on the probability of an investment cluster, which entails two conflicting phenomena. First, as depicted in Figure 1, increasing $\Delta$ leads to an increased willingness to postpone investment. However, a second effect is that increasing $\Delta$ raises the likelihood of a substantial market demand growth prior to $t_1 + \Delta$. Figure 7 illustrates that while the former effect is dominant for short time-to-build delays, the latter

\(^{20}\)Note that expanding our model to a general equilibrium context would deliver null probabilities of clusters: indeed, unless marginal utility from consumption is bounded at zero, markets could not clear if both firms were simultaneously retooling. However, we would expect strictly positive probabilities of investment clustering in economies populated by three or more firms, as in those of Buois, Huisman and Kort (2006), since simultaneous investment by all firms would not constitute an equilibrium.
prevails for large values of $\Delta$ (e.g., $\Delta = 2.5$). However, the nuanced reversal in the shape of the above curves appears only at low levels of the quality-improvement factor (e.g., $\lambda = 1.2, 1.3$); when the quality-improvement upon replacement is expected to be large (e.g., $\lambda = 1.4, 1.5$), the follower’s decision to initiate its capital-replacement process is insensitive to the length of the time-to-build delay (see Figure 1). For instance, assuming a 20% quality enhancement upon capital replacement ($\lambda = 1.2$), the probability of the follower’s rapidly emulating the leader diminishes from 77.5% to 74.9% as $\Delta$ varies from six months to one year, and rises thereafter up to 76.1% ($\Delta = 2.5$).

Thus, we conclude that while a higher expected quality advancement makes it more likely for the follower to exercise its investment option promptly, a longer time-to-build phase impacts the probability of investment clusters non-monotonically. In Figure 8 we consider the effects of a higher expected demand growth rate, $\alpha$, and a higher market demand volatility, $\sigma$, on the probability of an investment cluster.

**Figure 8**

Probabilities of Attaining the Trigger Boundary Prior to $t_1 + \Delta$ Varying the Drift and Volatility of the Stochastic Demand Process

![Graph showing probabilities of attaining the trigger boundary](image)

**Calibration**: The investment horizon is discretized into $D = 2000$ annual time steps. The benchmark parameters are $I = 100$, $\lambda = 1.2$, $\Delta = 1$, $r = .05$, $\gamma = .8$, $\kappa = 1000$. 30,000 paths are simulated.

Figure 8 shows that the higher the expected demand growth rate the more likely the occurrence of an investment cluster. This monotonic increase in probabilities reflects the shape of the surface displayed in Figure 2: if the follower expects higher market growth, it perceives the investment opportunity as more attractive at a lower current demand level. Assuming a market demand volatility $\sigma$ equal to 0.05, the probability of an investment cluster grows from 66.9% ($\alpha = 0.01$) to 90.7% ($\alpha = 0.0495$). The effect of a rise in demand volatility
is twofold: first, higher market uncertainty increases the follower’s reluctance to invest, as it is shown in Figure 2, and second, increasing volatility improves the chances of the market demand’s growing substantially over the time-to-build interval. The "smile" shape of the contour curves in Figure 8 is indicative of the dominance of the former (latter) effect for low (high) degrees of market uncertainty. For instance, when market demand is expected to grow at a rate \( \alpha \) equal to 0.01, the probability of a cluster falls from 98.7% (\( \sigma = 0.01 \)) to 66.9% (\( \sigma = 0.05 \)) and it rises afterwards up to 71% (\( \sigma = 0.15 \)). Hence, we conclude that the probability of the follower’s emulating the leader prior to \( t_1 + \Delta \) is enhanced by a higher expected growth in market demand and it is convex and non-monotonic in market uncertainty.

Table 4 summarizes the comparative statics on the probability that the follower mimics the leader’s decision to invest within the leader’s time-to-build window, as quantified in Figures 7 and 8.

<table>
<thead>
<tr>
<th>Model Parameters</th>
<th>Sensitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quality-Improvement Factor</td>
<td>( \frac{\partial Pr(X_{t_1} \geq B_1 \text{ for some } t: t_1 \leq t &lt; t_1 + \Delta)}{\partial \alpha} &gt; 0 )</td>
</tr>
<tr>
<td>Time-to-Build Delay</td>
<td>( \frac{\partial Pr(X_{t_1} \geq B_1 \text{ for some } t: t_1 \leq t &lt; t_1 + \Delta)}{\partial \Delta} \leq 0 )</td>
</tr>
<tr>
<td>Drift of Demand Process</td>
<td>( \frac{\partial Pr(X_{t_1} \geq B_1 \text{ for some } t: t_1 \leq t &lt; t_1 + \Delta)}{\partial \alpha} &gt; 0 )</td>
</tr>
<tr>
<td>Volatility of Demand Process</td>
<td>( \frac{\partial Pr(X_{t_1} \geq B_1 \text{ for some } t: t_1 \leq t &lt; t_1 + \Delta)}{\partial \sigma} \leq 0 )</td>
</tr>
</tbody>
</table>

We conclude our probabilistic analysis of investment clusters by contrasting our results to Grenadier’s (1996). Grenadier develops a strategic real options game, which he solves via the partial differential equation (PDE) approach. In Section 4 of his article, Grenadier conducts comparative statics to study the occurrence of development cascades in the real estate market. He finds that the net effect of increasing demand volatility is a decrease in the median time between the leader and the follower’s investment executions. This result is only partially consistent with our findings in Figure 8. We attribute this discrepancy to Grenadier’s assumption of immediate harm effects to the follower (conditional on the leader’s exercise of its option), which does not produce a time-dependent trigger boundary such as those we obtain in Figures 5 and 6.
5 Conclusion

In this paper we consider an economy populated by two symmetric firms, each holding a unique capital-replacement option over an infinite horizon. Until one of the two firms exercises its capital-replacement option, both firms operate the same technology. Upon exercising its investment option, a firm is endowed with a full stock of higher-quality productive capital.

The contribution of this paper to the existing real options literature is twofold, economic and methodological. The economic contribution is the recognition that, under fixed costs of investment and time-to-build, a firm’s exercise of its capital-replacement option leads to a significant temporary reallocation of the firm’s revenues to its competitor. This results in a competitive advantage for the second mover, whose opportunity cost of investment is inflated by the prospect of transitory monopoly profits.

By incorporating this effect we depart from the previous literature, which has exclusively modeled the degree of "harm" to the follower’s revenues caused by the leader’s adoption of the improved technology, and we provide a new perspective on understanding competitive behavior among firms. The combination of stochastic firm revenues with this demand transfer mechanism renders the standard partial differential equations (PDE) approach intractable, calling for a more powerful solution methodology. Our methodological contribution consists of demonstrating the superiority of the EEP representation for the characterization of optimal exercise policies in real options games.

We find that, conditional on the leader’s having completed its capital-replacement process, the investment trigger boundary for the follower is proportional to the adjusted fixed cost of investment, which we define as the actual fixed cost of investment weighted by the present value of future revenues per unit of market demand. This result enables us to conduct comparative statics on the follower’s investment initiation boundary, which demonstrate that the follower’s willingness to delay investment is enhanced by a longer time-to-build and a more volatile market demand, while it is weakened by a higher quality improvement upon replacement as well as by a higher expected growth in market demand. In addition, we show that the value of the follower’s capital-replacement option monotonically increases with current market demand, regardless of the technological enhancement achieved upon investment and it becomes more valuable as the quality of the newly installed capital rises, this latter effect being magnified by higher market demand levels. We devote special attention to the near linearity of the option’s value along the market demand dimension, which at first may appear at variance with the well-established convexity properties of plain-vanilla contingent claims. This peculiarity reflects the relative contribution of the present value of the firm’s revenues prior to investment to the overall value of the investment option.

During the interim in which the leader does not operate in the market, the follower’s optimal investment policy solves a non-linear recursive integral equation, which we derive in pseudo closed form. We obtain an atypically convex and monotonically decreasing pattern for the follower’s boundary over time, which finds its justification in the temporary reallocation of the leader’s revenues to its competitor. This, in turn, motivates the follower’s increasing willingness to initiate its capital-replacement process as the opportunity cost of investment, in terms of foregone monopoly profits, declines over time.

Finally, we solve the leader’s problem: because all the expressions relevant to the pricing of the leader’s investment option are identical to those of the follower, we find that the
leader’s trigger boundary as well as the value of its option to invest coincide with those of the follower.

The competitive advantage for the second mover referred to above engenders the salient question: what is the probability that the follower mimics the leader’s decision within the leader’s time-to-build window? We define this probability as the probability of an investment cluster. We conclude that, while a higher quality advancement upon investment and a higher growth rate in market demand make it more likely for the follower to exercise its investment option promptly, a higher market uncertainty and a longer time-to-build alter the probability of an investment cluster non-monotonically.

Future venues of research extending the focus of this paper may include the introduction of capital depreciation, first-mover advantages as well as repeated capital upgrades and oligopolistic markets.
References


6 Appendix

The Section collects a detailed description of the ex-ante values of the leader and the follower as well as all lengthy mathematical derivations and proofs used in the body of the article.

6.1 Ex-ante Values of the Leader and the Follower

In this Subsection, we denote the ex-ante values of the leader and the follower by $V_{0,L}$ and $V_{0,F}$ respectively. These values obtain from Eq. (25) for the leader and from Eqs. (7) and (19) for the follower. Their functional forms are given by

\[ V_{0,L} = G_0(EP, H, t_1) \]

and

\[
V_{0,F} = E_0 \left[ \int_0^{t_1} e^{(-rt)} dH + e^{(-rt_1)} F_{t_1}(EP, H, t_2) \right] \\
= 2^{-\gamma} R^{1-\gamma} X_0 E_0 \left[ \int_0^{t_1} e^{(\alpha-rt_1)\sigma^2} dv \right] + E_0 \left[ e^{(-rt_1)} \right] F_{t_1}(EP, H, t_2),
\]

in which the second equality in the expression for $V_{0,F}$ follows from the definition of the $X$ process and from the non-stochastic nature of $F_{t_1}(EP, H, t_2)$, a direct consequence of the leader’s optimal behavior, as derived in Section 4. In addition, the level of initial demand that equates the values of the two firms is indicated by $X_{0}^{(V_{0,L}=V_{0,F})}$. Based on our benchmark calibration, we identify two regions in the space of initial values for the $X$ process

\[
\begin{align*}
V_{0,L} < V_{0,F} & \quad \text{if} \quad X_0 < X_{0}^{(V_{0,L}=V_{0,F})} \\
V_{0,L} > V_{0,F} & \quad \text{if} \quad X_0 > X_{0}^{(V_{0,L}=V_{0,F})}
\end{align*}
\]

Our numerical simulations deliver a value for $X_{0}^{(V_{0,L}=V_{0,F})} \in (B_{t_1,L}, B_{t_1,F})$, where $B_{t_1,L}$ ($= B_{\infty}$) and $B_{t_1,F}$ are the leader and the follower’s trigger boundaries at $t_1$, respectively. This allows us to further decompose the space of initial values for the $X$ process to distinguish between sequential and simultaneous option exercise. In the first region, where $X_0 < X_{0}^{(V_{0,L}=V_{0,F})}$, two outcomes may occur: either $X_0 < B_{t_1,L}$, in which case the leader would invest at time $t_1 > 0$, or $X_0 \geq B_{t_1,L}$, in which case the leader would invest exactly at time $t_1 = 0$. In both cases, since $X_0 < B_{t_1,F}$, the equilibrium of the game would be sequential. In the second region, where $X_0 > X_{0}^{(V_{0,L}=V_{0,F})}$, the leader firm would invest at the game’s inception ($t_1 = 0$), and the equilibrium of the game could be either sequential, if $X_0 < B_{t_1,F}$, or simultaneous, if $X_0 \geq B_{t_1,F}$.

Hence, unless $X_0 = X_{0}^{(V_{0,L}=V_{0,F})}$, a criterion is needed to resolve the inequality in firm values and to assign the more valuable position to either one of the two firms. The preferred position is granted to the firm willing to pay the difference $|V_{0,L} - V_{0,F}|$. Because both firms would spend this amount to gain the privileged position, we suggest that a public auction may fulfill the role of a tie breaker. The outcome of the auction is an indisputable assignment of market roles to which the firms are committed. Alternatively, the price to be paid to win
the desired position may take the form of advertisement costs and other forms of brand-name building and marketing. Once leadership is unequivocally established, the model is developed as in Sections 2, 3, and 4.

6.2 Derivation of the Follower’s $EP_t$

$EP_t$ is the value of the future flow of revenues earned by firm 2 beyond time $t_2 + \Delta$ when the option to replace the firm’s capital is exercised at time $t = t_2$. Standard manipulations yield

\[
EP_t = E_t \left[ \int_{t+\Delta}^{\infty} b_{t,v}k_{2,v} \left[ \lambda X_t D(\Sigma_{i=1}^{2} k_{i,v}) \right] dv \right]
\]

\[
= \lambda X_t E_t \left[ \int_{t+\Delta}^{\infty} e^{\{-(r-v)\}} \kappa e^{\left\{ \left[ a - \frac{1}{2} \sigma^2 \right] (v-t) + \sigma (z_v - z_t) \right\}} (2\kappa)^{-\gamma} dv \right]
\]

\[
= \lambda 2^{-\gamma} \kappa^{1-\gamma} X_t E_t \left[ \int_{t+\Delta}^{\infty} e^{\left\{ [a - \frac{1}{2} \sigma^2] (v-t) + \sigma (z_v - z_t) \right\}} dv \right]
\]

\[
= \lambda 2^{-\gamma} \kappa^{1-\gamma} X_t \int_{t+\Delta}^{\infty} e^{\{(a-r)(v-t)\}} dv
\]

\[
= \lambda 2^{-\gamma} \kappa^{1-\gamma} X_t \frac{e^{\{(a-r)\Delta\}}}{r - \alpha}.
\]
6.3 Proposition 1

The follower’s early exercise premium, $EEP_t$, is equal to

$$EEP_t = 2^{-\gamma} \kappa^{1-\gamma} \left( \lambda e^{(\alpha-r)\Delta} - 1 \right) X_t \left[ \int_t^T e^{(\alpha-r)(v-t)} \Phi(d_1) dv \right]$$

$$-rI \left[ \int_t^T e^{-(v-t)} \Phi(d_1 - \sigma \sqrt{v-t}) dv \right],$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function and

$$d_1 = \frac{\ln \left( \frac{X_t}{B_v} \right) + (\alpha + \frac{1}{2} \sigma^2)(v-t)}{\sigma \sqrt{v-t}}.$$

**Proof.** We derive a pseudo closed-form solution for the value of the early exercise premium as a function of the unknown exercise boundary $B_v$ as in Kim (1990), Jacka (1991), and Carr-Jarrow-Myneni (1992). For any particular $T$, we have that

$$EEP_t = E_t \left[ \int_t^T e^{-(v-t)} 1_{\{X_v \geq B_v\}} \phi_v dv \right]$$

$$= \int_t^T e^{-(v-t)} \int_{\ln B_v}^\infty \frac{\phi_v}{\sqrt{2\pi \sigma^2(v-t)}} e^{\left( -\frac{1}{2\sigma^2(v-t)} \left[ \ln X_v - \ln X_t + (\alpha - \frac{1}{2} \sigma^2)(v-t) \right]^2 \right)} d\ln X_v dv$$

$$= 2^{-\gamma} \kappa^{1-\gamma} \left( \lambda e^{(\alpha-r)\Delta} - 1 \right)$$

$$\cdot \int_t^T e^{-(v-t)} \int_{\ln B_v}^\infty \frac{X_v}{\sqrt{2\pi \sigma^2(v-t)}} e^{\left( -\frac{1}{2\sigma^2(v-t)} \left[ \ln X_v - \ln X_t + (\alpha - \frac{1}{2} \sigma^2)(v-t) \right]^2 \right)} d\ln X_v dv$$

$$-rI \int_t^T e^{-(v-t)} \int_{\ln B_v}^\infty \frac{1}{\sqrt{2\pi \sigma^2(v-t)}} e^{\left( -\frac{1}{2\sigma^2(v-t)} \left[ \ln X_v - \ln X_t + (\alpha - \frac{1}{2} \sigma^2)(v-t) \right]^2 \right)} d\ln X_v dv$$

$$= 2^{-\gamma} \kappa^{1-\gamma} \left( \lambda e^{(\alpha-r)\Delta} - 1 \right) X_t \int_t^T e^{(\alpha-r)(v-t)} \Phi \left( \frac{\ln \left( \frac{X_t}{B_v} \right) + (\alpha + \frac{1}{2} \sigma^2)(v-t)}{\sigma \sqrt{v-t}} \right) dv$$

$$-rI \int_t^T e^{-(v-t)} \Phi \left( \frac{\ln \left( \frac{X_t}{B_v} \right) + (\alpha - \frac{1}{2} \sigma^2)(v-t)}{\sigma \sqrt{v-t}} \right) dv.$$
6.4 Proposition 2

The value of the European counterpart to the American contingent claim \( W_t(EP, H, t_2) \) is

\[
\lim_{T \to \infty} E_t \left[ \int_t^T e^{-(r(v-t))} dH_v \right] + \lim_{T \to \infty} E_t \left[ e^{-(r(T-t))} \left[ EP_T - I \right]^+ \right] = \frac{2^{-\gamma} \kappa^{1-\gamma}}{r - \alpha} X_t,
\]

which in turn determines the value of firm 2’s capital-replacement option to be

\[
W_t(EP, H, t_2) = \frac{2^{-\gamma} \kappa^{1-\gamma}}{r - \alpha} X_t
\]

\[+ 2^{-\gamma} \kappa^{1-\gamma} \left( \lambda e^{((\alpha-r)\Delta)} - 1 \right) X_t \lim_{T \to \infty} \left[ \int_t^T e^{((\alpha-r)(v-t))} \Phi(d_1) dv \right]
\]

\[-r I \lim_{T \to \infty} \left[ \int_t^T e^{-(r(v-t))} \Phi(d_1 - \sigma \sqrt{v-t}) \right] dv \],

where \( d_1 \) is as in Proposition 1.

**Proof.** We begin by evaluating the expectation in the first term of Eq. (14). This yields

\[
E_t \left[ \int_t^T b_{t,v} dH_v \right] = E_t \left[ \int_t^T e^{-(r(v-t))} 2^{-\gamma} \kappa^{1-\gamma} X_v dv \right]
\]

\[= 2^{-\gamma} \kappa^{1-\gamma} X_t \int_t^T E_t \left[ e^{\left((\alpha-r)\frac{1}{2} \sigma^2(v-t)+\sigma(z_v-z_t)\right)} \right] dv
\]

\[= 2^{-\gamma} \kappa^{1-\gamma} X_t \int_t^T e^{((\alpha-r)(v-t))} dv
\]

\[= 2^{-\gamma} \kappa^{1-\gamma} X_t \frac{e^{((\alpha-r)(T-t))} - 1}{(\alpha - r)},
\]

which, in the limit as \( T \to \infty \), is equal to

\[
\lim_{T \to \infty} E_t \left[ \int_t^T e^{-(r(v-t))} dH_v \right] = \frac{2^{-\gamma} \kappa^{1-\gamma}}{r - \alpha} X_t.
\]

We now consider the second term in Eq. (14).

\[
e^{-r(T-t)} E_t \left[ [EP_T - I]^+ \right] = e^{-r(T-t)} E_t \left[ \frac{\lambda 2^{-\gamma} \kappa^{1-\gamma} e^{((\alpha-r)\Delta)}}{r - \alpha} X_T - I \right]^+
\]

\[= \lambda 2^{-\gamma} \kappa^{1-\gamma} e^{((\alpha-r)\Delta)} e^{-r(T-t)} E_t \left[ \frac{I}{\lambda 2^{-\gamma} \kappa^{1-\gamma} e^{((\alpha-r)\Delta)}} - X_T \right]^+.
\]

(27)
Letting \( \bar{I} = \frac{I}{\lambda^{2-\gamma}K^{1-\gamma}e^{(a-r)\Delta}} \), we can simplify Eq. (27) and we obtain

\[
\lambda^{2-\gamma}K^{1-\gamma}e^{(a-r)\Delta} \int_{\ln I}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \left[ X_T - \bar{I} \right] e^{\left( -\frac{\ln X_T - \ln X_t - (a - \frac{1}{2}\sigma^2)(T-t)}{2\sigma^2(T-t)} \right)^2} d\ln X_T
\]

\[
= \lambda^{2-\gamma}K^{1-\gamma}e^{(a-r)\Delta} \int_{\ln I}^{\infty} \frac{X_T}{\sqrt{2\pi\sigma^2(T-t)}} e^{\left( -\frac{\ln X_T - \ln X_t - (a - \frac{1}{2}\sigma^2)(T-t)}{2\sigma^2(T-t)} \right)^2} d\ln X_T
\]

\[-\lambda^{2-\gamma}K^{1-\gamma}e^{(a-r)\Delta} \int_{\ln I}^{\infty} \frac{\bar{I}}{\sqrt{2\pi\sigma^2(T-t)}} e^{\left( -\frac{\ln X_T - \ln X_t - (a - \frac{1}{2}\sigma^2)(T-t)}{2\sigma^2(T-t)} \right)^2} d\ln X_T
\]

\[
= \lambda^{2-\gamma}K^{1-\gamma}e^{(a-r)\Delta} \left[ X_t e^{(a-r)(T-t)} \Phi (d_1) - e^{-r(T-t)} \bar{I} \Phi (d_1 - \sigma \sqrt{T-t}) \right],
\]  

(28)
in which

\[
\bar{d}_1 = \frac{\ln \left( \frac{X_t}{T} \right) + (\alpha + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}.
\]

In the limit as \( T \to \infty \), Eq. (28) becomes

\[
\lim_{T \to \infty} E_t \left[ e^{-(r(T-t))} \left[ E_{P_T} - I \right]^+ \right] = \lim_{T \to \infty} \lambda^{2-\gamma}K^{1-\gamma}e^{(a-r)\Delta} \left[ X_t e^{(a-r)(T-t)} \Phi (d_1) - e^{-r(T-t)} \bar{I} \Phi (d_1 - \sigma \sqrt{T-t}) \right]
\]

\[
= 0.
\]

Therefore the value of the European component is

\[
\lim_{T \to \infty} E_t \left[ \int_t^T e^{-(r(v-t))} dH_v \right] + \lim_{T \to \infty} E_t \left[ e^{-(r(T-t))} \left[ E_{P_T} - I \right]^+ \right] = \frac{2^{-\gamma}K^{1-\gamma}}{r - \alpha} X_t,
\]

which, when combined with the result of Proposition 1, provides us with the value of the American option below.

\[
W_t(EP, H, t_2) = \frac{2^{-\gamma}K^{1-\gamma}}{r - \alpha} X_t
\]

\[
+ 2^{-\gamma}K^{1-\gamma} (\lambda e^{(a-r)\Delta} - 1) X_t \lim_{T \to \infty} \left[ \int_t^T e^{(a-r)(v-t)} \Phi (d_1) dv \right]
\]

\[
- r I \lim_{T \to \infty} \left[ \int_t^T \left[ e^{-(r(v-t))} \Phi (d_1 - \sigma \sqrt{v-t}) \right] dv \right].
\]
6.5 Theorem 1

The immediate exercise boundary for the follower beyond \( t_1 + \Delta \) is given by \( B_\infty = \bar{I} \frac{\beta}{\beta + 1} \), in which

\[
\bar{I} = \frac{I}{I_{(\alpha-r)^2} - I} = \frac{\sqrt{2 \pi \sigma^2 + \rho^2 - \rho}}{\sigma^2}
\]

and \( \rho = \alpha - \frac{1}{2} \sigma^2 \).

**Proof.** In what follows, we rely on the proof of Corollary 35 in Detemple (2006, p.76) and on the proof of Eqs. (8) and (9) in Kim (1990, pp. 569-570).

The immediate exercise boundary solves the recursive non-linear integral equation

\[
\left( \lambda^{2-\gamma} \kappa^{1-\gamma} e^{(\alpha-r)\Delta} \right) \frac{e^{(\alpha-r)\Delta}}{r - \alpha} B_t - I
\]

\[
= \frac{2^{2-\gamma} \kappa^{1-\gamma}}{r - \alpha} B_t
\]

\[
+ 2\gamma \kappa^{1-\gamma} \left( \lambda e^{(\alpha-r)\Delta} - 1 \right)
\]

\[
\cdot B_t \lim_{T \to \infty} \int_t^T e^{(\alpha-r)(v-t)} \Phi \left( \frac{\ln \left( \frac{B_t}{B_v} \right) + (\alpha + \frac{1}{2} \sigma^2)(v-t)}{\sigma \sqrt{v-t}} \right) dv
\]

\[
- rI \lim_{T \to \infty} \int_t^T e^{(-r(v-t))} \Phi \left( \frac{\ln \left( \frac{B_t}{B_v} \right) + (\alpha - \frac{1}{2} \sigma^2)(v-t)}{\sigma \sqrt{v-t}} \right) dv.
\]

(29)

When \( T \) approaches \( \infty \) the immediate exercise boundary becomes time-independent: \( B = B_\infty \). It follows that

\[
d_1 = \frac{(\alpha + \frac{1}{2} \sigma^2)(v-t)}{\sigma \sqrt{v-t}},
\]

which is independent of \( B_\infty \).

Accordingly, we can rewrite the first integral in Eq. (29) as

\[
\int_t^\infty e^{(\alpha-r)(v-t)} \Phi \left( \frac{\ln \left( \frac{B_t}{B_v} \right) + (\alpha + \frac{1}{2} \sigma^2)(v-t)}{\sigma \sqrt{v-t}} \right) dv
\]

\[
= \Phi \left( \frac{(\alpha + \frac{1}{2} \sigma^2)}{\sigma} \sqrt{v-t} \right) e^{(\alpha-r)(v-t)} \left[ \frac{\ln \left( \frac{B_t}{B_v} \right) + (\alpha - \frac{1}{2} \sigma^2)(v-t)}{(\alpha - r)} \right]_{t}^{\infty} - \int_t^\infty e^{(\alpha-r)(v-t)} \frac{1}{\sqrt{2 \pi}} \frac{\lambda^{2-\gamma} \kappa^{1-\gamma} e^{(\alpha-r)\Delta} \left( \frac{(\alpha + \frac{1}{2} \sigma^2) \sqrt{v-t}}{\sigma^2} \right)^2}{\frac{(\alpha + \frac{1}{2} \sigma^2)}{\sigma^2} \sqrt{2 \pi}} dv
\]

\[
= 1 - \frac{1}{2(\alpha - r)} - \frac{1}{\sqrt{2 \pi}} \int_t^\infty e^{(\alpha-r)(v-t)} \frac{1}{(\alpha - r)} \frac{-\frac{1}{2} \sqrt{\frac{(\alpha + \frac{1}{2} \sigma^2) \sqrt{v-t}}{(\alpha + \frac{1}{2} \sigma^2)}}}{\sigma^2} dv.
\]
Letting $u = v - t$, the above becomes

$$
\frac{1}{2 (r - \alpha)} - \int_0^\infty e^{(\alpha - r)u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{(\alpha - r)^2}{\sigma^2} \right)} (\alpha + \frac{1}{2} \sigma^2) du
$$

$$
= \frac{1}{2 (r - \alpha)} + \frac{1}{2 \sigma (r - \alpha)} \left( \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(-au)} \frac{du}{\sqrt{u}} \right)
$$

$$
= \frac{1}{2 (r - \alpha)} + \frac{1}{2 \sigma (r - \alpha)} \left( \frac{1}{\sqrt{2 (r - \alpha) f \left( \frac{(\alpha + \frac{1}{2} \sigma^2)}{\sigma} \right)^2}} \right),
$$

in which we use the identity

$$
\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(-au)} \frac{1}{\sqrt{u}} du = \sqrt{2a} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi} a}, \quad (30)
$$

resulting from the change of variable $x = \sqrt{2au}$ where $a = (r - \alpha) + \frac{1}{2} \left( \frac{(\alpha + \frac{1}{2} \sigma^2)}{\sigma} \right)^2$.

Similarly, for the second integral becomes

$$
\int_t^\infty e^{(r-vt)} \Phi \left( \frac{(\alpha - \frac{1}{2} \sigma^2)(v - t)}{\sigma \sqrt{v - t}} \right) dv
$$

$$
= \Phi \left( \frac{(\alpha - \frac{1}{2} \sigma^2)(v - t)}{\sigma \sqrt{v - t}} \right) \left|_t^- \right. - \int_t^\infty e^{(r-vt)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{(\alpha - \frac{1}{2} \sigma^2)^2}{\sigma^2} \right)} (\alpha - \frac{1}{2} \sigma^2) \sigma \sqrt{v - t} dv.
$$

With the same change of variables as above, $u = v - t$, we obtain

$$
\Phi \left( \frac{(\alpha - \frac{1}{2} \sigma^2)(0 - t)}{\sigma \sqrt{-t}} \right) \cdot 0 + \frac{1}{2r} - \int_0^\infty e^{(-ru)} \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2} \left( \frac{(\alpha - \frac{1}{2} \sigma^2)^2}{\sigma^2} \right)} (\alpha - \frac{1}{2} \sigma^2) \sigma 2u du
$$

$$
= \frac{1}{2r} + \frac{1}{2 \sigma r} \left( \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(-\tilde{a}u)} \frac{du}{\sqrt{u}} \right),
$$

in which $\tilde{a} = r + \frac{1}{2} \left( \frac{(\alpha - \frac{1}{2} \sigma^2)^2}{\sigma^2} \right)$. Applying the identity in Eq. (30) yields

$$
\frac{1}{2r} + \frac{(\alpha - \frac{1}{2} \sigma^2)}{2 \sigma r} \left( \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{(-\tilde{a}u)} \frac{du}{\sqrt{u}} \right) = \frac{1}{2r} + \frac{(\alpha - \frac{1}{2} \sigma^2)}{2 \sigma r} \left( \frac{1}{\sqrt{2r + \left( \frac{(\alpha - \frac{1}{2} \sigma^2)^2}{\sigma^2} \right)}} \right).
$$
Therefore, Eq. (29) becomes

\[
\left(\lambda 2^{-\gamma} k^{1-\gamma} e^{\{(\alpha-r)\Delta\}} \right) B_\infty - I = \frac{2^{-\gamma} k^{1-\gamma}}{r-\alpha} B_\infty \\
+ 2^{-\gamma} k^{1-\gamma} \left( \lambda e^{\{(\alpha-r)\Delta\}} - 1 \right) B_\infty \left( \frac{1}{2 (r-\alpha)} + \frac{(\alpha + \frac{1}{2} \sigma^2)}{2 \sigma (r-\alpha) \sqrt{2 (r-\alpha) + \left(\frac{(\alpha + \frac{1}{2} \sigma^2)}{\sigma}\right)^2}} \right) \\
- r I \left( \frac{1}{2} + \frac{(\alpha - \frac{1}{2} \sigma^2)}{2 \sigma \sqrt{2 r + \left(\frac{(\alpha - \frac{1}{2} \sigma^2)}{\sigma}\right)^2}} \right),
\]

which can be rearranged as

\[
B_\infty \left[ \left(\lambda 2^{-\gamma} k^{1-\gamma} e^{\{(\alpha-r)\Delta\}} \right) - \frac{2^{-\gamma} k^{1-\gamma}}{r-\alpha} \right] = 2^{-\gamma} k^{1-\gamma} \left( \lambda e^{\{(\alpha-r)\Delta\}} - 1 \right) \left( \frac{1}{2} + \frac{(\alpha + \frac{1}{2} \sigma^2)}{2 \sigma \sqrt{2 (r-\alpha) + \left(\frac{(\alpha + \frac{1}{2} \sigma^2)}{\sigma}\right)^2}} \right)
\]

\[
= I \left( \frac{1}{2} - \frac{(\alpha - \frac{1}{2} \sigma^2)}{2 \sigma \sqrt{2 r + \left(\frac{(\alpha - \frac{1}{2} \sigma^2)}{\sigma}\right)^2}} \right),
\]

producing the solution

\[
B_\infty = \frac{I \left( 1 - \frac{(\alpha - \frac{1}{2} \sigma^2)}{\sigma \sqrt{2 r + \left(\frac{(\alpha - \frac{1}{2} \sigma^2)}{\sigma}\right)^2}} \right)}{2^{-\gamma} k^{1-\gamma} \left( \lambda e^{\{(\alpha-r)\Delta\}} - 1 \right) \left( 1 - \frac{(\alpha + \frac{1}{2} \sigma^2)}{\sigma \sqrt{2 (r-\alpha) + \left(\frac{(\alpha + \frac{1}{2} \sigma^2)}{\sigma}\right)^2}} \right)}
\]

\[
= \frac{I \left( 1 - \frac{(\alpha - \frac{1}{2} \sigma^2)}{\sqrt{2 r + \left(\frac{(\alpha - \frac{1}{2} \sigma^2)}{\sigma}\right)^2}} \right)}{2^{-\gamma} k^{1-\gamma} \left( \lambda e^{\{(\alpha-r)\Delta\}} - 1 \right) \left( 1 - \frac{(\alpha + \frac{1}{2} \sigma^2)}{\sqrt{2 (r-\alpha) + \left(\frac{(\alpha + \frac{1}{2} \sigma^2)}{\sigma}\right)^2}} \right)},
\]

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Denoting $\rho = \alpha - \frac{1}{2}\sigma^2$, we conclude that

$$B_{\infty} = I \frac{2^{-\gamma}\kappa^{1-\gamma}}{(r-\alpha)} \left( \frac{1}{\sqrt{2r\sigma^2+\rho^2}} \right) \left( 1 - \frac{\rho}{\sqrt{2r\sigma^2+\rho^2}} \right)$$

$$= \frac{I}{2^{-\gamma}\kappa^{1-\gamma}} \frac{(\lambda e^{((\alpha-r)\Delta}) - 1)}{\lambda e^{((\alpha-r)\Delta)}} \left( 1 - \frac{\rho+\sigma^2}{\sqrt{2r\sigma^2+\rho^2}} \right) \frac{\sqrt{2r\sigma^2+\rho^2-\rho}}{\sigma^2}$$

$$= \tilde{I} \frac{\beta}{\beta - 1},$$

where $\tilde{I} = \frac{I}{2^{-\gamma}\kappa^{1-\gamma}} \frac{1}{(\lambda e^{((\alpha-r)\Delta}) - 1)}$ and $\beta = \frac{\sqrt{2r\sigma^2+\rho^2-\rho}}{\sigma^2}$. \hfill \blacksquare
6.6 Corollary 1

The value of the follower’s capital-replacement option at time \( t \geq t_1 + \Delta \) is given by

\[
W_t(EP, H, \infty) = \begin{cases} 
\frac{2^{-\gamma k^{1-\gamma}}}{r - \alpha} X_t + \frac{2^{-\gamma k^{1-\gamma} (\lambda e^{(\alpha - \gamma)\Delta}) - 1}}{r - \alpha} \left[ \frac{\bar{I}}{\bar{I} - 1} \left( \frac{X_t(\beta - 1)}{\bar{I} \beta} \right)^\beta \right] & \text{if } X_t < B_\infty \\
EP_t - I & \text{if } X_t \geq B_\infty
\end{cases}
\]

**Proof.** The value of the follower’s replacement option is obtained by replacing the \( B_v \) with \( B_\infty = \bar{I}^\beta \), as derived in Theorem 1, in Eq. (15), which yields

\[
W_t(EP, H, \infty) = \frac{2^{-\gamma k^{1-\gamma}}}{r - \alpha} X_t + \frac{2^{-\gamma k^{1-\gamma} (\lambda e^{(\alpha - \gamma)\Delta}) - 1}}{r - \alpha} \left[ \int_t^\infty (r - \alpha) X_t e^{-(r - \alpha)(v - t)} \frac{\ln \left( \frac{X_t}{B_\infty} \right) + (\alpha + \frac{1}{2} \sigma^2)(v - t)}{\sigma \sqrt{v - t}} dv \right] \\
\]

\[
- r \int_t^\infty e^{-(r - \alpha)(v - t)} \frac{\ln \left( \frac{X_t}{B_\infty} \right) + (\alpha - \frac{1}{2} \sigma^2)(v - t)}{\sigma \sqrt{v - t}} dv \\
= \frac{2^{-\gamma k^{1-\gamma}}}{r - \alpha} X_t + \frac{2^{-\gamma k^{1-\gamma} (\lambda e^{(\alpha - \gamma)\Delta}) - 1}}{r - \alpha} \left[ \int_t^\infty (r - \alpha) X_t e^{-(r - \alpha)(v - t)} \frac{\ln \left( \frac{X_t}{B_\infty} \right) + (\alpha + \frac{1}{2} \sigma^2)(v - t)}{\sigma \sqrt{v - t}} dv \right] \\
\]

\[
- r \int_t^\infty e^{-(r - \alpha)(v - t)} \frac{\ln \left( \frac{X_t}{B_\infty} \right) + (\alpha - \frac{1}{2} \sigma^2)(v - t)}{\sigma \sqrt{v - t}} dv \\
= \frac{2^{-\gamma k^{1-\gamma}}}{r - \alpha} X_t + \frac{2^{-\gamma k^{1-\gamma} (\lambda e^{(\alpha - \gamma)\Delta}) - 1}}{r - \alpha} \left[ \int_t^\infty (r - \alpha) X_t e^{-(r - \alpha)(v - t)} \frac{\ln \left( \frac{X_t}{B_\infty} \right) + (\alpha + \frac{1}{2} \sigma^2)(v - t)}{\sigma \sqrt{v - t}} dv \right] \\
- \frac{2^{-\gamma k^{1-\gamma} (\lambda e^{(\alpha - \gamma)\Delta}) - 1}}{r - \alpha} \int_t^\infty r \bar{I} e^{-(r - \alpha)(v - t)} \frac{\ln \left( \frac{X_t}{B_\infty} \right) + (\alpha - \frac{1}{2} \sigma^2)(v - t)}{\sigma \sqrt{v - t}} dv.
\]
Letting $u = v - t$, we obtain

$$
\frac{2^{-\gamma}k^{1-\gamma}}{r - \alpha} X_t
+ \frac{2^{-\gamma}k^{1-\gamma}(\lambda e^{((\alpha-r)\Delta)} - 1)}{r - \alpha} \int_0^\infty (r - \alpha) X_t e^{-(r-\alpha)u} \Phi \left( \frac{\ln \left( \frac{X_t}{B_\infty} \right) + (\alpha + \frac{1}{2}\sigma^2)u}{\sigma \sqrt{u}} \right) du
- \frac{2^{-\gamma}k^{1-\gamma}(\lambda e^{((\alpha-r)\Delta)} - 1)}{r - \alpha} \int_0^\infty \tilde{I} e^{-(ru)} \Phi \left( \frac{\ln \left( \frac{X_t}{B_\infty} \right) + (\alpha - \frac{1}{2}\sigma^2)u}{\sigma \sqrt{u}} \right) du.
$$

This equation is analogous to the expression $V(s)$ on page 570 of Kim (1990). This allows us to apply the widely-known result for the value of the perpetual American option (Merton, 1973, Kim, 1990, and Detemple, 2006)

$$
\int_0^\infty \left[ (r - \alpha) X_t e^{-(r-\alpha)u} \Phi \left( \frac{\ln \left( \frac{X_t}{B_\infty} \right) + (\alpha + \frac{1}{2}\sigma^2)u}{\sigma \sqrt{u}} \right) - \tilde{I} e^{-(ru)} \Phi \left( \frac{\ln \left( \frac{X_t}{B_\infty} \right) + (\alpha - \frac{1}{2}\sigma^2)u}{\sigma \sqrt{u}} \right) \right] du
= \frac{\tilde{I}}{\beta - 1} \left( \frac{X_t(\beta - 1)}{\tilde{I} \beta} \right)^\beta.
$$

Therefore, we conclude that the value of the follower’s capital-replacement option at time $t$ is equal to

$$
W_t(EP, H, \infty) = \begin{cases} 
\frac{2^{-\gamma}k^{1-\gamma}}{r - \alpha} X_t & \text{if } X_t < B_\infty, \\
\frac{2^{-\gamma}k^{1-\gamma}(\lambda e^{((\alpha-r)\Delta)} - 1)}{r - \alpha} \left[ \frac{\tilde{I}}{\beta - 1} \left( \frac{X_t(\beta - 1)}{\tilde{I} \beta} \right)^\beta \right] & \text{if } X_t \geq B_\infty.
\end{cases}
$$
6.7 Proposition 3

The value of the follower’s capital-replacement option at time \( t_1 \leq t \leq t_1 + \Delta \) is given by

\[
F_t(EP,H,t_2) = \frac{\kappa^{1-\gamma}X_t 1 - e^{((\alpha-r)(t_1+\Delta-t))}}{r-\alpha} + \frac{2^{-\gamma}\kappa^{1-\gamma}b_{t,t_1+\Delta}X_t e^{(\alpha t)}}{r-\alpha} + \frac{2^{-\gamma}\kappa^{1-\gamma}(\lambda e^{((\alpha-r)\Delta)} - 1)}{r-\alpha} \frac{\bar{I}}{\beta - 1} \left( \frac{\beta - 1}{I\beta} \right)^{\beta} b_{t,t_1+\Delta} \nonumber \\
\cdot \Phi \left( -\ln \frac{X_t}{B_\infty} + \frac{(\alpha + (\beta - \frac{1}{2})\sigma^2)\tau}{\sigma \sqrt{\tau}} \right) X_t e^{\left(\frac{\beta(2\alpha + (\beta-1)\sigma^2)\tau}{2}\right)} \nonumber \\
+ b_{t,t_1+\Delta} \Phi \left( \ln \frac{X_t}{B_\infty} + \frac{(\alpha + (\beta - \frac{1}{2})\sigma^2)\tau}{\sigma \sqrt{\tau}} \right) \left[ \frac{2^{-\gamma}\kappa^{1-\gamma}}{r-\alpha} X_t e^{(\alpha t)} (\lambda e^{((\alpha-r)\Delta)} - 1) - 1 \right] \nonumber \\
+ \kappa^{1-\gamma} (\lambda 2^{-\gamma}e^{((\alpha-r)\Delta)} - 1) X_t \left[ \int_{t}^{t_1+\Delta} e^{((\alpha-r)(v-t))} \Phi (d_1) dv \right] \nonumber \\
- rI \left[ \int_{t}^{t_1+\Delta} e^{(\beta(v-t))} \Phi (d_1 - \sigma \sqrt{v-t}) dv \right],
\]

in which

\[
d_1 = \ln \left( \frac{X_t}{B_\infty} \right) + (\alpha + \frac{1}{2}\sigma^2)(v-t) \frac{\sigma \sqrt{\tau}}{\sigma \sqrt{\tau} - t}.\]

**Proof.** Recall from Eq. (17) that the follower’s capital-replacement option prior to \( t_1 + \Delta \) is worth

\[
F_t(EP,H,t_2) = E_t \left[ \int_{t}^{t_1+\Delta} b_{t,v} dH_v + b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,t_2) \right] + EEP_t.
\]

We begin by solving for the explicit value of the option’s European component

\[
E_t \left[ \int_{t}^{t_1+\Delta} b_{t,v} dH_v + b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,\infty) \right] = E_t \left[ \int_{t}^{t_1+\Delta} b_{t,v} dH_v \right] + E_t [b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,\infty)].
\]

Using the result for \( dH_t \) in Eq. (18), it can be readily shown that

\[
E_t \left[ \int_{t}^{t_1+\Delta} b_{t,v} dH_v \right] = \kappa^{1-\gamma} X_t 1 - e^{((\alpha-r)(t_1+\Delta-t))}.\]

In order to solve for the second term, we partition the range of values that the market demand can take at time \( t_1 + \Delta \). This allows us to reexpress \( E_t [b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,\infty)] \) as

\[
E_t [b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,\infty) \mid X_{t_1+\Delta} < B_\infty] \Pr (X_{t_1+\Delta} < B_\infty) \\
+ E_t [b_{t,t_1+\Delta} W_{t_1+\Delta}(EP,H,\infty) \mid X_{t_1+\Delta} \geq B_\infty] \Pr (X_{t_1+\Delta} \geq B_\infty),
\]

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in which

\[
\Pr(X_{t_1+\Delta} < B_\infty) = \Pr(\ln X_{t_1+\Delta} < \ln B_\infty) \\
= \Pr\left(\ln X_t + \left(\alpha - \frac{1}{2}\sigma^2\right)(t_1 + \Delta - t) + \sigma(z_{t_1+\Delta} - z_t) < \ln B_\infty\right) \\
= \Pr\left(\frac{(z_{t_1+\Delta} - z_t)}{\sqrt{t_1 + \Delta - t}} < \frac{\ln B_\infty - \ln X_t - \left(\alpha - \frac{1}{2}\sigma^2\right)(t_1 + \Delta - t)}{\sigma\sqrt{t_1 + \Delta - t}}\right) \\
= \Phi\left(-\frac{\ln \frac{X_t}{B_\infty} + \left(\alpha - \frac{1}{2}\sigma^2\right)(t_1 + \Delta - t)}{\sigma\sqrt{t_1 + \Delta - t}}\right) \\
= 1 - \Phi\left(\frac{\ln \frac{X_t}{B_\infty} + \left(\alpha - \frac{1}{2}\sigma^2\right)(t_1 + \Delta - t)}{\sigma\sqrt{t_1 + \Delta - t}}\right),
\]

Let us consider the first conditional expectation.

\[
E_t[b_{t,t_1+\Delta}W_{t_1+\Delta}(EP, H, \infty) \mid X_{t_1+\Delta} < B_\infty]
= E_t\left[b_{t,t_1+\Delta} \left(\begin{array}{c}
2^{-\gamma} \kappa^{1-\gamma}X_{t_1+\Delta} \\
\frac{2^{-\gamma} \kappa^{1-\gamma} \left(\lambda e^{((\alpha-r)\Delta)} - 1\right)}{r - \alpha} \left(\frac{I}{\beta - 1} \left(\frac{X_{t_1+\Delta} (\beta - 1)}{I \beta}\right)^\beta\right) \right) \mid X_{t_1+\Delta} < B_\infty\right]
= \frac{2^{-\gamma} \kappa^{1-\gamma}}{r - \alpha} b_{t,t_1+\Delta} E_t[X_{t_1+\Delta} \mid X_{t_1+\Delta} < B_\infty]
+ \frac{2^{-\gamma} \kappa^{1-\gamma} \left(\lambda e^{((\alpha-r)\Delta)} - 1\right)}{r - \alpha} \frac{I}{\beta - 1} \left(\frac{\beta - 1}{I \beta}\right)^\beta b_{t,t_1+\Delta} E_t[X_{t_1+\Delta} \mid X_{t_1+\Delta} < B_\infty].
\]

In the above formula, we solve the two conditional expectations sequentially. The first expectation is

\[
E_t[X_{t_1+\Delta} \mid X_{t_1+\Delta} < B_\infty]
= \frac{1}{\Pr(X_{t_1+\Delta} < B_\infty) \sqrt{2\pi\sigma^2(t_1 + \Delta - t)}} \int_{-\infty}^{\ln B_\infty} X_{t_1+\Delta} e^{-\frac{1}{2} \left(\frac{\ln X_{t_1+\Delta} - \ln X_t - \left(\alpha - \frac{1}{2}\sigma^2\right)(t_1 + \Delta - t)}{\sigma\sqrt{t_1 + \Delta - t}}\right)^2} d\ln X_{t_1+\Delta},
\]

to which we apply the transformations

\[
Z_{t,t_1+\Delta} = \ln X_{t_1+\Delta} - \ln X_t \text{ and } \tau = t_1 + \Delta - t.
\]
We then rewrite the above expression as
\[
E_t \left[ X_{t+\Delta} \mid X_{t+\Delta} < B_\infty \right] = \frac{1}{\Pr(X_{t+\Delta} < B_\infty)} \frac{1}{\sqrt{2\pi \sigma^2 \tau}} \cdot X_t \int_{-\infty}^{\ln B_\infty - \ln X_t} e^{\left\{ Z_{t,t+\Delta} - \frac{1}{2} \left( \frac{Z_{t,t+\Delta} - (\alpha + \frac{1}{2} \sigma^2 \tau)}{\sigma \sqrt{\tau}} \right)^2 \right\}} dZ_{t,t+\Delta}
\]
\[
= \frac{1}{\Pr(X_{t+\Delta} < B_\infty)} \frac{1}{\sqrt{2\pi \sigma^2 \tau}} \cdot X_t \int_{-\infty}^{\ln B_\infty - \ln X_t} e^{\left\{ \frac{2Z_{t,t+\Delta} - Z_{t,t+\Delta} - 2Z_{t,t+\Delta} + 2Z_{t,t+\Delta} + \Delta (\alpha - \frac{1}{2} \sigma^2 \tau - (\alpha - \frac{1}{2} \sigma^2 \tau))^2}{2\sigma^2 \tau} \right\}} dZ_{t,t+\Delta}
\]
and, by adding and subtracting \(2\sigma^2 \alpha \tau^2\), we complete the square inside the exponential, which yields
\[
E_t \left[ X_{t+\Delta} \mid X_{t+\Delta} < B_\infty \right] = \frac{1}{\Pr(X_{t+\Delta} < B_\infty)} \frac{1}{\sqrt{2\pi \sigma^2 \tau}} \cdot X_t e^{\left\{ \frac{-Z_{t,t+\Delta} + \Delta - \frac{1}{2} Z_{t,t+\Delta} - (\alpha + \frac{1}{2} \sigma^2 \tau + (\alpha + \frac{1}{2} \sigma^2 \tau)^2}{2\sigma^2 \tau} \right\}} dZ_{t,t+\Delta}
\]
Letting
\[
\varepsilon_{t,t+\Delta} = \frac{Z_{t,t+\Delta} - (\alpha + \frac{1}{2} \sigma^2 \tau)}{\sigma \sqrt{\tau}},
\]
we obtain the closed-form expression
\[
E_t \left[ X_{t+\Delta} \mid X_{t+\Delta} < B_\infty \right] = \frac{1}{\Pr(X_{t+\Delta} < B_\infty)} \frac{X_t e^{\left\{ \frac{-Z_{t,t+\Delta} + \Delta - \frac{1}{2} Z_{t,t+\Delta} - (\alpha + \frac{1}{2} \sigma^2 \tau)}{2\sigma^2 \tau} \right\}}}{\sqrt{2\pi}} \cdot X_t e^{\left\{ \frac{-Z_{t,t+\Delta} + \Delta - \frac{1}{2} Z_{t,t+\Delta} - (\alpha + \frac{1}{2} \sigma^2 \tau)}{2\sigma^2 \tau} \right\}} d\varepsilon_{t,t+\Delta}
\]
\[
= \frac{1}{\Pr(X_{t+\Delta} < B_\infty)} X_t e^{\left\{ \frac{-Z_{t,t+\Delta} + \Delta - \frac{1}{2} Z_{t,t+\Delta} - (\alpha + \frac{1}{2} \sigma^2 \tau)}{2\sigma^2 \tau} \right\}} \Phi \left( \frac{\ln X_t - (\alpha + \frac{1}{2} \sigma^2 \tau)}{\sigma \sqrt{\tau}} \right)
\]
\[
= \Phi \left( \frac{\ln X_t - (\alpha + \frac{1}{2} \sigma^2 \tau)}{\sigma \sqrt{\tau}} \right) X_t e^{\left\{ \frac{-Z_{t,t+\Delta} + \Delta - \frac{1}{2} Z_{t,t+\Delta} - (\alpha + \frac{1}{2} \sigma^2 \tau)}{2\sigma^2 \tau} \right\}}.
\]

The second expectation, \( E_t \left[ X_{t+\Delta}^2 \mid X_{t+\Delta} < B_\infty \right] \), can be handled in a similar fashion:
applying the same change of variables as we did to the first expectation and completing the square inside the exponential by adding and subtracting $\beta \sigma^2 (2\alpha + (\beta - 1) \sigma^2)$, we obtain

$$E_t \left[ X_{t_1 + \Delta}^\beta \mid X_{t_1 + \Delta} < B_\infty \right] = \frac{\Phi \left( - \frac{\ln \frac{X_t}{B_\infty} + (\alpha + (\beta - \frac{1}{2}) \sigma^2) \tau}{\sigma \sqrt{\tau}} \right) X_t^\beta e^{\left\{ \frac{\beta (2\alpha + (\beta - 1) \sigma^2) \tau}{2} \right\}}}{\Phi \left( - \frac{\ln \frac{X_t}{B_\infty} + (\alpha - \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}} \right)}.$$ 

Before we consider the case of $X_{t_1 + \Delta} \geq B_\infty$, we summarize the above results in Eqs. (32) and (33) below.

$$E_t \left[ b_{t,t_1+\Delta} W_{t_1+\Delta} (EP, H, \infty) \mid X_{t_1+\Delta} < B_\infty \right] = \frac{2^{-r - \alpha} b_{t_1+\Delta}}{\gamma^r} \left( \frac{\Phi \left( - \frac{\ln \frac{X_t}{B_\infty} + (\alpha + (\beta - \frac{1}{2}) \sigma^2) \tau}{\sigma \sqrt{\tau}} \right) X_t e^{\{\alpha \tau\}}}{\Phi \left( - \frac{\ln \frac{X_t}{B_\infty} + (\alpha - \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}} \right)} \right)$$

$$+ \frac{2^{-r - \alpha} \left( \lambda e^{(\alpha-r)\Delta} - 1 \right)}{\gamma^r} \frac{I}{\beta - 1} \left( \frac{\beta - 1}{I \beta} \right)^\beta b_{t,t_1+\Delta}$$

$$\cdot \Phi \left( - \frac{\ln \frac{X_t}{B_\infty} + (\alpha + (\beta - \frac{1}{2}) \sigma^2) \tau}{\sigma \sqrt{\tau}} \right) X_t^\beta e^{\left\{ \frac{\beta (2\alpha + (\beta - 1) \sigma^2) \tau}{2} \right\}} \right) \right)$$

We now turn to the case of $X_{t_1 + \Delta} \geq B_\infty$. In particular,

$$E_t \left[ b_{t,t_1+\Delta} W_{t_1+\Delta} (EP, H, \infty) \mid X_{t_1+\Delta} \geq B_\infty \right] = E_t \left[ b_{t_1+\Delta} (EP_{t_1+\Delta} - I) \mid X_{t_1+\Delta} \geq B_\infty \right]$$

$$= E_t \left[ b_{t_1+\Delta} EP_{t_1+\Delta} \mid X_{t_1+\Delta} \geq B_\infty \right] - E_t \left[ b_{t_1+\Delta} I \mid X_{t_1+\Delta} \geq B_\infty \right]$$

$$= E_t \left[ b_{t_1+\Delta} \lambda 2^{-r - \alpha} X_{t_1+\Delta} e^{(\alpha-r)\Delta} \mid X_{t_1+\Delta} \geq B_\infty \right] - b_{t_1+\Delta} I$$

$$= b_{t_1+\Delta} \lambda 2^{-r - \alpha} e^{(\alpha-r)\Delta} E_t \left[ X_{t_1+\Delta} \mid X_{t_1+\Delta} \geq B_\infty \right] - b_{t_1+\Delta} I.$$
subtracting $2\sigma^2 \alpha \tau^2$, which yields

$$E_t [X_{t_1+\Delta} \mid X_{t_1+\Delta} \geq B_{\infty}] = \frac{\Phi \left( \frac{\ln \frac{X_t}{B_{\infty}} + (\alpha + \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}} \right)}{\Phi \left( \frac{\ln \frac{X_t}{B_{\infty}} + (\alpha - \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}} \right)} X_t e^{(\alpha \tau)}$$

and, weighted by the appropriate probability,

$$E_t [b_{t,t_1+\Delta} W_{t_1+\Delta} (EP, H, \infty) \mid X_{t_1+\Delta} \geq B_{\infty}] \Pr (X_{t_1+\Delta} \geq B_{\infty})$$

(34)

$$= b_{t,t_1+\Delta} \lambda 2^{-\gamma} \kappa^{1-\gamma} e^{((\alpha-r)\Delta)} \frac{2^{-\gamma} \kappa^{1-\gamma} (\lambda e^{((\alpha-r)\Delta)} - 1)}{r - \alpha} \frac{I}{\beta - 1} \left( \frac{\beta - 1}{I \beta} \right) b_{t,t_1+\Delta}$$

$$\Phi \left( -\frac{\ln \frac{X_t}{B_{\infty}} + (\alpha + (\beta - \frac{1}{2}) \sigma^2) \tau}{\sigma \sqrt{\tau}} \right) X_t e^{\left\{ \frac{\sqrt{\beta(2(r+(\beta-1)\sigma^2)) \tau}}{2} \right\}}$$

$$+ b_{t,t_1+\Delta} \Phi \left( -\frac{\ln \frac{X_t}{B_{\infty}} + (\alpha + \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}} \right) \left[ 2^{-\gamma} \kappa^{1-\gamma} \frac{2^{-\gamma} \kappa^{1-\gamma}}{r - \alpha} X_t e^{(\alpha-r)\Delta} \left( \lambda e^{((\alpha-r)\Delta)} - 1 \right) - I \right].$$

The last step in the determination of the $F_t(EP, H, t_2)$ option consists in evaluating the gains from early exercise, $EEP_t$. More specifically, using the expressions for $dH_t$ in Eq. (18) and that for $dA_v$ in Eq. (11),

$$EEP_t = E_t \left[ \int_t^{t_1+\Delta} b_{t,v} 1_{\{r_v=v\}} \left( \kappa^{1-\gamma} (\lambda 2^{-\gamma} e^{((\alpha-r)\Delta)} - 1) X_v - r I \right) dv \right] .$$

Following the proof of Proposition 1, we integrate the above and we obtain

$$EEP_t = \kappa^{1-\gamma} (\lambda 2^{-\gamma} e^{((\alpha-r)\Delta)} - 1) X_t \left[ \int_t^{t_1+\Delta} e^{((\alpha-r)(v-t))} \Phi \left( d_1 \right) dv \right]$$

$$- r I \left[ \int_t^{t_1+\Delta} e^{((\alpha-r)(v-t))} \Phi \left( d_1 - \sigma \sqrt{v-t} \right) dv \right] ,$$

in which

$$d_1 = \frac{\ln \left( \frac{X_t}{B_t} \right) + (\alpha + \frac{1}{2} \sigma^2) (v-t)}{\sigma \sqrt{v-t}} .$$
This allows us to state the value of the American option $F_t(EP, H, t_2)$ as

\[ F_t(EP, H, t_2) = \kappa^{1-\gamma}X_t \left( 1 - e^{\{(\alpha-r)(t_1+\Delta-t)\}} \right) \]

\[ + \frac{2^{-\gamma} \kappa^{1-\gamma}}{r-\alpha} b_{t,t_1+\Delta} X_t e^{(\alpha r)} + \frac{2^{-\gamma} \kappa^{1-\gamma} (\lambda e^{\{(\alpha-r)\Delta\}} - 1)}{r-\alpha} \frac{\bar{I}}{\beta - 1} b_{t,t_1+\Delta} \]

\[ \Phi \left( \ln \frac{X_t}{B_\infty} + \frac{(\alpha + (\beta - 1) \sigma^2) \tau}{\sigma \sqrt{\tau}} \right) X_t e^{\left\{ \beta \left( \frac{2\alpha + (\beta - 1) \sigma^2}{2} \right) \right\} r} \]

\[ + b_{t,t_1+\Delta} \Phi \left( \ln \frac{X_t}{B_\infty} + \frac{(\alpha + \frac{1}{2} \sigma^2) \tau}{\sigma \sqrt{\tau}} \right) \left[ \frac{2^{-\gamma} \kappa^{1-\gamma}}{r-\alpha} X_t e^{(\alpha r)} (\lambda e^{\{(\alpha-r)\Delta\}} - 1) - 1 \right] \]

\[ + \kappa^{1-\gamma} \left( \lambda 2^{-\gamma} e^{\{(\alpha-r)\Delta\}} - 1 \right) X_t \left[ \int_{t}^{t_1+\Delta} e^{\{(\alpha-r)(v-t)\}} \Phi (d_1) dv \right] \]

\[ - r I \left[ \int_{t}^{t_1+\Delta} e^{(\alpha-r)(v-t)} \Phi (d_1 - \sigma \sqrt{v-t}) dv \right].\]
6.8 Lemma 1

The terminal condition to the recursive integral equation is given by

\[ B_{(t_1+\Delta)} = \bar{I} \max \left( \frac{r}{r-\alpha}, \frac{\beta}{\beta-1} \right) = \frac{\bar{I} \beta}{\beta - 1} = B_\infty. \]

**Proof.** The proof of Lemma 1 results from a few consecutive steps. The first two steps consist of determining two distinct lower bounds for the boundary \( B_{(t_1+\Delta)} \). These are \( B_{(t_1+\Delta)} \geq \frac{\bar{I} \beta}{\beta - 1} \) and \( B_{(t_1+\Delta)} \geq \frac{r}{r-\alpha} \). The former follows from the optimal exercise policy at time \( t_1 + \Delta \). Since

\[
\max [EP_{t_1+\Delta} - I, W_{t_1+\Delta}(EP, H, t_2)] = W_{t_1+\Delta}(EP, H, t_2),
\]

as shown in Eq. (17), if the \( F \) option remains unexercised up to an instant before \( t_1 + \Delta \), the follower will optimally exchange it with the \( W \) option at time \( t_1 + \Delta \) exactly. Furthermore, Theorem 1 states that the optimal exercise policy associated with the \( W \) option is \( B_\infty = \frac{\bar{I} \beta}{\beta - 1} \). Because the immediate exercise boundary, \( B_t \), is non-decreasing in time-to-maturity (e.g., Detemple, 2006, p.59), \( B_\infty \) provides a lower bound for \( B_{(t_1+\Delta)} \). The latter results from the fact that immediate exercise cannot be optimal if

\[ 2^{-\gamma} \kappa^{1-\gamma} \left( \lambda e^{((\alpha-r)\Delta)} - 1 \right) X_t - rI < 0, \]

as can be seen in Eq. (12). Thus, a necessary condition for immediate exercise is

\[ B_{(t_1+\Delta)} \geq \frac{rI}{2^{-\gamma} \kappa^{1-\gamma} \left( \lambda e^{((\alpha-r)\Delta)} - 1 \right)} = \frac{\bar{I} \beta}{\beta - 1} \frac{r}{r-\alpha}, \]

where the last equality follows from the definition of \( \bar{I} \).

In order for both lower bounds to be satisfied, \( B_{(t_1+\Delta)} \) must be greater or equal to \( \bar{I} \max \left( \frac{r}{r-\alpha}, \frac{\beta}{\beta-1} \right) \). Equality is then proved by contradiction. Suppose that \( B_{(t_1+\Delta)} \) were strictly greater than \( \bar{I} \max \left( \frac{r}{r-\alpha}, \frac{\beta}{\beta-1} \right) \) and consider a point \( b \) between \( B_{(t_1+\Delta)} \) and \( \bar{I} \max \left( \frac{r}{r-\alpha}, \frac{\beta}{\beta-1} \right) \). Then, since immediate exercise is suboptimal at \( b \), \( F_{(t_1+\Delta)}(EP(b), H, t_2) > EP(b) - I \). On the other hand, by the continuity of the option value function \( F_{(t_1+\Delta)}(EP(b), H, t_2) = F_{(t_1+\Delta)}(EP(b), H, t_2) = EP(b) - I \), a contradiction. We conclude that \( B_{(t_1+\Delta)} = \bar{I} \max \left( \frac{r}{r-\alpha}, \frac{\beta}{\beta-1} \right) \).

It remains to be shown that \( \bar{I} \max \left( \frac{r}{r-\alpha}, \frac{\beta}{\beta-1} \right) = \frac{\bar{I} \beta}{\beta - 1} \). This is proved by contradiction by considering the equality \( \bar{I} \max \left( \frac{r}{r-\alpha}, \frac{\beta}{\beta-1} \right) = \frac{\bar{I} \beta}{\beta - 1} \). Because the \( EEP \) associated with the \( W \) option is identical to that of the \( F \) option, \( \bar{I} \frac{r}{r-\alpha} \) also constitutes a lower bound on the immediate exercise boundary beyond \( t_1 + \Delta \). This, in turn, implies that \( B_\infty < \frac{\bar{I} r}{r-\alpha} \) belongs to the interior of the continuation region, a contradiction of Theorem 1.

Finally we note that since \( B_{t_1+\Delta} = B_{(t_1+\Delta)} = B_\infty \), from Eq. (17) and Theorem 1, and \( B_{(t_1+\Delta)} = B_\infty \), from the above demonstration, the recursive boundary is continuous at \( t_1 + \Delta \).
6.9 Proposition 4

The value of the flow of future revenues earned beyond time $t_1 + \Delta$ when the option to replace the firm’s capital is exercised at time $t = t_1$ is equal to $2^{-\gamma} \lambda \kappa^{1-\gamma} X_t e^{(\alpha-r)\Delta}$.

**Proof.** Rearranging the expression for $EP_t$ in a suitable form for tractable analysis, we obtain

$$
EP_t = E_t \left[ \int_{t+\Delta}^{\infty} b_{t,v} k_{1,v} \left[ \lambda X_v D \left( \sum_{i=1}^{2} \kappa_{i} \right) \right] dv \right] 
= E_t \left[ \int_{t+\Delta}^{\infty} \left[ 1_{\{t_2 \geq v\}} + 1_{\{v \geq t_2 + \Delta\}} + \frac{(1-1_{\{t_2 \geq v\}})(1-1_{\{v \geq t_2 + \Delta\}})}{2^{-\gamma}} \right] b_{t,v} \left( \lambda 2^{-\gamma} \kappa^{1-\gamma} \right) X_v dv \right] 
= E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{t_2 \geq v\}} b_{t,v} \left( \lambda 2^{-\gamma} \kappa^{1-\gamma} \right) X_v dv \right] + E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{v \geq t_2 + \Delta\}} b_{t,v} \left( \lambda 2^{-\gamma} \kappa^{1-\gamma} \right) X_v dv \right] 
+ E_t \left[ \int_{t+\Delta}^{\infty} \frac{1-1_{\{t_2 \geq v\}} - 1_{\{v \geq t_2 + \Delta\}} + 1_{\{t_2 \geq v\}} 1_{\{v \geq t_2 + \Delta\}}}{2^{-\gamma}} b_{t,v} \left( \lambda 2^{-\gamma} \kappa^{1-\gamma} \right) X_v dv \right] 
= \frac{1}{2^{-\gamma}} E_t \left[ \int_{t+\Delta}^{\infty} b_{t,v} \left( \lambda 2^{-\gamma} \kappa^{1-\gamma} \right) X_v dv \right] + \frac{1}{2^{-\gamma}} E_t \left[ \int_{t+\Delta}^{\infty} (2^{-\gamma} - 1) 1_{\{t_2 \geq v\}} b_{t,v} \left( \lambda 2^{-\gamma} \kappa^{1-\gamma} \right) X_v dv \right] 
+ \frac{1}{2^{-\gamma}} E_t \left[ \int_{t+\Delta}^{\infty} (2^{-\gamma} - 1) 1_{\{v \geq t_2 + \Delta\}} b_{t,v} \left( \lambda 2^{-\gamma} \kappa^{1-\gamma} \right) X_v dv \right] 
+ \frac{1}{2^{-\gamma}} E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{t_2 \geq v\}} 1_{\{v \geq t_2 + \Delta\}} b_{t,v} \left( \lambda 2^{-\gamma} \kappa^{1-\gamma} \right) X_v dv \right] 
= \lambda \kappa^{1-\gamma} E_t \left[ \int_{t+\Delta}^{\infty} b_{t,v} X_v dv \right] + (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{t_2 \geq v\}} b_{t,v} X_v dv \right] 
+ (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{v \geq t_2 + \Delta\}} b_{t,v} X_v dv \right].
$$

We consider the three expectations separately. The first expectation is readily solved and is equal to

$$
E_t \left[ \int_{t+\Delta}^{\infty} b_{t,v} X_v dv \right] = X_t e^{(\alpha-r)\Delta}.
$$

The second expectation, as well as the third expectation are solved via methods often applied to price exotic barrier options (e.g., Shreve, 2004). These derivations are somewhat intricate, which motivates our detailed proof. Consider the second expectation

$$
E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{t_2 \geq v\}} b_{t,v} X_v dv \right] = \int_{t+\Delta}^{\infty} E_t \left[ b_{t,v} 1_{\{t_2 \geq v\}} X_v \right] dv,
$$

to which we apply the change of variables $Y_s = \frac{X_s}{B_s}$. Thus, we can rewrite

$$
\Pr(t_2 \geq v) \iff \Pr(Y_s < 1, \forall s \in [t, v])
$$

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The payoffs are expressed as

\[ Y_v = \frac{X_v}{B_v} = \frac{X_t e^{(\alpha - \frac{1}{2} \sigma^2)(v-t)+\sigma(z_v-z_t)}}{B_v} = Y_t e^{\frac{1}{\sigma} \left\{ \ln \left( \frac{B_t}{B_v} \right) + \left( \alpha - \frac{1}{2} \sigma^2 \right) \right\} \mu_{t,v} \left( v-t \right) + \left( z_v-z_t \right)} \equiv Y_t e^{\sigma (\hat{z}_v - \hat{z}_t)}, \]

where

\[ \hat{z}_v - \hat{z}_t = \mu_{t,v} \left( v-t \right) + \left( z_v-z_t \right). \]

We now define \( \hat{M}_v \) as the maximum of the Brownian motion difference \( \hat{z}_v - \hat{z}_t \) over the interval \([t, v]\), i.e.,

\[ \hat{M}_v = \max_{t \leq s \leq v} (\hat{z}_v - \hat{z}_t). \]

The payoffs are expressed as

\[
E_t \left[ b_{t,v} 1_{\{t_2 \geq v\}} X_v \right] = B_v E_t \left[ b_{t,v} 1_{\{t_2 \geq v\}} Y_v \right] = B_v E_t \left[ b_{t,v} 1_{\{M_v \leq b, (\hat{z}_v - \hat{z}_t) \geq k\}} Y_v \right],
\]

in which

\[ b = \frac{1}{\sigma} \ln \left( \frac{1}{Y_t} \right) = -\frac{1}{\sigma} \ln \left( Y_t \right) > 0 \]

and \( k = \frac{1}{\sigma} \ln \left( \frac{0}{Y_t} \right) = -\infty \).

Following Shreve (2004, pp.304-308) and denoting \( u = v - t \), we manage to re-express our expectation as

\[
E_t \left[ b_{t,v} 1_{\{M_v \leq b, (\hat{z}_v - \hat{z}_t) \geq k\}} Y_v \right] = \int_{-\infty}^{b} \int_{z^+}^{b} e^{-ru} Y_t e^{\sigma z} \frac{2 \left( 2m - z \right)}{u \sqrt{2\pi u}} e^{\left( \mu_{t,v} z - \frac{1}{2} \mu^2_{t,v} u - \frac{1}{4u \sigma^2} \right)} \frac{m=b}{dmdz} \]

\[ = - \int_{-\infty}^{b} e^{-ru} Y_t e^{\sigma z} \frac{1}{\sqrt{2\pi u}} e^{\left( \mu_{t,v} z - \frac{1}{2} \mu^2_{t,v} u - \frac{1}{4u \sigma^2} \right)} \frac{m=b}{dmdz} \]

\[ = - \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} Y_t e^{\sigma z} e^{-ru + \mu_{t,v} z - \frac{1}{2} \mu^2_{t,v} u - \frac{1}{4u \sigma^2} \left( 2b-z \right)^2} dz \]

\[ + \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} Y_t e^{\sigma z} e^{-ru + \mu_{t,v} z - \frac{1}{2} \mu^2_{t,v} u - \frac{1}{4u \sigma^2} \left( 2b-z \right)^2} dz \]

\[ = Y_t \left[ -I_1 + I_2 \right], \]

in which

\[ I_1 = \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} e^{\sigma z - ru + \mu_{t,v} z - \frac{1}{2} \mu^2_{t,v} u - \frac{1}{4u \sigma^2} \left( 2b-z \right)^2} dz \]

and

\[ I_2 = \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} e^{\sigma z - ru + \mu_{t,v} z - \frac{1}{2} \mu^2_{t,v} u - \frac{1}{4u \sigma^2} \left( 2b-z \right)^2} dz. \]
A few manipulations allow us to write $I_1$ as

\[
I_1 = \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} e^{\sigma z - ru + \mu_{t,v} z - \frac{1}{2} \mu_{t,v}^2 u - \frac{1}{2\pi} (2b-z)^2} dz
\]

\[
= \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} e^{\sigma z - ru + \mu_{t,v} z - \frac{1}{2} \mu_{t,v}^2 u - \frac{2b^2}{u} - \frac{2b}{u} z + \frac{2b}{u}} dz
\]

\[
= \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} e^{\left\{ -\frac{1}{2} \mu_{t,v}^2 u - \frac{2b^2}{u} - ru + \left( \sigma + \mu_{t,v} + \frac{2b}{u} \right) z - \frac{1}{2\pi} z^2 \right\}} dz
\]

\[
= \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} e^{\left\{ -\frac{1}{2} \mu_{t,v}^2 u - \frac{2b^2}{u} + \frac{2b}{u} \right\} z - \frac{1}{2\pi} z^2} dz
\]

Similarly $I_2$ can be rearranged as follows.

\[
I_2 = \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} e^{\left\{ -\frac{1}{2} \mu_{t,v}^2 u - \frac{2b^2}{u} + \frac{2b}{u} \right\} z - \frac{1}{2\pi} z^2} dz
\]

\[
= \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} e^{\left\{ -\frac{1}{2} \mu_{t,v}^2 u - \frac{2b^2}{u} + \frac{2b}{u} \right\} z - \frac{1}{2\pi} z^2} dz
\]

\[
= \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{b} e^{\left\{ -\frac{1}{2} \mu_{t,v}^2 u - \frac{2b^2}{u} + \frac{2b}{u} \right\} z - \frac{1}{2\pi} z^2} dz
\]

which, using $y_1 = \frac{z - \eta_1 u}{\sqrt{u}}$, becomes

\[
I_1 = e^{\left\{ -\frac{1}{2} \mu_{t,v}^2 u - \frac{2b^2}{u} + \frac{2b}{u} \right\} z - \frac{1}{2\pi} z^2} dz
\]

\[
I_1 = e^{\left\{ -\frac{1}{2} \mu_{t,v}^2 u - \frac{2b^2}{u} + \frac{2b}{u} \right\} z - \frac{1}{2\pi} z^2} dz
\]

\[
I_1 = e^{\left\{ -\frac{1}{2} \mu_{t,v}^2 u - \frac{2b^2}{u} + \frac{2b}{u} \right\} z - \frac{1}{2\pi} z^2} dz
\]
which, using $y_2 = \frac{x - \eta u}{\sqrt{u}}$, becomes

$$
I_2 = e^{\omega_2 + \frac{1}{2}\eta^2 u} \int_{-\infty}^{b - \eta_2 u} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{y^2}{2}\right)} dy_2
$$

$$
= e^{\omega_2 + \frac{1}{2}\eta^2 u} \Phi \left( \frac{b - \eta_2 u}{\sqrt{u}} \right)
$$

Therefore, the second expectation, is equal to

$$
E_t \left[ b_{t,v} 1_{\{t_2 \geq v\}} X_v \right] = -B_v Y_t \left[ e^{\omega_2 + \frac{1}{2}\eta^2 u} \Phi \left( -\frac{1}{\sigma} \ln (Y_t) + \eta_1 u \right) - e^{\omega_2 + \frac{1}{2}\eta^2 u} \Phi \left( -\frac{1}{\sigma} \ln (Y_t) + \eta_2 u \right) \right].
$$

We now turn our attention to the third expectation

$$
E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{v \geq t_2 + \Delta\}} b_{t,v} X_v dv \right].
$$

The payoffs are

$$
E_t \left[ 1_{\{v \geq t_2 + \Delta\}} b_{t,v} X_v \right] = B_v E_t \left[ 1_{\{v \geq t_2 + \Delta\}} b_{t,v} X_v \right]
$$

$$
= B_v E_t \left[ b_{t,v} 1_{\{\hat{M}_v \geq b, (\hat{v}_v - \hat{z}_t) \geq k\}} \right] Y_v
$$

$$
= B_v E_t \left[ b_{t,v} 1_{\{\hat{M}_v \geq b, (\hat{v}_v - \hat{z}_t) \geq k\}} \right] Y_v
$$

$$
= B_v E_t \left[ b_{t,v} 1_{\{\hat{M}_v \geq b, (\hat{v}_v - \hat{z}_t) \geq k\}} \right] Y_v - B_v E_t \left[ b_{t,v} 1_{\{\hat{M}_v \leq b, (\hat{v}_v - \hat{z}_t) \geq k\}} \right] Y_v,
$$

where

$$
b = \frac{1}{\sigma} \ln \left( \frac{1}{Y_t} \right) = -\frac{1}{\sigma} \ln (Y_t) > 0 \text{ and } k = \frac{1}{\sigma} \ln \left( \frac{0}{Y_t} \right) = -\infty.
$$

In the above payoff representation, the crucial step consists in recognizing the equivalence between the indicator function $1_{\{\hat{M}_v \geq b, (\hat{v}_v - \hat{z}_t) \geq k\}}$ and the difference between the two indicators $1_{\{\hat{v}_v - \hat{z}_t \geq k\}}$ and $1_{\{\hat{M}_v \leq b, (\hat{v}_v - \hat{z}_t) \geq k\}}$. The derivations for the second expectation provides us with the solution to the second term in Eq. (36), which we recall being

$$
E_t \left[ b_{t,v} 1_{\{t_2 \geq v\}} X_v \right] = -B_v Y_t \left[ e^{\omega_2 + \frac{1}{2}\eta^2 u} \Phi \left( -\frac{1}{\sigma} \ln (Y_t) + \eta_1 u \right) - e^{\omega_2 + \frac{1}{2}\eta^2 u} \Phi \left( -\frac{1}{\sigma} \ln (Y_t) + \eta_2 u \right) \right].
$$
Lastly, the first term in Eq. (36) is easily shown to be

\[ B_v E_t \left[ b_{t,v} 1_{\{ \hat{z}_t - \hat{z}_v \geq k \}} Y_v \right] \]

\[ = B_v E_t \left[ b_{t,v} Y_v \right] \]

\[ = E_t \left[ b_{t,v} X_v \right] \]

\[ = X_t e^{(a-r)u}. \]

Hence, the value of future revenue upon exercise is equal to

\[
EP_t = \lambda \kappa^{1-\gamma} E_t \left[ \int_{t+\Delta}^{\infty} b_{t,v} X_v dv \right] + (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{ t_2 \geq v \}} b_{t,v} X_v dv \right]
\]

\[ + (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} E_t \left[ \int_{t+\Delta}^{\infty} 1_{\{ t_2 + \Delta \geq v \}} b_{t,v} X_v dv \right] \]

\[ = \lambda \kappa^{1-\gamma} X_t \frac{e^{(a-r)\Delta}}{r-\alpha}
\]

\[ - (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} \int_{t+\Delta}^{\infty} B_v Y_t e^{\left\{ \omega_1 + \frac{1}{2} \eta_1^2 u \right\}} \Phi \left( -\frac{1}{\sigma} \ln (Y_t) + \eta_1 u \right) \frac{1}{\sqrt{u}} dv
\]

\[ + (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} \int_{t+\Delta}^{\infty} B_v Y_t e^{\left\{ \omega_2 + \frac{1}{2} \eta_2^2 u \right\}} \Phi \left( -\frac{1}{\sigma} \ln (Y_t) + \eta_2 u \right) \frac{1}{\sqrt{u}} dv
\]

\[ + (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} \int_{t+\Delta}^{\infty} X_t e^{(a-r)u} dv] \]

\[ + (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} \int_{t+\Delta}^{\infty} B_v Y_t e^{\left\{ \omega_1 + \frac{1}{2} \eta_1^2 u \right\}} \Phi \left( -\frac{1}{\sigma} \ln (Y_t) + \eta_1 u \right) \frac{1}{\sqrt{u}} dv
\]

\[ - (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} \int_{t+\Delta}^{\infty} B_v Y_t e^{\left\{ \omega_2 + \frac{1}{2} \eta_2^2 u \right\}} \Phi \left( -\frac{1}{\sigma} \ln (Y_t) + \eta_2 u \right) \frac{1}{\sqrt{u}} dv
\]

\[ = \lambda \kappa^{1-\gamma} X_t \frac{e^{(a-r)\Delta}}{r-\alpha} + (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} \int_{\Delta}^{\infty} X_t e^{(a-r)u} du
\]

\[ = \lambda \kappa^{1-\gamma} X_t \frac{e^{(a-r)\Delta}}{r-\alpha} + (2^{-\gamma} - 1) \lambda \kappa^{1-\gamma} X_t e^{(a-r)\Delta} \frac{e^{(a-r)\Delta}}{r-\alpha}
\]

\[ = 2^{-\gamma} \lambda \kappa^{1-\gamma} X_t \frac{e^{(a-r)\Delta}}{r-\alpha}. \]
7 Matlab Code

This Section contains sample Matlab algorithms implemented to produce the numerical results presented in the body of this article. Subsection 7.1.1 contains the program necessary to generate Figure 1 and Figure 2. Subsection 7.1.2’s output is displayed in Figure 3. In Subsection 7.2 we report the function and its outer shell for the obtention of the boundary prior to $t_1 + \Delta$, as depicted in Figure 5, as well as the graphical summary of the equilibrium strategies of both firms, as in Figure 6. The electronic Matlab files can be downloaded from the authors’ webpages at http://people.bu.edu/rdoriana/lumpsandclusterscode/index.html and http://people.bu.edu/jtreussa/lumpsandclusterscode-1/index.html. The reader is invited to contact the authors for further details.

7.1 Follower’s Optimal Investment Policy beyond $t_1 + \Delta$

7.1.1 Immediate Exercise Boundary and Related Comparative Statics

function [B]=InfBound(I,alpha,sigma,r,gamma,kappa,lambda,Delta)
% Computes the trigger boundary for the follower beyond t1+Delta
% on the following inputs:
% I = Fixed cost of investment
% alpha = Drift of the demand process
% sigma = Volatility of the demand process
% r = Discount rate
% gamma = Concavity parameter
% kappa = Full stock of capital
% lambda = Quality-improvement factor
% Delta = Time to build
% Please note that the use of this code is not restricted in any way.
% However, referencing the authors of the code would be appreciated.
% To run this program, simply use the function defined in the first line.
% http://people.bu.edu/rdoriana
% http://people.bu.edu/jtreussa
% Doriana Ruffino and Jonathan Treussard (December 2006)
% Num denotes the numerator of the boundary
Num=I*((1/2)-((alpha-(1/2)*sigma^2)/(2*sigma*(2*r+((alpha-(1/2)*sigma^2)/sigma)^2)^(.5))));
% Den denotes the denominator of the boundary
Den1=(2^(-gamma))*(kappa^(1-gamma))/(r-alpha);
Den2=(lambda*exp((alpha-r)*Delta)-1);
Den3=((1/2)-((alpha+(1/2)*sigma^2)/(2*sigma*(2*(r-alpha)+((alpha+(1/2)*sigma^2)/sigma)^2)^(.5))));
Den=Den1*Den2*Den3;
% B denotes the value of the boundary
B=Num/Den;
% Conducts comparative statics on the boundary beyond t1+Delta with respect
% to the quality-improvement factor and the time-to-build interval
% on the following inputs:
% I = Fixed cost of investment
% alpha = Drift of the demand process
% sigma = Volatility of the demand process
% r = Discount rate
% gamma = Concavity parameter
% kappa = Full stock of capital
% lambda = Quality-improvement factor
% Delta = Time to build
% Please note that the use of this code is not restricted in any way.
% However, referencing the authors of the code would be appreciated.
% http://people.bu.edu/rdoriana
% http://people.bu.edu/jtreussa
% Doriana Ruffino and Jonathan Treussard (December 2006)
I=100;
alpha=.02;
sigma=.05;
r=.05;
gamma=.8;
kappa=1000;
lambda=[1.1:01:1.5];
Delta=[.5:.05:2];
Ilength=length(I);
alphalength=length(alpha);
sigmalength=length(sigma);
rlength=length(r);
gammalength=length(gamma);
kappalength=length(kappa);
lambdalength=length(lambda);
Deltalength=length(Delta);
B=zeros(lambdalength,Deltalength);
for a=1:lambdalength
for s=1:Deltalength
B(a,s)=InfBound(I,alpha,sigma,r,gamma,kappa,lambda(a),Delta(s));
end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Conducts comparative statics on the boundary beyond t1+Delta with respect
% to the drift and the volatility of the demand process
% on the following inputs:
% I = Fixed cost of investment
% alpha = Drift of the demand process
% sigma = Volatility of the demand process
% r = Discount rate
% gamma = Concavity parameter
% kappa = Full stock of capital
% lambda = Quality-improvement factor
% Delta = Time to build
% Please note that the use of this code is not restricted in any way.
% However, referencing the authors of the code would be appreciated.
% http://people.bu.edu/rdoriana
% http://people.bu.edu/jtreussa
% Doriana Ruffino and Jonathan Treussard (December 2006)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

I=100;
alpha=[.01:.0025:.0495];
sigma=[.01:.005:.15];
r=.05;
gamma=.8;
kappa=1000;
lambda=1.2;
Delta=1;
Ilength=length(I);
alphalength=length(alpha);
sigmalength=length(sigma);
rlength=length(r);
gammalength=length(gamma);
kappalength=length(kappa);
lambdalength=length(lambda);
Deltalength=length(Delta);
Ilength=length(I);
B=zeros(alphalength,sigmalength);
for a=1:alphalength
for s=1:sigmalength
B(a,s)=InfBound(I,alpha(a),sigma(s),r,gamma,kappa,lambda,Delta);
end
end

7.1.2 Follower's Investment Option

function [W,Im,B,beta,CompOption]=AmericanOptionW(X,I,alpha,sigma,r,gamma,kappa,lambda,Delta)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Computes the follower's investment option value beyond t1+Delta
% on the following inputs:
% X = Stochastic demand process
% I = Fixed cost of investment
% alpha = Drift of the demand process
% sigma = Volatility of the demand process
% r = Discount rate
% gamma = Concavity parameter
% kappa = Full stock of capital
% lambda = Quality-improvement factor
% Delta = Time to build

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B=zeros(1,1);
delta=r-alpha;
rho=alpha-(1/2)*sigma^2;
beta=((2*r*sigma^2+rho^2)^(1/2)-rho)/sigma^2;
W1=((2^(-gamma)*kappa^(1-gamma))/delta)*(lambda*exp(-delta*Delta)-1);
Ical=I/(W1);
B=beta*Ical/(beta-1);
EP=lambda*exp(-delta*Delta)*(((2^(-gamma)*kappa^(1-gamma))/delta)*X);
W2=((2^(-gamma)*kappa^(1-gamma))/delta)*X;
W3=Ical/(beta-1)*((beta-1)/(Ical*beta))^(beta);
Slope=W2/X+W1*Ical/(beta-1)*((beta-1)/(Ical*beta))^(beta)*(beta*X^(beta-1));
Im=EP-I;
CompOption=W1*W3;
if X < B
    W=W2+W1*W3;
else
    W=Im;
end

% Conducts comparative statics on the follower's investment option value beyond t1+Delta
% with respect to the quality-improvement factor and the stochastic demand process
% on the following inputs:
% X = Stochastic demand process
% I = Fixed cost of investment
% alpha = Drift of the demand process
% sigma = Volatility of the demand process
% r = Discount rate
% gamma = Concavity parameter
% kappa = Full stock of capital
% lambda = Quality-improvement factor
% Delta = Time to build
% Please note that the use of this code is not restricted in any way.
% However, referencing the authors of the code would be appreciated.
% http://people.bu.edu/rdoriana
% http://people.bu.edu/jtreussa
% Doriana Ruffino and Jonathan Treussard (December 2006)
7.2 Follower’s Optimal Investment Policy prior to $t_1 + \Delta$ and Equilibrium Strategies

Following Kallast and Kivinukk (2003), we implement the Newton-Raphson procedure to produce a sequence of values for the recursive boundary $\{B^{i+1,k}(i) : k = 0, 1, \ldots\}$ at the $i^{th}$ time step away from maturity. We run the iterative equation

$$B^{i+1,k+1}(i) = B^{i+1,k}(i) - \chi \left[ \frac{\partial \text{Diff}(B^{i+1,k}(i))}{\partial B^{i+1,k}(i)} \right]^{-1} \text{Diff}(B^{i+1,k}(i)), \quad (37)$$

in which $\text{Diff}(B^{i+1,k}(i))$ is the difference between the left and the right-hand sides of Eq. (20) and $\frac{\partial \text{Diff}(B^{i+1,k}(i))}{\partial B^{i+1,k}(i)}$ is its derivative. Letting $k^*(i)$ be the iteration at which convergence is achieved at step $i$, the initial value for the $(i + 1)^{th}$ step is set at $B^{i+1,0}(i) = B^{i,k^*(i)}(i - 1)$. Due to the convex nature of the recursive boundary, we introduce a constant factor $\chi < 1$ to prevent divergence.\(^21\) At the $i^{th}$ time step away from maturity, the iterative procedure is terminated when the difference between subsequent iterations is smaller than a preselected tolerance level, $\epsilon$.

\[^{21}\text{In contexts where the recursive boundary displays a concave pattern}, \chi = 1 \text{ guarantees convergence, even with a small number of iterations.}\]
B4=normcdf(B4ins,0,1);
P11=B1*B2*(1/D);
P12=B3*B4*(1/D);
DP6=normpdf(P6ins,0,1)*normpdf(P8ins,0,1)/(sigma*(tau^-1/2));
DP9=P4*exp(alpha*tau);

% OUTPUT:
% Diff denotes the value of the RHS minus the LHS of the integral equation
% evaluated at the current value of B
% DerDiff denotes the first derivative of Diff evaluated at the current value of B

Di¤=(-P1*B+I+P2*B+P3*B+P4*P5*P6*(B^(beta))*P7+P8*P9+P10*B*P11-r*I*P12);
DerDi¤=(-P1+P2+P3+P4*P5*P6*P7*P8*beta*(B-(beta-1))+(B-beta)*DP6+DP8*P9+P8*DP9+P10*(P11+B*DP11)-r*P12);

% Computes the follower's recursive boundary prior to t1+Delta
% and produces a summary of the equilibrium of the game with sample paths
% on the following inputs:
% lambda = Quality-improvement factor
% kappa = Full stock of capital
% gamma = Concavity factor
% alpha = Drift of the demand process
% r = Discount rate
% Delta = Time to build
% I = Fixed cost of investment
% sigma = Volatility of the demand process
% D = Annual number of time steps
% Please note that the use of this code is not restricted in any way.
% However, referencing the authors of the code would be appreciated.
% http://people.bu.edu/rdoriana
% http://people.bu.edu/jtreussa
% Doriana Ruffino and Jonathan Treussard (December 2006)

lambda=1.2;
kappa=1000;
gamma=.8;
alpha=.02;
r=.05;
Delta=1;
I=100;
sigma=.05;
D=2000;
delta=r-alpha;
P1=lambda*(2^(-gamma))*(kappa^(1-gamma))*((exp((-delta)*Delta))/(delta));
Ical=I/((2^(-gamma))*(kappa^(1-gamma))/(delta)*exp((-delta)*Delta-1));
rho=alpha-(1/2)*sigma^2;
beta=(((2*r*(sigma^2)+(rho^2))^(1/2))-rho)/(sigma^2);
Binf=beta*Ical/(beta-1);
P10=(kappa*(2^-gamma))/(lambda*exp(-delta*Delta-1));
FINAL_OUT=zeros(Delta*D,1);
BhistBegin=[Binf];
for i=2:D*Delta
    It=100*i/(D*Delta)
    L=length(BhistBegin);
    tau=L/D;
P2=(kappa^-1)*(1-exp((-delta)*tau))/((delta));
P3=(kappa^-1)*exp((-delta)*tau);
P5=(Ical/((1/Binf)+exp(-tau)));
P7=exp((1/2)*beta*(2*alpha+beta-1)*(sigma^-2)*tau);
CumTS=[1:1:D]/D;
B1=exp(-delta*CumTS);
B3=exp(-r*CumTS);
B_in=BhistBegin(1,1);
epsilon=10;
Iter=1;
while ((epsilon>0.0005) & (Iter<10000))
    [Diff_in,DerDiff_in]=BoundaryNR(B_in,BhistBegin,lambda,kappa,gamma,alpha,r,Delta,I,sigma,D,L,tau,Delta,
P1,P2,P3,P4,Ical,tau,beta,Binf,P5,P7,P10,CumTS,B1,B3);
    B_out=B_in-0.1*(Diff_in/DerDiff_in);
    epsilon=abs(B_out-B_in);
    Iter=Iter+1;
end
BhistEnd=[B_out BhistBegin];

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BhistBegin=BhistEnd;
BhistEnd=0;

% FINAL_OUT denotes the time series for the recursive boundary
FINAL_OUT=BhistBegin;

% The following creates the summary diagram
OverallBound=[Binf*ones(Delta*1.5*D,1)' FINAL_OUT' Binf*ones(Delta*1.25*D,1)']';
Q=length(OverallBound);
X=zeros(Q,1);
X(Delta*1.5*D,1)=Binf;
Wb=randn(1,1);
for i=(Delta*1.5*D+1):Q
X(i,1)=X(i-1,1)+X(i-1,1)*(alpha*(1/D)+sigma*Wb*((1/D)^(1/2)));
We=randn(1,1);
Wb=We;
end
for j=1:(Delta*1.5*D-1)
if (X(Delta*1.5*D+1-j,1)+X(Delta*1.5*D+1-j,1)*(alpha*(1/D)+sigma*Wb*((1/D)^(1/2))))<Binf
X(Delta*1.5*D-j,1)=X(Delta*1.5*D+1-j,1)+X(Delta*1.5*D+1-j,1)*(-alpha*(1/D)+sigma*Wb*((1/D)^(1/2)));
else
X(Delta*1.5*D-j,1)=X(Delta*1.5*D+1-j,1)-X(Delta*1.5*D+1-j,1)*(-alpha*(1/D)+sigma*Wb*((1/D)^(1/2)));
end
We=randn(1,1);
Wb=We;
end
Y=zeros(Q,1);
Y(Delta*1.5*D,1)=Binf;
Wb=randn(1,1);
for i=(Delta*1.5*D+1):Q
Y(i,1)=Y(i-1,1)+Y(i-1,1)*(alpha*(1/D)+sigma*Wb*((1/D)^(1/2)));
We=randn(1,1);
Wb=We;
end
Y(1:Delta*1.5*D-1)=X(1:Delta*1.5*D-1);
figure(1)
plot(OverallBound)
hold on
plot(X,'r')
plot(Y,'g')
hold off

7.3 Probabilities of Investment Clusters

function [Prob]=ProbCal(lambda,Delta,alpha,sigma,Factor,Epsilon)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Computes the probability of an investment cluster
% on the following inputs:
% lambda = Quality-improvement factor
% Delta = Time to build
% alpha = Drift of the demand process
% sigma = Volatility of the demand process
% Factor = Adjustment factor to Newton-Raphson step
% Epsilon = Iteration termination criterion
% Please note that the use of this code is not restricted in any way.
% However, referencing the authors of the code would be appreciated.
% To run this program, simply use the function defined in the first line.
% http://people.bu.edu/rdoriana
% http://people.bu.edu/jtreussa
% Doriana Ruffino and Jonathan Treussard (December 2006)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Load Boundary
kappa=1000;
gamma=.8;
r=.05;
l=100;
D=2000;
delta=r-alpha;
P1=lambda*(2^(-gamma))*(kappa^(-1+gamma))*((exp((-delta)*Delta))/(delta));
P4=P1-((2^(-gamma))*(kappa^(-1+gamma)))/(delta);
Ical=I/((2^(-gamma))*(kappa^(-1+gamma))*(lambda*exp(-delta*Delta)-1));
rho=alpha-1/2*sigma^2;
beta=((2*sqrt(sigma^2)+(rho-2)^(-1/2)-rho)/sigma^2);
Binf=beta*Ical/2;
P10=(kappa^(-1+gamma))*((lambda^2-2*sigma^2)*exp(-delta*Delta)-1);
% Initialize

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BhistBegin=[Binf];
FINAL_OUT=zeros(Delta*D,1);

% Loop
for i=2:D*Delta
    It=100*i/(D*Delta)
    L=length(BhistBegin);
    tau=L/D;
    P2=(kappa^(1-gamma))*(1-exp((-delta)*tau))/(delta);
    P3=((-2^(-gamma))*(kappa^(1-gamma))/(delta))*exp((-delta)*tau);
    P5=(Ical/(beta-1))*((1/Binf)^(beta))*exp(-r*tau);
    P7=exp((1/2)*beta*(2*alpha+(beta-1)*sigma^2)*tau);
    CumTS=[1:1:L]'/D;
    B1=exp(-delta*CumTS);
    B3=exp(-r*CumTS);
    B_in=BhistBegin(1,1);
    epsilon=10;
    Iter=1
    while ((epsilon>Epsilon) & (Iter<10000))
        [Diff_in,DerDiff_in]=BoundaryNR(B_in,BhistBegin,lambda,kappa,gamma,alpha,r,Delta,I,sigma,D,L,tau,delta,P1,P2,P3,P4,Ical,rho,beta,Binf,P5,P7,P10,CumTS,B1,B3);
        B_out=B_in-Factor*(Di" in/DerDi" in);
        epsilon=abs(B_out-B_in);
        Iter=Iter+1;
        B_in=B_out;
    end
    BhistEnd=[B_out BhistBegin]' ;
    BhistBegin=BhistEnd;
    BhistEnd=0;
end
FINAL_OUT=BhistBegin;

% Simulates paths of the underlying market demand process
Horizon=Delta;
P=30000;
Wb=randn(P,1);
XB=Binf*ones(P,1);
YB=zeros(P,1);
Ind=zeros(P,1);
for i=2:Horizon*D
    At=100*i/(Horizon*D)
    XE=XB+XB.*(alpha*(1/D)+sigma*Wb*((1/D)^(1/2)));
    Ind((XE./FINAL_OUT(i,1))>1)=1;
    YE=YB+Ind;
    We=randn(P,1);
    Wb=We;
    XB=XE;
    YB=YE;
    XE=0;
    YE=0;
    Ind=zeros(P,1);
end
Y=YB;
Ind(Y>=1)=1;
Prob=sum(Ind)/P;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Conducts comparative statics on the probability of an investment cluster
% on the following inputs:
% lambda = Quality-improvement factor
% Delta = Time to build
% alpha = Drift of the demand process
% sigma = Volatility of the demand process
% Factor = Adjustment factor to Newton-Raphson step
% Epsilon = Iteration termination criterion
% Please note that the use of this code is not restricted in any way.
% However, referencing the authors of the code would be appreciated.
% http://people.bu.edu/rdoriana
% http://people.bu.edu/jtreussa
% Doriana Ruffino and Jonathan Treussard (December 2006)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

7.4 Ex-Ante Value of the Firms

This code determines the initial value of the underlying stochastic demand process that makes the two firms identical in value at time 0.

```matlab
function [DIFF_LF]=diffunc(IV,IVHist,dHHist,DD,P,TIME,lambda,kappa,gamma,alpha,r,Delta,I,sigma,Binf);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Computes the value of the leader and the follower at time zero
% on the following inputs:
% IV = Initial value of the stochastic demand process
% IVHist = History of simulated demand processes
% dHHist = History of simulated present value of profits prior to investment
% DD = Annual number of time steps
% P = Number of simulated paths
% TIME = Total number of time steps (to horizon truncation)
% lambda = Quality-improvement factor
% kappa = Full stock of capital
% gamma = Concavity factor
% alpha = Drift of the demand process
% r = Discount rate
% Delta = Time to build
% I = Fixed cost of investment
% sigma = Volatility of the demand process
% Binf = Trigger boundary for the leader
% Please note that the use of this code is not restricted in any way.
% However, referencing the authors of the code would be appreciated.
% To run this program, simply use the function defined in the first line.
% http://people.bu.edu/rdoriana
% http://people.bu.edu/jtreussa
% Doriana Ruffino and Jonathan Treussard (December 2006)

load F.mat %(loads the scalar value of the follower at time t1 as in BoundaryNR.m)
IVHistNEW=IVHist*IV;
IND=zeros(P,TIME);
IND(IVHistNEW>Binf)=1;
IND(:,TIME)=ones(P,1);
t1=zeros(P,1);
b=zeros(P,1);
DH=zeros(P,1);
for p=1:P
  f=zeros(1,1);
  f=find(IND(p,:)==1);
  t1(p,1)=f(1)/DD;
  b(p,1)=exp(-r*t1(p,1));
  DH(p,1)=IV*dHHist(p,f(1));
end
PVF=F*b+DH;
EF0=sum(PVF)/P;
%
% Value of the leader
% [W,Im,B,beta,CompOption]=AmericanOptionW(IV,I,alpha,sigma,r,gamma,kappa,lambda,Delta);
% Difference between leader and follower
DIFF_LF=W-EF0;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

7.4 Ex-Ante Value of the Firms

This code determines the initial value of the underlying stochastic demand process that makes the two firms identical in value at time 0.
% the values of the leader and the follower are equal
% on the following inputs:
% IV = Initial value of the stochastic demand process
% IVHist = History of simulated demand processes
% dHHist = History of simulated present value of profit flows prior to investment
% DD = Annual number of time steps
% P = Number of simulated paths
% TIME = Total number of time steps (to horizon truncation)
% lambda = Quality-improvement factor
% kappa = Full stock of capital
% gamma = Concavity factor
% alpha = Drift of the demand process
% r = Discount rate
% Delta = Time to build
% I = Fixed cost of investment
% sigma = Volatility of the demand process
% Binf = Trigger boundary for the leader
% Please note that the use of this code is not restricted in any way.
% However, referencing the authors of the code would be appreciated.
% http://people.bu.edu/rdoriana
% http://people.bu.edu/jtreussa
% Doriana Ruffino and Jonathan Treussard (December 2006)

% clear;
lambda=1.2;
kappa=1000;
gamma=.8;
alphea=.02;
r=.05;
Delta=1;
I=100;
sigma=.05;
DD=100;
TIME=DD*50;
P=1000;
delta=r-alpha;
Icalc=1/(((2^(-gamma))*(kappa^(1-gamma))/delta)*(lambda*exp(-delta*Delta)-1));
rho=alpha-(1/2)*sigma^2;
betct=(((2*r*(sigma^2)+(rho^2))^(1/2))-rho)/(sigma^2);
Binf=betct/Icalc/(betct-1);
IVHist=zeros(P,TIME);
IVHist(:,1)=ones(P,1);
dHHist=zeros(P,TIME);
Wb=randn(P,1);
for i=2:TIME
  IVHist(:,i)=IVHist(:,i-1)+IVHist(:,i-1).*(alpha*(1/DD)+sigma*Wb*((1/DD)^(1/2))));
dHHist(:,i)=dHHist(:,i-1)+exp(-r*i/DD)*((2^(-gamma))*(kappa^(1-gamma)).*IVHist(:,i-1)*(1/DD));
We=randn(P,1);
Wb=We;
end
% IV_out is the value of the stochastic demand process that solves (value of leader - value of follower = 0)
[IV_out]=fzero(@(IV)diff(IV,IVHist,dHHist,DD,P,TIME,lambda,kappa,gamma,alpha,r,Delta,I,sigma,Binf),Binf);