Risk Sharing under Limited Commitment*

ÁRPÁD ÁBRAHÁM[†] AND EVA CÁRCELES-POVEDA[‡]

November 22, 2006

ABSTRACT. In a model with capital accumulation, aggregate risk and competitive intermediaries, Ábrahám and Cárcelés-Poveda (2006) show that the constrained efficient allocations can be decentralized as a competitive equilibrium with endogenous borrowing limits that do not allow for default if one also imposes an upper limit on the intermediaries' capital holdings. Since it is difficult to find any empirical evidence of such restrictions, this paper characterizes the equilibrium with no capital accumulation constraints. In this case, we show that the borrowing limits that do not allow for default arise as an equilibrium outcome. We also find that capital accumulation is higher in the absence of accumulation constraints, since the intermediaries do not internalize that fact that a higher aggregate capital increases the incentives to default. In addition, agents may enjoy a higher welfare in the long run in spite of the fact that this allocation is not constrained efficient.

Keywords: Complete markets, Enforcement Constraints, Intermediation

JEL Classification: E44, D52, G12

*The authors thank the participants of the SED (2005), the AEA (2006) and the Midwest Macroeconomic meetings (2006) for their comments.

[†]*Correspondence:* Department of Economics, University of Rochester, Harkness Hall, P.O.Box 270156, Rochester NY 14627. *E-mail:* aabraham2@mail.rochester.edu.

[‡]*Correspondence:* Department of Economics, State University of New York, Stony Brook NY-11794-4384. *Email:* ecarcelespov@notes.cc.sunysb.edu. *Web:* <http://ms.cc.sunysb.edu/~ecarcelespov/>.

1. INTRODUCTION

In a model with capital accumulation, aggregate risk and competitive intermediaries, Ábrahám and Cárceles-Poveda (2006) show that the constrained efficient allocations can be decentralized as a competitive equilibrium with endogenous borrowing limits that do not allow for default if one also imposes an upper limit on the intermediaries' capital holdings. However, we are not aware of any empirical evidence of the presence of capital accumulation constraints and it is difficult to provide micro foundations for these type of restrictions. Further, since the intermediaries can make strictly positive profits with accumulation constraints, the borrowing limits that do not allow for default may not be supported in equilibrium. In the present paper, we characterize the equilibrium allocations in the absence of accumulation constraints.

We first show that the allocations with no capital accumulation constraints solve almost the same system of equations as the constrained efficient allocations. Moreover, we show that the endogenous borrowing constraints that do not allow for default arise in equilibrium if the intermediaries are allowed to choose them. These characterization results provide a relatively simple solution method for a potentially very complicated equilibrium problem.

The allocations with and without capital accumulation constraints are then compared numerically and we find that they are qualitatively similar. In particular, they exhibit perfect risk sharing in the long run with the benchmark calibration. On the other hand, important differences arise in the short run. First, the economy with no capital accumulation constraints accumulates more capital because the constraints bind occasionally. Second, since a higher capital accumulation increases the value of going into autarky and the incentives to default, the model with capital accumulation constraints leads to a bigger range of initial wealth distributions where the participation constraints are not binding in equilibrium. Finally, although agents can enjoy a higher consumption in the constrained optimal allocation, the fact that capital accumulation is lower affects their lifetime utilities negatively. We find that this last effect dominates for the more wealthy agents, since a higher consumption is less important for them. Given this, the allocation of the economy without capital accumulation constraints is not Pareto dominated by the constrained efficient allocation, where these constraints are imposed in equilibrium.

We also study the sensitivity of these results to alternative model formulations. First, we modify the autarky penalties by allowing agents to save in physical capital after default. We find that this modification does not alter any of the qualitative findings described above, although less risk sharing is obviously supported in this

case. Second, we choose a different calibration where agents are more impatient and where the weight of capital income in their total income is lower. Under this scenario, the long run equilibrium allocations are not characterized by perfect risk sharing any more but the short run differences that we have described also hold in the long run. This implies that capital accumulation in the stationary distribution tends to be higher in the economy without capital accumulation constraints. More surprisingly, we find that the economy without capital accumulation constraints is actually experiencing a higher expected (average) welfare in the stationary distribution due to the higher aggregate capital.

The paper is organized as follows. Section 2 introduces the model economy. Section 3 discusses the competitive equilibrium with endogenous borrowing limits and financial intermediaries that may be subject to capital accumulation constraints. Section 4 characterizes the constrained efficient allocations of the benchmark economy. In addition, Sections 5 and 6 compare the competitive equilibria with and without capital accumulation constraints analytically and numerically and Section 7 summarizes and concludes.

2. THE ECONOMY

We consider an infinite horizon economy with aggregate uncertainty, idiosyncratic risk, endogenous production and participation constraints.¹ Time is discrete and indexed by $t = 0, 1, 2, \dots$. Further, the resolution of uncertainty is represented by an information structure or event-tree N . Each node $s^t \in N$, summarizing the history until date t , has a finite number of immediate successors, denoted by $s^{t+1}|s^t$. We use the notation $s^r|s^t$ with $r \geq t$ to indicate that node s^r belongs to the sub-tree with root s^t . Further, with the exception of the unique root node s^0 at $t = 0$, each node has a unique predecessor, denoted by s^{t-1} . The probability of s^t as of period 0 is denoted by $\pi(s^t)$, with $\pi(s^0) = 1$. Moreover, $\pi(s^r|s^t)$ represents the conditional probability of s^r given s^t . For notational convenience, we let $\{x\} = \{x(s^t)\}_{s^t \in N}$ represent the entire state-contingent sequence of any variable x throughout the paper.

At each node s^t , there exists a spot market for a single consumption good $y(s^t) \in \mathbb{R}_+$, which is produced with the following aggregate technology:

$$y(s^t) = f(z(s^t), K(s^{t-1}), L(s^t)). \quad (1)$$

$K(s^{t-1}) \in \mathbb{R}_+$ and $L(s^t) \in \mathbb{R}_+$ denote the aggregate capital and labor respectively, with $K(s^{-1}) \in \mathbb{R}_{++}$ given. Further, $z(s^t) \in \mathbb{R}_{++}$ is a productivity shock that follows

¹Our model extends the economies in Kocherlakota (1996) and Alvarez and Jermann (2000) to a context with endogenous production.

a stationary Markov chain with N_z possible values. Given z , the production function $f(z, \cdot, \cdot) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is assumed to be continuously differentiable on the interior of its domain, strictly increasing, strictly concave in K , and homogeneous of degree one in the two arguments. Moreover, we assume that $f_{LK}(z, K, L) > 0$, $\lim_{K \rightarrow 0} f_K(z, K, L) = \infty$ and $\lim_{L \rightarrow \infty} f_L(z, K, L) = 0$ for all $K > 0$ and $L > 0$. Capital depreciates at a constant rate δ and we define $F(s^t) = y(s^t) + (1 - \delta)K(s^{t-1})$.

The economy is populated by two types of households that are indexed by $i \in \{1, 2\} \equiv I$, with a continuum of identical consumers within each type.² Households have additively separable preferences over sequences of consumption $\{c_i\}$ of the form:

$$U(\{c_i\}) = \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c_i(s^t)) = E_0 \sum_{t=0}^{\infty} \beta^t u(c_i(s^t)) \quad (2)$$

where $\beta \in (0, 1)$ is the subjective discount factor and E_0 denotes the expectation conditional on information at date $t = 0$. The period utility function u is strictly increasing, strictly concave, unbounded below and continuously differentiable, with $\lim_{c \rightarrow 0} u'(c) = \infty$ and $\lim_{c \rightarrow \infty} u'(c) = 0$.

At each date-state s^t , households receive a stochastic labour endowment $\epsilon_i(s^t)$ that follows a stationary Markov chain with N_ϵ possible values. Households supply labor inelastically, implying that $L(s^t) = \sum_{i \in I} \epsilon_i(s^t)$. Further, they have a potentially history dependent outside option of $V_i(s^t)$. Thus, they are subject to a participation constraint of the form:

$$\sum_{r=t}^{\infty} \sum_{s^r | s^t} \beta^{r-t} \pi(s^r) u(c_i(s^r)) \geq V_i(s^t) \text{ for all } i \in I \text{ and } s^t. \quad (3)$$

Finally, the resource constraint of the economy at s^t is given by:

$$\sum_{i \in I} c_i(s^t) + K(s^t) = F(s^t). \quad (4)$$

3. COMPETITIVE EQUILIBRIUM

This section defines a competitive equilibrium with endogenous borrowing limits and a competitive intermediation sector for the framework described in the previous section. To do this, we assume that the economy is populated by a representative firm that operates the production technology and by a risk neutral and competitive financial

²All the results in the paper hold for any arbitrary finite number of types, and the assumption of two types is therefore without loss of generality. On the other hand, it simplifies both the notation and the exposition.

intermediation sector that operates the investment technology. Since we will consider only symmetric equilibria where all intermediaries hold the same portfolio, we focus on the representative intermediary.

Each period, after observing the realization of the productivity shock, the representative firm rents labor from the households and physical capital from the intermediary to maximize the period profits:

$$\max_{K(s^{t-1}), L(s^t)} f(z(s^t), K(s^{t-1}), L(s^t)) - w(s^t) L(s^t) - r(s^t) K(s^{t-1}).$$

Profit maximization implies that factor prices are given by:

$$w(s^t) = f_L(s^t) \equiv f_L(z(s^t), K(s^{t-1}), L(s^t)) \quad \forall s^t \quad (5)$$

$$r(s^t) = f_K(s^t) \equiv f_K(z(s^t), K(s^{t-1}), L(s^t)) \quad \forall s^t. \quad (6)$$

The representative intermediary lives for two periods. An intermediary born at node s^t first decides how much capital $k(s^t)$ to purchase subject to the following capital accumulation constraint $k(s^t) \leq B(s^t)$. The capital is rented to the firm, earning a rental revenue of $r(s^{t+1})k(s^t)$ and a liquidation value of $(1 - \delta)k(s^t)$ the following period. To finance the capital purchases, the intermediary sells the future consumption goods in the spot market for one period ahead contingent claims, which are traded at price $q(s^{t+1}|s^t)$. At s^t , the intermediary solves:

$$\max_{k(s^t)} \left\{ -k(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) [r(s^{t+1}) + 1 - \delta] k(s^t) \right\} \text{ s.t.} \\ k(s^t) \leq B(s^t). \quad (7)$$

If $\psi(s^t)$ is the multiplier on the capital accumulation constraint in (7), optimality requires that:

$$1 = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) [r(s^{t+1}) + 1 - \delta] - \psi(s^t) \quad \forall s^t. \quad (8)$$

Here, it is important to note that $1 \leq \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) [r(s^{t+1}) + (1 - \delta)]$ due to the fact that $\psi(s^t) \geq 0$. In other words, if the savings constraint is not binding ($\psi(s^t) = 0$), the intermediary makes zero profits. Otherwise, the non-negative profits at node s^t are given by:

$$d(s^t) = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) [r(s^{t+1}) + (1 - \delta)] k(s^t) - k(s^t) = \psi(s^t) k(s^t). \quad (9)$$

We assume that profits are distributed to the households when they are realized, i.e. during the first period of the intermediary's life-cycle. The period before an intermediary starts its business, households own $\theta_i^0(s^{t-1})$ shares of it, which they can immediately trade at a price $p(s^t)$. This price represents the value of an intermediary that will pay dividends next period. At each s^t , households can also trade in a complete set of state contingent claims to one period ahead consumption. The budget constraint of household $i \in I$ at s^t is therefore given by:

$$\bar{c}_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) a_i(s^{t+1}) + p(s^t) \theta_i(s^t) \leq d(s^t) \theta_i(s^{t-1}) + a_i(s^t). \quad (10)$$

In the previous equation, $\bar{c}_i(s^t) = c_i(s^t) - p(s^t) \theta_i^0(s^{t-1}) - w(s^t) \epsilon_i(s^t)$ represents the individual consumption net of the value of initial shares in the intermediaries and of the labor income. In addition, $a_i(s^{t+1})$ and $\theta_i(s^t)$ represent the amount of state contingent claims and shares in the intermediary held by $i \in I$ at the end of period t .

Market clearing for the state contingent securities requires that the debt issued by the intermediaries matches the demand of the households, that is, $\sum_i a_i(s^{t+1}) = [r(s^{t+1}) + (1 - \delta)]K(s^t)$. Further, $\theta_i^0(s^{t-1})$ is given for $i = 1, 2$, while $\sum_i \theta_i(s^t) = \sum_i \theta_i^0(s^{t-1}) = 1$. If we denote by $\omega_i(s^t) \equiv d(s^t) \theta_i(s^{t-1}) + a_i(s^t)$ the initial asset wealth of the household, its optimization problem at s^t can be written as:

$$\max_{\{c_i, a_i, \theta_i\}} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c_i(s^t)) \quad \text{s.t.}$$

$$\bar{c}_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) \omega_i(s^{t+1}) \leq \omega_i(s^t) \quad (11)$$

$$\omega_i(s^{t+1}) \geq A_i(s^{t+1}). \quad (12)$$

As reflected by the equation (12), the individual asset wealth is subject to a borrowing constraint of $A_i(s^{t+1})$. The equilibrium determination of these limits will be discussed later on. If $\gamma_i(s^{t+1}) \geq 0$ is the multiplier on this constraint, the necessary and sufficient first order conditions with respect to $a_i(s^{t+1})$ and $\theta_i(s^t)$ from the maximization problem of household $i \in I$ imply that:

$$q(s^{t+1}|s^t) = \beta \pi(s^{t+1}|s^t) \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\} + \frac{\gamma_i(s^{t+1})}{u'(c_i(s^t))} \quad \forall s^{t+1}|s^t \quad (13)$$

and

$$p(s^t) = \beta \sum_{s^{t+1}|s^t} \left\{ \pi(s^{t+1}|s^t) \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} d(s^{t+1}) + \frac{\gamma_i(s^{t+1})}{u'(c_i(s^t))} d(s^{t+1}) \right\} \quad \forall s^t.$$

Combining the above two first-order conditions yields the pricing equations for the shares of the intermediaries:

$$p(s^t) = \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) d(s^{t+1}) \quad \forall s^t. \quad (14)$$

The previous equation can also be obtained using no arbitrage arguments. Further, it allows us to rewrite the agents problem as if the decision variable was the next period wealth $\omega_i(s^{t+1})$ instead of $a_i(s^{t+1})$ and $\theta_i(s^t)$ separately. We will use this below in our definition of a competitive equilibrium. This result also implies that there are a continuum of possible combinations of $a_i(s^{t+1})$ and $\theta_i(s^t)$ that will yield the same allocations, since the share in the intermediaries is a “redundant” asset in spite of markets being endogenously incomplete. Finally, the transversality condition in terms of wealth is given by:

$$\lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \pi(s^t) u'(c_i(s^t)) [\omega_i(s^t) - A_i(s^t)] \leq 0 \quad \forall s^t. \quad (15)$$

Definition 1. A competitive equilibrium with borrowing constraints $\{A_i\}_{i \in I}$, capital accumulation constraints $\{B\}$ and initial conditions $K(s^{-1})$ and $\{\omega_i(s^0)\}_{i \in I}$ is a vector of quantities $\{(c_i, \omega_i)_{i \in I}, k, K, d\}$ and prices $\{w, r, q\}$ such that (i) given prices, $\{c_i, \omega_i\}$ solves the problem for each household $i \in I$; (ii) the factor prices $\{w, r\}$ satisfy the optimality conditions of the firm; (iii) q, r and d satisfy the optimality condition of the intermediary; (iv) all markets clear, i.e., for all $s^t \in N$, $k(s^t) = K(s^t)$, $\sum_i \omega_i(s^{t+1}) = [r(s^{t+1}) + 1 - \delta]K(s^t) + d(s^t)$, $\sum_i \epsilon_i(s^t) = L(s^t)$ and $\sum_{i \in I} c_i(s^t) + K(s^t) = F(s^t)$.

As stated in the previous section, each household has an outside option of $V_i(s^t)$. In the present setting, we assume that households can leave the risk sharing arrangement at any date-state to go to financial autarky. In this case, they will only be able to consume their labour income, while they are excluded from financial markets forever.³ Given this, we choose limits that are *not too tight*, in the sense that a looser limit would imply that an agent with that level of debt prefers to leave the trading arrangement. To determine these limits, we define the value of the trading arrangement recursively. The state vector of household $i \in I$ is represented by $S_i(s^t) = (\epsilon_i(s^t); \epsilon_{-i}(s^t), z(s^t), K(s^{t-1}))$, where $(\epsilon_i(s^t); \epsilon_{-i}(s^t), z(s^t))$ is the vector of exogenous states and $K(s^{t-1})$ is an endogenous state that is determined in equilibrium. Using this notation, the value of the trading arrangement at s^t can be written

³A different outside option where households are excluded from trade in Arrow securities but can still save by accumulating physical capital is considered later on.

as follows:

$$W^{ce}(\omega_i(s^t), S_i(s^t)) = u(c_i(s^t)) + \beta \sum_{s^{t+1}} \pi(s^{t+1}|s^t) W^{ce}(\omega_i(s^{t+1}), S_i(s^{t+1})) \quad (16)$$

Definition 2. *The borrowing constraints $\{A_i\}_{i \in I}$ are not too tight if they satisfy the following condition for all $i \in I$ and all nodes $s^t \in N$:*

$$W^{ce}(A_i(s^t), S_i(s^t)) = V^{ce}(S_i(s^t)) \quad (17)$$

where the value of the outside option at s^t is given by:

$$V^{ce}(S_i(s^t)) = \sum_{r=t}^{\infty} \sum_{s^r|s^t} \beta^{r-t} \pi(s^r|s^t) u(w(s^r) \epsilon_i(s^r)). \quad (18)$$

It is important to note that the value of staying in the trading arrangement W^{ce} is strictly increasing in wealth, whereas the autarky value V^{ce} is not a function of $\omega_i(s^t)$. This implies that the limits defined by (17) exist and they are unique under our assumptions on the utility function. Moreover, since $W^{ce}(0, S_i(s^t)) \geq V^{ce}(S_i(s^t))$ and W^{ce} is increasing in ω_i , equation (17) implies that $A_i(s^t) \leq 0$. Intuitively, no agent would default with a positive level of wealth, since he could then afford a higher current consumption than in autarky and at least as high of a life-time utility as in autarky from next period on.

4. CONSTRAINED EFFICIENT ALLOCATIONS

This section characterizes the constrained efficient allocations of the economy described earlier. As usual, the optimal allocations solve a central planning problem where the planner takes into account both the resource constraint and the participation constraints of the two households. If α_i is the initial Pareto weight assigned by the planner to each household, the problem of the planner at s^t can be written as follows:

$$\max_{\{c_i\}_{i \in I, K}} \sum_{i \in I} \alpha_i \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t u(c_i(s^t)) \quad \text{s.t.} \quad (19)$$

$$\sum_{i \in I} c_i(s^t) + K(s^t) = F(s^t) \quad (20)$$

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi(s^r) u(c_i(s^r)) \geq V(S_i(s^t)) \quad \text{for } i \in I \text{ and } s^t. \quad (21)$$

Note that we have set $V_i(s^t) = V(S_i(s^t))$ by assuming that the outside option value for $i \in I$ depends on $S_i(s^t) = (\epsilon_i(s^t); \epsilon_{-i}(s^t), z(s^t), K(s^{t-1}))$. Whereas standard dynamic

programming is inapplicable to the previous setup, we can follow Marcet and Marimon (1999) and rewrite the Lagrangian of the above problem as follows:

$$\inf_{\{\gamma_i\}} \sup_{\{c_i, K\}} H \equiv \sum_{i \in I} \sum_{t=0}^{\infty} \sum_{s^t} \pi(s^t) \beta^t \{u(c_i(s^t))(\mu_i(s^t) + \alpha_i) - \gamma_i(s^t) V(S_i(s^t))\}.$$

where $\beta^t \gamma_i(s^t)$ is the Lagrange multiplier of the time t participation constraint for household $i \in I$. Further, $\mu_i(s^t)$ is a pseudo state variable that is defined recursively as follows:

$$\mu_i(s^t) = \mu_i(s^{t-1}) + \gamma_i(s^t), \mu_i(s^{-1}) = 0 \text{ for } i = 1, 2. \quad (22)$$

It is easy to see that the solution to the previous problem can be characterized by the resource and participation constraints in (20)-(21) and by the following first order conditions:⁴

$$\frac{u'(c_1(s^t))}{u'(c_2(s^t))} = \lambda(s^t) = \frac{(1 + v_2(s^t))}{(1 + v_1(s^t))} \lambda(s^{t-1}) \quad (23)$$

$$\begin{aligned} 1 &= \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} (1 + v_i(s^{t+1})) F_K(s^{t+1}) \right\} \\ &\quad - \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \sum_{j=1,2} \frac{v_j(s^{t+1})}{u'(c_j(s^t))} V_K(S_j(s^{t+1})) \right\} \text{ for } i \in I. \end{aligned} \quad (24)$$

The terms $F_K(s^{t+1}) = f_K(z(s^{t+1}), K(s^t), L(s^{t+1})) + 1 - \delta$ and $\{V_K(S_i(s^{t+1}))\}_{i \in I}$ on the right hand side of the previous equation represent the derivatives of total output F and of the outside option value V with respect to the aggregate capital stock K . Further, we have expressed the first order conditions in terms of the normalized multipliers λ and v_i , which simplify the system of equilibrium equations and are given by:

$$v_i(s^t) = \frac{\gamma_i(s^t)}{\mu_i(s^{t-1}) + \alpha_i} \text{ for } i \in I \quad (25)$$

$$\lambda(s^t) = \frac{\mu_2(s^t) + \alpha_2}{\mu_1(s^t) + \alpha_1}, \text{ with } \lambda(s^{-1}) = \frac{\alpha_2}{\alpha_1}. \quad (26)$$

Several remarks are worth noting. First, since $\mu_i(s^{t-1}) + \alpha_i > 0$, it follows that $v_i(s^t) > 0$ only if $\gamma_i(s^t) > 0$. This implies that $v_i(s^t)$ is positive only when the participation constraint of type $i \in I$ is binding. Second, λ represents a the time varying relative Pareto weight of type 2 households relative to type 1 households.

⁴The above first-order conditions for this problem are only necessary but not sufficient in general. For a detailed discussion of this issue see Ábrahám and Cárceles-Poveda (2006).

Thus, as usual in models with endogenously incomplete markets, condition (23) implies that the relative consumptions of the two types are determined by their time varying relative Pareto weights. Third, as in other models with commitment (see e.g. Thomas and Worrall (1988) and Kocherlakota (1996)) whenever households of type 1 have a binding participation constraint ($v_1(s^t) > 0$), λ will decrease, and their relative Pareto weight will therefore increase. The opposite happens when the participation constraint of type 2 household is binding. Finally, since the aggregate technology and the idiosyncratic income shocks are Markovian, the optimal allocation of this problem is recursive in $(\epsilon_1, \epsilon_2, z, K, \lambda)$.

As reflected by the Euler equation in (24), when the participation constraints are not binding for any household at any continuation history $s^{t+1}|s^t$, implying that $v_i(s^{t+1}) = 0$ for $i = 1, 2$, the equation reduces to the standard capital Euler condition of the stochastic growth model. On the other hand, the presence of binding enforcement constraints at s^{t+1} introduces two additional effects on the inter-temporal allocation of consumption and capital.

First, it increases the planner's marginal rate of substitution between period t and $t + 1$ goods, raising the benefits of a higher aggregate capital at $t + 1$, since this increases future consumption and decreases the default incentives. This is reflected by *the presence of $v_i(s^{t+1})$* on the first part of the right hand side of the equation. Second, it tightens the enforcement constraints through an increase in the autarky value, reducing the benefits of more capital at $t + 1$. This is reflected by *the autarky effects* on the second part of the right hand side of the equation.

In what follows, we focus on allocations that have high implied interest rates, in the sense that their present value is finite when discounted with the appropriate present value prices.⁵ We say that an allocation $\{c\} \equiv \{c_i + c_{-i}\}$ has high implied interest rates if:

$$\sum_{t=0}^{\infty} \sum_{s^t} Q_p(s^t|s^0) c(s^t) < \infty,$$

where

$$q_p(s^{t+1}|s^t) = \max_{i=1,2} \beta \pi(s^{t+1}|s^t) \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\} \quad (27)$$

and

$$Q_p(s^t|s^0) = q_p(s^t|s^{t-1})q_p(s^{t-1}|s^{t-2})\dots q_p(s^1|s^0). \quad (28)$$

⁵This assumption is not very restrictive in the present setting, since it will be satisfied whenever consumption is bounded away from zero.

5. CHARACTERIZATION OF THE COMPETITIVE EQUILIBRIUM

As shown by Ábrahám and Cárceles-Poveda (2006), a decentralization of the constrained efficient allocations with borrowing constraints that are not too tight is possible in the presence of endogenous borrowing constraints and financial intermediaries if the intermediaries are also subject to capital accumulation constraints. However, we are not aware of any evidence for the latter constraints in the data, and it is difficult to imagine how these upper bounds would arise as an equilibrium outcome. Given this, the present section characterizes the equilibrium allocations with borrowing constraints that are not too tight and with no binding capital accumulation constraints.

In this case, the intermediaries always make zero profits, implying that $d(s^t) = 0$ and $p(s^t) = 0$ for all $s^t \in N$. Hence, households only trade in Arrow securities subject to the following budget and portfolio constraints:

$$c_i(s^t) + \sum_{s^{t+1}|s^t} q(s^{t+1}|s^t) a_i(s^{t+1}) \leq a_i(s^t) + w_i(s^t) \quad (29)$$

$$a_i(s^{t+1}) \geq A_i(s^{t+1}) \quad (30)$$

Since the portfolio constraint in (30) can only be binding for one of the two households, it follows that $\gamma_i(s^{t+1}) = 0$ for at least one household. Given this, equations (8) and (13) of the competitive equilibrium can be rewritten as:

$$q(s^{t+1}|s^t) = \beta \pi(s^{t+1}|s^t) \max_{i=1,2} \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\} \quad (31)$$

$$1 = \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \beta \max_{i=1,2} \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right\} F_K(s^{t+1}) \quad (32)$$

where we have substituted for $r(s^{t+1})$ from (6). An equilibrium for this case is defined in what follows.

Definition 3. A competitive equilibrium with borrowing constraints $\{A_i\}_{i \in I}$ and initial conditions $K(s^{-1})$ and $\{a_i(s^0)\}_{i \in I}$ is a vector of quantities $\{(c_i, a_i)_{i \in I}, k, K\}$ and prices $\{w, r, q\}$ such that (i) given prices, $\{c_i, a_i\}$ maximizes the utility for each household $i \in I$ subject to (29) and (30); (ii) the factor prices $\{w, r\}$ satisfy the optimality conditions of the firm in (5) and (6); (iii) q and r satisfy the optimality condition of the intermediary in (32); (iv) all markets clear, i.e., for all $s^t \in N$, $k(s^t) = K(s^t)$, $\sum_i a_i(s^{t+1}) = [r(s^{t+1}) + 1 - \delta]K(s^t)$, $\sum_i c_i(s^t) = L(s^t)$ and $\sum_{i \in I} c_i(s^t) + K(s^t) = F(s^t)$.

We now show that the competitive equilibrium allocations of the model with borrowing constraints that are not too tight but no capital accumulation constraints

satisfy the same system of equations as the constrained efficient problem except the Euler condition in (24), which is replaced by:

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} (1 + v_i(s^{t+1})) F_K(s^{t+1}) \right\}. \quad (33)$$

This result is stated by Propositions 2 and 3. In addition, Proposition 1 below shows that, if we allow the intermediaries to set the borrowing constraints on the households, they will choose the ones which are not too tight, providing an equilibrium foundation for our endogenous borrowing limits.⁶

Proposition 1. *(i) The CE with borrowing constraints that are not too tight remains a competitive (Nash) equilibrium if the intermediaries are allowed to set the borrowing limits. (ii) No symmetric competitive (Nash) equilibrium exists for limits that are looser than the ones that are not too tight.*

The previous proposition shows first that no intermediary has incentives to loosen or tighten the limits individually when they are not too tight, since these deviations are not profitable. This implies that these constraints arise as an equilibrium decision of the intermediaries. Intuitively, since the intermediaries make zero profits with any limits which do not allow for default, they have no incentive to tighten them. On the other hand, since they are price-takers, they cannot break even with looser limits. Further, the proposition also shows that no symmetric equilibrium exists where some or all of the limits are looser than the ones dictated by (17). This result is due to the fact that, if there is default, the intermediaries can always increase their profits by not buying Arrow securities from households with a positive probability of default next period.

It is important to note that, in the model with binding capital accumulation constraints, we have to assume that the intermediaries take the limits as given, since Proposition 1 is not necessarily true. In this case, intermediaries may still make positive profits by entering the market and by setting borrowing limits that allow for some default. This finding provides further motivation for studying the model with no capital accumulation constraints. We are now ready to state our equivalence results.

Proposition 2. *Let $\{c_1, c_2, K\}$ be a solution to equations (20), (21), (23), (25), (26) and (33) where $\{c\} = \sum_i \{c_i\}$ has high implied interest rates. Then, this alloca-*

⁶It is important to note that Proposition 3 implies the existence of symmetric equilibria with no default borrowing constraints and the non-existence of symmetric equilibria with equilibrium default. However, we cannot rule out non-symmetric equilibria where default can be an equilibrium outcome.

tion can be decentralized as a competitive equilibrium with borrowing constraints that are not too tight and no capital accumulation constraints.

Proposition 3. *Let $\{(c_i, a_i)_{i \in I}, K, q, r, w\}$ be a competitive equilibrium with borrowing constraints $\{A_i\}_{i \in I}$ that are not too tight and no capital accumulation constraints. Then $\{(c_i)_{i \in I}, K\}$ is a solution to equations (20), (21), (23), (25), (26) and (33). Further, $c = \sum_i c_i$ satisfies the high implied interest rates condition with respect to the price $Q(s^t|s^0)$ defined by:*

$$Q(s^t|s^0) = q(s^t|s^{t-1})q(s^{t-1}|s^{t-2})\dots q(s^1|s^0).$$

Propositions 2 and 3 provide a useful characterization of the equilibrium with no capital accumulation constraints, since they show that the equilibrium allocations solve a system of equations that is very similar to constrained efficient allocation. However, we should note that this equilibrium allocation is different from the optimal allocation only due to the fact that it ignores the autarky effects. In other words, as opposed to the social planner, the financial intermediaries do not internalize the effect of capital accumulation on the agents' autarky valuations.

Nevertheless, we think that the propositions are particularly important, since they characterize an empirically more plausible competitive equilibrium which can be used to analyze several applied questions where capital accumulation and limited commitment are both important. As an example, one could study consumption and wealth inequality along the growth path, where capital accumulation can play an important role in determining the incentives to default. The computation of competitive equilibrium for this type of non-optimal economies is potentially very demanding. On the other hand, one important implication of the above propositions is that, in these cases, this computation would not require any extra burden as compared to the relatively easy computation of the optimal solution.

6. QUANTITATIVE COMPARISON OF THE ALLOCATIONS

This section compares the two competitive equilibrium allocations (with and without capital accumulation constraints) numerically. The parameters of the economy are calibrated following the asset pricing and real business cycle literature. The time period is assumed to be one quarter, and the discount factor and depreciation rate are therefore set to $\beta = 0.99$ and $\delta = 0.025$. The first parameter is chosen to generate an annual average interest rate of approximately 4% in the stationary distribution, whereas the second replicates the US average investment to capital ratio during the postwar period.

Concerning the functional forms, we assume that the production function is Cobb-Douglas, with a constant capital share of $\alpha = 0.36$. Further, the utility function of the households is assumed to be $u(c) = \log(c)$. Finally, the exogenous shock processes are assumed to be independent. In particular, the aggregate technology shock follows a two state Markov chain with $z \in \{z_l, z_h\} = \{0.99, 1.01\}$, and its transition matrix is given by:

$$\Pi_z = \begin{bmatrix} \pi_{ll} & \pi_{lh} \\ \pi_{hl} & \pi_{hh} \end{bmatrix} = \begin{bmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{bmatrix}.$$

The aggregate labor supply is constant and we normalize it to 1. As to the idiosyncratic income process, it is assumed to follow a seven state Markov chain. The values and transition matrix are obtained by using the Hussey and Tauchen (1991) procedure to discretize the following process:

$$\epsilon^{i'} = (1 - \psi_\epsilon)\mu_\epsilon + \psi_\epsilon\epsilon^i + u, \quad u \sim N(0, \sigma_u^2).$$

The shock parameters are set to $\psi_\epsilon = 0.956$ and $\sigma_u^2 = 0.082$, corresponding to quarterly adjusted estimates from annual idiosyncratic earnings data. Further, since a constant aggregate labor supply implies that $\epsilon^{-i} = 1 - \epsilon^i$, the values for ϵ^1 were chosen to be symmetric around $\mu_\epsilon = 0.5$. This implies that the idiosyncratic productivity of the two types follows the same process and the shocks are perfectly negatively correlated across the two types. Finally, note that our characterization results provide us with a relatively easy and analogous computational method for both models.

In what follows, we let $s_1 = [\epsilon, \lambda; z, K]$ and $s_2 = [1 - \epsilon, 1/\lambda; z, K]$. Under our Markovian assumption on the shocks, the previous set of equations implies that we can describe the optimal allocations in both models by the consumption functions $\{c_i(s_i)\}_{i=1,2}$, the normalized multipliers on the participations constraints $\{\nu_i(s_i)\}_{i=1,2}$ and the laws of motion for the relative wealth $\lambda'(s_1)$ and the aggregate capital $K'(s_1)$. To solve for these functions, we use policy functions iterations in both models.

Our numerical results for this benchmark parametrization are presented in Figures 1 to 6 of Appendix 1. All the optimal policies are conditioned on the low aggregate technology shock $z = 0.99$ and on $K = 38.6$, which is the mean of the stationary distribution of capital, but similar pictures can be obtained for the high technology shock. For expositional convenience, we have plotted the results for only three levels of the labour endowment, where ϵ_1 is the lowest and ϵ_7 is the highest labor endowment. Recall that type 2 households have the highest labor endowment when type 1 households have the lowest. Note also that both types have equal endowments when $\epsilon_4 = 1 - \epsilon_4 = 0.5$.

Figure 1 displays $\lambda' \equiv \lambda(s^{t+1})$ as a function of $\lambda \equiv \lambda(s^t)$ for the three different levels of the idiosyncratic income shocks. The first important observation based on this figure is that agents enjoy permanent perfect risk sharing in the long run in both models. To see this, assume first that our initial λ is inside its ergodic set, which is equal to $\lambda \in [0.8368, 1.195]$ and $\lambda \in [0.8366, 1.1953]$ for the models without and with the savings constraint respectively. As we see on the graph, $\lambda' = \lambda$ inside this region, independently of the labor income shocks. However, this can only happen if neither agent's participation constraint is binding. In addition, the ratio of marginal utilities remains constant over time. The last result, however, is the defining feature of a perfect risk sharing allocation.

Assume now that we start with $\lambda > 2.5$, implying that type 1 households hold significantly lower initial assets, and they are therefore entitled to less consumption than type 2 households. In this case, Figure 1 implies that λ' depends on the idiosyncratic income of the agent, and that it will drop to a new level depending on the shock realization. In particular, the higher the idiosyncratic income, the lower will be the new level of the relative wealth λ' . This is due to the fact that type 1 agents will then enjoy a higher autarky value and require therefore a higher compensation for staying in the risk sharing arrangement.

Here, it is important to note that, whenever λ jumps, type 1 agents' participation constraint is binding, and this new level of λ' pins down the borrowing constraint A_i of the competitive equilibrium faced by type 1 households in the previous period. This process will go on until the highest income (ϵ^7) is experienced by the type 1 agents. In this case, λ will enter the stationary distribution⁷ ($\lambda = 1.195$) and remain constant forever. Thus, agents will enjoy permanent perfect risk sharing from that period on. In addition, a symmetric argument implies that whenever $\lambda < 0.83$, λ will become 0.83 and remain constant forever after finite number of periods. Finally, whereas agents will obtain full insurance in the long-run for any initial wealth distribution, note that the economy may experience movements in consumption and in λ in the short run.

The second important observation is that two economies are qualitatively very similar. As stated above, the long-run behavior is practically identical, in the sense that there is perfect risk sharing in the long run. In addition, if $\lambda(s^0) \in [0.8368, 1.195]$, the long-run allocations will be identical. This is due to the fact that the borrowing

⁷We use the terms ergodic set and the stationary distribution loosely in this paper. Notice, however that we defined these sets as the possible values of λ in the long run. In fact, the initial condition λ_0 will pin down a unique long-run value for the relative wealth, that is, for any given initial value, the long run distribution is degenerate.

constraints (and therefore the savings constraint of the intermediary) will never bind in this case. Thus, the individual consumptions will be determined by $\lambda(s^0)$ and the capital accumulation will be (unconstrained) efficient. On the other hand, if $\lambda(s^0)$ is outside the above interval, the long-run allocations will be somewhat different due to the fact that the bounds of the stationary distribution are slightly different in the two models. As we see, the model with savings constraint allows for a slightly wider range of λ (the wealth distribution) where the participation constraints are not binding. As we will see below, this is the consequence of the different capital accumulation pattern in the two economies.

Figure 2, shows the optimal consumption of type 1 households in the two economies as a function of λ for different levels of the labor endowment. Obviously, as the relative wealth of type 1 households decreases (λ increases) their consumption decreases. Also, since we have perfect risk sharing in the stationary distribution, consumption does not depend on the idiosyncratic labour endowment there. For the same reason, the optimal consumption allocations are identical across the two models in this range. Outside the stationary distribution, as expected, consumption is increasing in the labour endowment. We also observe that the model with autarky effects allows for a higher consumption for every λ and ϵ outside the stationary distribution. As explained below, this is the consequence of higher capital accumulation in the economy with capital accumulation constraints.

Figure 3 displays the next period's aggregate capital K' as a function of λ and ϵ . Again, aggregate capital is independent of both the wealth distribution and the labour endowments in the stationary distribution, where it is at its efficient level. On the other hand, markets are effectively incomplete outside the stationary distribution, where we see a higher capital accumulation. This result is well-documented in models with exogenously incomplete markets (see e.g. Aiyagari (1994) for a model without aggregate uncertainty and Ábrahám and Cárceles-Poveda (2005) for a model with a similar set-up but trade in physical capital only). As reflected by the figure, a similar behavior arises in the present setting. In particular, capital accumulation is higher when the low idiosyncratic labour endowment coincides with low wealth (high λ). This is the case for type 1 households on the upper right corner of the figure and for type 2 households in the upper left corner.

To see why this happens, we can look at Figure 1 and at the Euler equation of the constrained efficient problem. It is clear from Figure 1 that, when type 1 households have a labour endowment of ϵ_7 and low λ (high wealth), the participation constraint of type 2 households is going to be binding in many continuation states ($v_i(s^{t+1}) > 0$).

In turn, this implies that the return of investment is higher, and more capital will be accumulated.

In the decentralized problem this is equivalent to an increase of most of the Arrow security prices $q(s^{t+1}|s^t)$, implying that intermediaries have to pay a lower return to the agents and can therefore invest more. This is the only effect in the model without autarky effects. On the other hand, this over accumulation is mitigated by the autarky effects in the constrained efficient allocation. In this case, the planner internalizes that a higher capital will increase the autarky values, leading to a lower capital accumulation than in the economy with no capital accumulation constraints. In the decentralized optimal solution, this is internalized with a binding upper limit on capital accumulation, which deters intermediaries from excessively overinvesting. In this case, households will also have less incentives to default, since the value of their outside option is lower due to a lower capital accumulation. As a consequence, we obtain perfect risk sharing for a higher range of the wealth distribution (a higher range of λ) in the model with capital accumulation constraints.

Using the results stated in Proposition 1 in Ábrahám and Cárceles-Poveda (2006), we have also depicted the individual consumptions c_i and the next period capital stock K' as a function of the initial Arrow security holdings a_1 and the same levels of idiosyncratic shocks in Figures 4 and 5.⁸ As already documented above, Figure 5 illustrates that capital accumulation is always higher in the economy with no capital accumulation constraints. In particular, capital accumulation is the highest when the low idiosyncratic shock for the type 1 households ϵ_1 is combined with a low level of initial asset holdings a_1 , or when the high idiosyncratic shock for the type 1 households ϵ_7 is combined with a high level of initial asset holdings a_1 . In this latter case, the borrowing constraint will be binding for the type 2 households. We also note that the difference between the two economies is significant. In terms of the average investment, the economy without autarky effects invest 15% more than the one with autarky effects when the lowest wealth coincides with the lowest income. Consequently, consumption will be higher in the constrained efficient allocation, especially with these combinations of idiosyncratic income and initial asset holdings. This is reflected in figure 4.

Finally, Figure 6 shows the life-time utilities of the agents for different initial wealth levels. Obviously, welfare is identical across the two economies in the stationary distribution, since the allocations are identical. Outside the stationary distribution, however, agents gain some utility in the allocation with no capital accumulation constraints compared to the allocation with autarky effects if they are relatively wealthy ($a_1 > 30$),

⁸For these calculations, we have set $\theta_1^0(s^{t-1}) = \theta_2^0(s^{t-1}) = 0.5$.

and they lose some utility when they are less wealthy ($a_1 < 10$). The reason for the utility loss in the constrained efficient allocation is that, although agents can enjoy a higher current consumption, there is also less capital accumulation, affecting their life-time utility negatively. Since the higher consumption is more important in utility terms for the low wealth agents, this second effect dominates only for relatively wealthy households.

Overall, we conclude that both economies have very similar allocations in the long run (stationary distribution), and they exhibit some important differences in the short run. As we have seen, the model without capital accumulation constraints leads to higher short run capital accumulation and consequently to a lower current consumption. A key question is how robust these properties are to some key features of our model and calibration. In order to check this, we investigate several variations of the above model and calibration in what follows.

Relaxing the Autarky Punishment. In the first experiment, we allow agents to accumulate physical capital in autarky, increasing the value of the outside option and limiting the scope of risk sharing in both economies. Formally, the autarky value at state-date s^t solves the following problem:

$$V^{CE}(s^t) \equiv \max_{\{c_i(s^{t+\tau}), \kappa_i(s^{t+\tau})\}_{\tau=0}^{\infty}} \sum_{\tau=0}^{\infty} \sum_{s^{t+\tau}} \pi(s^t) \beta^t u(c_i(s^{t+\tau})) \quad \text{s.t.}$$

$$c_i(s^{t+\tau}) + \kappa_i(s^{t+\tau}) \leq w(s^{t+\tau}) \epsilon_i(s^{t+\tau}) + r(s^{t+\tau}) \kappa_i(s^{t+\tau-1}) \quad \text{for } \forall \tau \geq 0 \quad (34)$$

$$\kappa_i(s^{t+\tau}) \geq 0 \quad \text{for } \forall \tau \geq 0 \quad \text{and } \kappa_i(s^{t-1}) \equiv 0. \quad (35)$$

where $\kappa_i(s^{t+\tau})$ represents the individual capital holdings of type $i \in I$ households. Note that the budget constraint in (34) implies that households face (exogenously) incomplete asset markets after default. Further, the first constraint in (35) reflects that households can only save but not borrow (short-sell) physical capital after default. Finally, we assume that they take the aggregate capital accumulation and therefore the current and future prices ($w(s^{t+\tau})$ and $r(s^{t+\tau})$) as given. Since we only consider individual (Nash) deviations and there is no default in equilibrium, these expectations are indeed rational.

Whereas we obtain a narrower range of λ in the stationary distribution, all the key qualitative findings of our original model are robust to this extension. In particular, we still find a perfect risk sharing in the long-run in both economies, while there is higher capital accumulation and a lower consumption in the short run with no capital

accumulation constraints.⁹ We can therefore conclude that neither the qualitative differences between the two equilibria nor the long-run perfect risk sharing property is a consequence of the tight autarky penalty that we have assumed in the benchmark model.

Using Different Parameterizations. To see if our results are robust to different parameter values, we have also studied a significantly different parametrization of the benchmark model. First, it is clear that a lower individual discount factor will make default more attractive in this environment. For this reason, we have set β to 0.65. This relatively low value of the discount factor was used by Alvarez and Jermann (2001), who study asset pricing implications of limited commitment in an endowment economy. Since this parametrization is more consistent with an annual model, we have also increased δ to 0.1. Second, it is clear that our economy is approaching a pure exchange economy as the one studied by Alvarez and Jermann (2000) as α goes to 0. In addition, the higher α is, the more important capital income becomes for the determination of the agents' consumption. In other words, a lower capital share will make default ceteris paribus more attractive. Given this, we have reduced α to 0.20.¹⁰

Some of the key results resulting from this parametrization are shown on Figures 7 to 9. As shown by Figure 7, the long-run stationary distribution of λ is not degenerate with the new parameterization, implying that the individual shares of aggregate consumption are fluctuating in the long run. First, this shows that the full risk sharing result obtained with the benchmark parametrization is due to the specific parameter values we have chosen before. On the other hand, our results illustrate that the qualitative differences between the equilibria (with and without capital accumulation constraints) remain the same with the new parameterization. In particular, the competitive equilibrium without capital constraints is accumulating more capital, whereas the constrained efficient economy (with capital accumulation constraints) does not Pareto dominate the economy without accumulation constraints. Since this last economy does not exhibit full risk sharing in the long run, we can also study the differences between the two equilibria in the stationary distribution.

Figure 8 displays the path for the aggregate capital stock in the stationary distribution and along some (artificial) business cycle simulations. On the second panel of the figure, the aggregate productivity shock alternates between 10 low and 10 high values. At the same time, we draw 1000 independent samples of the idiosyncratic process of

⁹More detailed results are available from the authors upon request.

¹⁰This value is actually consistent with the estimates of Lustig (2004), who classifies proprietor's income from farms and partnerships as labor income.

the agents for the same time horizon and we average out the results across these independent samples. Both the time series and the “business cycle” figures show that the aggregate capital stock is indeed higher in the economy without capital accumulation constraints. Finally, Figure 9 shows how the expected welfare of an agent changes during these artificial business cycles. Note that, by the law of large numbers, this expected welfare can be interpreted as the aggregate (social) welfare in the stationary distribution that arises if we assign equal weights to both types. Strikingly, we see that welfare is higher under the no capital accumulation constraint equilibrium throughout the business cycle. This result suggests that, on average, the higher income in this economy due to a higher capital accumulation offsets the welfare loss due to less risk sharing. Of course, since this allocation is not constrained efficient but satisfies the constraints of the planner’s problem by construction, agents will suffer welfare losses during the transition towards the higher capital accumulation that will more than offset the long run gains.

7. CONCLUSIONS

This paper studies an economy with capital accumulation and aggregate risk where households are subject to borrowing constraints that do not allow for default and where the financial intermediation sector may or may not be subject to capital accumulation constraints. The allocations with capital accumulation constraints are constrained efficient and one can therefore solve a planner’s problem.

We first show that the allocations without capital accumulation constraints solve a similar system of equations to the one of constrained efficient outcome, a characterization that considerably simplifies the equilibrium computation. In addition, we show that the borrowing limits that do not allow for default arise as an equilibrium outcome if we assume that the intermediaries are able to set them. This provides a motivation for studying such a setup.

The allocations with and without accumulation constraints are compared numerically and we find that they behave qualitatively very similar, but capital accumulation is higher in the absence of accumulation constraints. This result is robust to alternative autarky penalties and different calibrations of the model. In addition, the higher capital accumulation implies that welfare in the long run is higher in the model with no accumulation constraints in spite of the fact that this allocation is not constrained efficient. This welfare result has several important implications. First, it is related to the results of Davila et al. (2005) who study exogenously incomplete market economies with heterogeneous agents and show that these economies may benefit from a higher

capital accumulation in the long run for similar general equilibrium reasons. Further, this result indicates that less risk sharing can have non-trivial benefits in production economies due to precautionary capital accumulation.

APPENDIX 1

Proof of Proposition 1. (i) We first show that there are no profitable deviations from the equilibrium allocation with limits that are tighter or looser than the ones defined by (17). To see this, first notice that tightening the limits will not increase the profits of any intermediary. Further, we now show that no intermediary can make positive profits by loosening the limits, that is, by setting $\bar{A}_i(s^t) \leq A_i(s^t) < 0$ for all s^t . To do this, assume (without a loss of generality) that $\bar{A}_1(\hat{s}) < A_1(\hat{s})$ for some $\hat{s}|\tilde{s}$ where the borrowing constraint is binding for type 1 agents at the level of wealth $A_1(\hat{s})$. Under the original prices $q(s^{t+1}|\tilde{s})$, this implies that type 1 agents would default next period if node $\hat{s}|\tilde{s}$ occurs. Since type 1 households would choose $a_1(\hat{s}) < A_i(\hat{s}) < 0$ and default if $\hat{s}$ occurs, it is easy to see that the intermediary would make negative profits. First define $\bar{a}_1(s^{t+1}|\tilde{s})$ as the asset decision of type 1 households under the new limits and observe that $\bar{a}_1(\hat{s}) < A_i(\hat{s}) \leq 0$ under $q(\hat{s}|\tilde{s})$. Then, default of type 1 households imply that the profits of the intermediary are given by:

$$\begin{aligned} \bar{d}(\tilde{s}) &= -k(\tilde{s}) + \sum_{s^{t+1}|\tilde{s}} q(s^{t+1}|\tilde{s})[r(s^{t+1}) + (1 - \delta)]k(\tilde{s}) + q(\hat{s}|\tilde{s})\bar{a}_1(\hat{s}) \\ &< -k(\tilde{s}) + \sum_{s^{t+1}|\tilde{s}} q(s^{t+1}|\tilde{s})[r(s^{t+1}) + (1 - \delta)]k(\tilde{s}) = 0. \end{aligned}$$

The second equality follows from the equilibrium condition of the intermediaries in (8).

(ii) We now show that there does not exist any symmetric equilibrium with limits that are looser than the limits that are not too tight. To do this, we assume there exists an equilibrium with prices q and limits $\{A_i\}_{i=1,2}$ such that agents of type 1 would default under some continuation history $s^{t+1}|s^t = \hat{s}|s^t$ if the current history is $s^t = \tilde{s}$. First, notice that perfect competition would still require that intermediaries will make zero profits, which would be given by:

$$d(\tilde{s}) = -k(\tilde{s}) + \sum_{s^{t+1}|\tilde{s}} q(s^{t+1}|\tilde{s})[r(s^{t+1}) + (1 - \delta)]k(\tilde{s}) + q(\hat{s}|\tilde{s})a_1(\hat{s}) = 0.$$

Since a household would only default at node $\hat{s}$ if $a_1(\hat{s}) < 0$, the previous equation implies that:

$$-k(\tilde{s}) + \sum_{s^{t+1}|\tilde{s}} q(s^{t+1}|\tilde{s})[r(s^{t+1}) + (1 - \delta)]k(\tilde{s}) > 0.$$

Thus, in any symmetric equilibrium with default, it must be the case that:

$$\sum_{s^{t+1}|\tilde{s}} q(s^{t+1}|\tilde{s})[r(s^{t+1}) + (1 - \delta)] - 1 > 0.$$

The previous condition implies that any intermediary could make arbitrarily positive profits by trading only with agents of type 2 and by demanding arbitrary large amounts of total deposits ($\sum_{s^{t+1}|\tilde{s}} q(s^{t+1}|\tilde{s})a_2(s^{t+1}|\tilde{s})$) from them. However, this contradicts the fact that the original portfolio was optimal for the intermediaries under $q(s^{t+1}|s^t)$. ■

Proof of Proposition 2. The factor prices $w(s^t)$ and $r(s^t)$ that satisfy the optimality conditions of the firm in the competitive equilibrium can be constructed from the capital allocation of the planner's problem using equations (5)-(6). Given the consumption allocations $\{c_i\}_{i=1,2}$ that solve equations (20), (21), (23), (25), (26) and (33), we can use equations (27) and (28) to define the prices $q(s^{t+1}|s^t) = q_p(s^{t+1}|s^t)$ and $Q(s^{t+1}|s^t) = Q_p(s^{t+1}|s^t)$. Since the high implied interest rate condition holds, we can then use these prices and the consumption allocations in order to construct the asset holdings $\{a_i\}_{i=1,2}$ that satisfy the budget constraint of the households in the competitive equilibrium (equation (29)). These are equal to:

$$a_i(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \bar{c}_i(s^{t+n})$$

and

$$a_i(s^0) = \sum_{t=0}^{\infty} \sum_{s^t|s^0} Q(s^t|s^0) \bar{c}_i(s^t).$$

where $\bar{c}_i = c_i - w_i$. Note that we can choose the initial Pareto weights α_i for $i = 1, 2$ so that we exactly recover the initial asset levels in the competitive equilibrium.

Since there are no capital accumulation constraints, the profits of the intermediary are always equal to zero and $\psi = 0$. It is then easy to see that an allocation that satisfies condition (33) also satisfies the competitive equilibrium condition of the intermediary in (8) with $\psi = 0$. Concerning the trading limits, if $v_i(s^t) = 0$ for agent i , we first set $A_i(s^{t+1}) = -\sum_{n=1}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) w_i(s^{t+n})$ and we will redefine this limit later. Further, if $v_i(s^t) > 0$, we set $A_i(s^{t+1}) = a_i(s^{t+1})$, implying that it will be binding when the participation constraint in (21) is binding. To make sure that the optimality conditions of the households are satisfied, we can first use $q(s^{t+1}|s^t)$ to define the multiplier $\gamma_i(s^{t+1})$ so that the Euler condition in (13) holds. It is easy to check that the

multiplier will have the desired properties. In particular, if $v_i(s^{t+1}) = 0$, $\gamma_i(s^{t+1}) = 0$. Further, if $v_i(s^{t+1}) > 0$, it follows that $\gamma_i(s^{t+1}) > 0$.

The transversality condition is satisfied, since:

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \pi(s^t) u'(c_i(s^t)) [a_i(s^t) - A_i(s^t)] \\
& \leq \lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \pi(s^t) u'(c_i(s^t)) \left[\sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) c_i(s^{t+n}) \right] \\
& \leq u'(c_i(s^0)) \lim_{t \rightarrow \infty} \sum_{s^t} \beta^t \pi(s^t) \frac{u'(c_i(s^t))}{u'(c_i(s^0))} \left[\sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \sum_i c_i(s^{t+n}) \right] \\
& \leq u'(c_i(s^0)) \lim_{t \rightarrow \infty} \sum_{s^t} Q(s^t|s^0) \left[\sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \sum_i c_i(s^{t+n}) \right] = 0.
\end{aligned}$$

The first inequality follows from the fact that $[a_i(s^t) - A_i(s^t)]$ is equal to zero if the participation constraint is binding and it is equal to $\sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) c_i(s^{t+n}) \geq 0$ otherwise, since $a_i(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) \bar{c}_i(s^{t+n})$. The second follows from the fact that $c_i(s^t) \leq \sum_i c_i(s^t)$. The third inequality follows from the definition of $Q(s^t|s^0)$ and from the fact that $Q(s^t|s^0) \geq \beta^t \pi(s^t) \frac{u'(c_i(s^t))}{u'(c_i(s^0))}$ by construction. Finally, the last equality follows from the high implied interest rate condition.

Finally, we can construct the value functions $W(a_i(s^t); S_i(s^t))$ and $V(S_i(s^t))$ from the value functions of the planner's problem and redefine the borrowing constraints on Arrow security holdings so that they satisfy $W(A_i(s^{t+1}); S_i(s^{t+1})) = V(S_i(s^{t+1}))$ at every node. Since these limits do not bind for the originally unconstrained consumers, the constructed allocations are still feasible and optimal. ■

Proof of Proposition 3. To prove the proposition, we first note that the resource constraint in (20) is satisfied by the competitive equilibrium allocations. Since the asset holdings are subject to portfolio restrictions $\{A_i\}_{i \in I}$ that are not too tight, the value functions in the competitive equilibrium satisfy:

$$W^{ce}(a_i(s^t), S_i(s^t)) \geq V^{ce}(S_i(s^t))$$

for all $i = 1, 2$ and all $s^t \in N$, where $W^{ce}(a_i(s^t), S_i(s^t)) = \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi(s^r|s^t) u(c_i(s^r))$ and $V^{ce}(S_i(s^t)) = \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi(s^r|s^t) u(w(s^r) \epsilon_i(s^r))$. Given this, the functions defined by $W(S_i(s^t)) = W^{ce}(a_i(s^t), S_i(s^t))$ and $V(S_i(s^t)) = V^{ce}(S_i(s^t))$ satisfy the participation constraints in (21). We also note that the competitive equilibrium allocations

still solve the same problem if the borrowing constraints on the Arrow securities of the unconstrained households are substituted for the natural borrowing limits defined by:

$$A_i(s^{t+1}) = - \sum_{n=1}^{\infty} \sum_{s^{t+n}|s^t} Q(s^{t+n}|s^t) w_i(s^{t+n}).$$

Optimality implies that the previous limit is finite.¹¹ In addition, since the shocks z and ϵ lie in a compact set, the present values of K and $f_L(s^t)$ are finite, we can use the resource constraint to show that the competitive equilibrium allocation satisfies the high implied interest rate condition.

To recover the multipliers $\{\lambda\}$ and $\{v_i\}_{i=1,2}$, we can first use the equilibrium consumption allocations to define $\lambda(s^t) = \frac{u'(c_1(s^t))}{u'(c_2(s^t))}$. Further, $\{v_i\}_{i=1,2}$ can be recovered as follows. If the portfolio constraint is not binding for household i at node s^t in the decentralized problem, we set $v_i(s^t) = 0$. Otherwise, if it is binding for agent two, we set $v_1(s^t) = 0$ and $v_2(s^t)$ is recovered from:

$$\frac{u'(c_1(s^t))}{u'(c_2(s^t))} = (1 + v_2(s^t)) \frac{u'(c_1(s^{t-1}))}{u'(c_2(s^{t-1}))}$$

Similarly, if it is binding for agent one, we set $v_2(s^t) = 0$ and $v_1(s^t)$ is recovered from:

$$\frac{u'(c_1(s^t))}{u'(c_2(s^t))} = \frac{1}{(1 + v_1(s^t))} \frac{u'(c_1(s^{t-1}))}{u'(c_2(s^{t-1}))}.$$

Clearly, this implies that equations (23) and (25)-(26) are satisfied. In addition, the zero profit condition in equation (8) of the decentralized solution with $\psi = 0$ can be rewritten as:

$$\begin{aligned} 1 &= \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \max_{i=1,2} \left[\frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} \right] F_K(s^{t+1}) \right\} \\ &= \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left\{ \left[\frac{u'(c_i(s^{t+1}))}{u'(c_i(s^t))} (1 + v_i(s^{t+1})) \right] F_K(s^{t+1}) \right\} \end{aligned}$$

Given this, equation (33) is also satisfied. ■

¹¹In an exchange economy context with sequential trade and potentially incomplete financial markets, Santos and Woodford (1997) show that the natural borrowing limit implied by the optimal allocations has to be finite. Otherwise, one can construct a portfolio that yields more utility than the optimal allocation. The same proof can be used in the present setup.

REFERENCES

- [1] Ábrahám Á. and E. Cárceles-Poveda (2005), “Endogenous Trading Constraints with Incomplete Asset Markets”, mimeo., University of Rochester and SUNY, Stony Brook.
- [2] Ábrahám Á. and E. Cárceles-Poveda (2006), “Endogenous Incomplete Markets, Enforcement Constraints and Intermediation”, *Theoretical Economics*, forthcoming.
- [3] Aiyagari D. R. (1994), “Uninsured Idiosyncratic Risk and Aggregate Saving”, *The Quarterly Journal of Economics*, Vol. 109(3), pp. 659-684.
- [4] Alvarez, F. and U. Jermann (2000), “Efficiency, Equilibrium, and Asset Pricing with Risk of Default”, *Econometrica* 68(4), pp. 775-797.
- [5] Davila, J., J. H. Hong, P. Krusell, and J. V. Rios Rull (2005, “Constrained Efficiency in the Neoclassical Growth Model with Uninsurable Idiosyncratic Shocks”, Unpublished Manuscript, University of Pennsylvania.
- [6] Hussey R. and G. Tauchen (1991), “Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models”, *Econometrica* 59, pp. 371-396.
- [7] Kehoe P. and F. Perri (2002a), “International Business Cycles with Endogenous Incomplete Markets”, *Econometrica*, 70(3), pp. 907-928.
- [8] Kehoe P. and F. Perri (2002b), “Competitive Equilibria with Limited Enforcement”, *NBER Working Paper 9077*.
- [9] Kehoe P. and F. Perri (2004), “Competitive Equilibria with Limited Enforcement”, *Journal of Economic Theory*, 119(1), pp. 184-206 and NBER Working Paper 9077.
- [10] Kocherlakota N. (1996), “Implication of Efficient Risk Sharing without Commitment”, *Review of Economic Studies*, 63(4), pp. 595-609.
- [11] Lustig H., (2004), “The Market Price of Aggregate Risk and the Wealth Distribution”, *NBER Working Paper* No. 11132.
- [12] Marcet A. and R. Marimon (1998), “Recursive Contracts”, Working Paper Eco 98/37, European University Institute.

Appendix 1: Figures

Figure 1: Next Period Wealth Distribution (λ') as a Function of λ and ϵ

Figure 2: Optimal Consumption (c_1) as a Function of λ and ϵ

Figure 3: Next Period Capital Stock (K') as a Function of λ and ϵ

Figure 4: Optimal Consumption (c_1) as a Function of a_1 and ϵ

Figure 5: Next Period Capital Stock (K') as a Function of a_1 and ϵ

Figure 6: Life-Time Utility (W_1) as a Function of a_1 and ϵ

Figure 7: Next Period Wealth Distribution (λ') as a Function of λ and ϵ

Figure 8: Next Period Capital Stock (K') from Time Series Simulations

Figure 9: Average Life-Time Utility (W_1) from Time Series Simulations