Export Subsidies, Productivity and Welfare under Firm-Level Heterogeneity *

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Abstract

In this paper we use the monopolistic competition model with heterogeneous firms to study the effect of different policies on productivity and welfare, and provide three particular policies, which allow to reach the first best allocation in the economy. We also show that an export subsidy generates an increase in productivity, but - if policy already deals with the mark-up distortion that arises in this context (for example, through a subsidy on consumption of domestic varieties) - its effect on welfare is negative due to combination of falling variety and adverse terms of trade changes.

1 Introduction

Governments all over the world encourage exports in different ways.1 There are, of course, several theoretical reasons why doing so may be in a country’s best interest. There may be rents associated with some export markets, and a subsidy may be effective in increasing a country’s share of those rents. This is the "strategic trade policy" argument for export subsidies (see Brander and Spencer (1985) and Eaton and Grossman (1986)). Alternatively, there may be positive externalities generated by exporting, such as in the presence of external "learning by exporting," although empirically it has been hard to verify the significance of such externalities (Clerides, Lach and Tybout (1998)). In this paper we explore a different idea associated with the recent models of trade with heterogenous firms. In these models firms that export are generally more productive than domestically-oriented firms, so it is conceivable that by reallocating resources from low productivity to high productivity firms, an export subsidy may increase aggregate productivity. This may be the reasoning behind the claim that, by promoting exporting firms rather than those oriented to the domestic market, industrial policy in East-Asian countries was better (or at least less distortionary) than in Latin America.

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1See, for example, World Trade Organization Secretariat background paper “Export subsidies” with the details on export subsidies imposed by WTO members on a wide range of the products: http://docsonline.wto.org/DDFDocuments/t/tn/ag/S8.doc.
We study the effects of several policies, namely, an export subsidy (or tax), an import tariff, and a consumption subsidy on productivity and welfare in a small economy. Our model is based on Melitz (2003) and features increasing returns, product differentiation, and productivity differences across firms. It is well known that in this kind of model trade policy affects welfare through several channels. To focus on the effect of such policy through reallocation and productivity, we construct a model in which some of these channels are “neutralized”. There is, of course, the possible effect of trade policy through the terms of trade, which in standard models generates a positive optimal tariff. This channel is neutralized in our model by considering a “small economy” that does not affect the demand function for the varieties it produces. Since individual exporters exploit their market power, then from the small economy’s perspective the prices they charge are optimal. Another issue arises because of the mark-up charged by monopolists in the domestic market. If consumers can buy imports at the country’s opportunity cost (or international price) but must pay mark-ups on their purchases of domestically produced varieties, this creates a distortion. In the first part of the paper, we show that a consumption subsidy on domestically produced varieties equal in size to the mark-up neutralizes this distortion and allows the economy to reach the first best allocation. Another way to reduce this distortion is to impose an import tariff or an export tax. We derive the exact values of the optimal import tariff and export tax and show that their effect on welfare is equivalent to that of the consumption subsidy. In other words, the first best allocation in the economy can be achieved using any of these three policies.

In the second part of the paper we take a closer look at the effect of an export subsidy. It is obvious that in the presence of the consumption subsidy, which neutralizes the mark-up distortion, an export subsidy would have a negative effect on welfare. But we show that the positive productivity effect mentioned above is present: an export subsidy leads to a reallocation of resources from less productive firms oriented to the domestic market to exporters, and this increases overall productivity. What other effects could then (more than) compensate for this positive productivity effect of export subsidies? We show that welfare can be decomposed into three components: productivity, terms of trade, and variety. We then prove that the product of the terms of trade and variety components falls with the export subsidy, and this effect always dominates the positive productivity effect.

The separate behavior of the terms of trade and the variety index is also interesting. There is, of course, a standard negative effect of export subsidies on the terms of trade. However, recall that we are considering a small economy that takes as given the demand function for its products. Moreover, exports may increase through the extensive margin (i.e., more varieties exported) and not only through the intensive margin (i.e., larger exports of each variety). In fact, we show that the terms of trade as we define them do not always fall with the export subsidy. Similarly, intuition would suggest that export subsidies would decrease domestic variety, as low productivity firms exit. But we show that the variety index that matters for utility must also take into account the quantity imported. Since the export subsidy increases imports, this may counteract the negative effect on domestic variety and allow the variety index to increase.

The rest of the paper is organized as follows. The model is laid out and the equilibrium conditions are derived in Section 2. Section 3 shows that the first best allocation in the economy can be reached through either a consumption subsidy, an export tax, or an import tariff. Section 4 demonstrates the effect of the export subsidy on the economy. Section 5 concludes. The details of the proofs are given in the Appendix.
2 The Model

In this section we lay out the model, which incorporates both the export subsidy (or tax) and the consumer subsidy.\footnote{For more detailed derivations see Appendix.} The import tariff can be modeled similarly, which is done in Appendix. Consider a small country with \( L \) identical agents. To simplify, we set \( L = 1 \). Each agent supplies one unit of labor and spends his income on a continuum of domestically produced goods indexed by \( v \) and an imported good. Domestic goods and the imported good are consumed in quantities \( q(v) \) and \( z \) by each agent. Preferences are given by

\[
U = \left( z^\rho + \int_{v \in \Omega} q(v)^\rho dv \right)^{1/\rho}, \quad 0 < \rho < 1, \tag{1}
\]

where \( \Omega \) is the set of available domestic varieties, and \( \sigma = \frac{1}{1-\rho} \) is the elasticity of substitution. We assume that there is a consumption subsidy \( 1 - \eta \geq 0 \) for domestic goods, so that consumers pay \( \eta p(v) \) given price \( p(v) \) charged by producers. Defining the price index \( P \) by

\[
P = \left( z + \int_{v \in \Omega}(\eta p(v))^{1-\sigma} dv \right)^{1/\sigma},
\]

then the domestic demand for any variety is

\[
q(v) = R P^{\sigma-1} (\eta p(v))^{-\sigma} \quad \text{and} \quad z = R P^{\sigma-1} p_z^{-\sigma}, \tag{2}
\]

where \( R \) denotes aggregate expenditure. Hereafter, we normalize the price of the imported variety \( p_z \) to 1.

There is only one factor of production, labor, used by a continuum of monopolistically competitive heterogenous firms. Each firm pays a fixed cost \( f_e \) to enter the market. After paying this cost, it derives its productivity draw according to the cumulative distribution function \( G(\varphi) \). To simplify the analysis, we assume that the productivity distribution is Pareto,

\[
G(\varphi) = 1 - \left( \frac{b}{\varphi} \right)^\beta \quad \text{for} \quad \varphi \geq b, \quad \text{with} \quad \beta > \sigma.\footnote{Note that compared to the similar assumption of \( \beta > \sigma - 1 \) in Melitz (2003), we assume \( \beta > \sigma \), which allows us to calculate the aggregate quantities produced for the home and foreign markets.}
\]

In addition, there is a probability \( \delta \) that in each period firms can be hit by a bad shock and forced to exit.

A firm with productivity level \( \varphi \) has a labor requirement \( f + \frac{q}{\varphi} \) to produce \( q \) units of variety \( v \) for the domestic market. Thus, it has a marginal cost \( \frac{w}{\varphi} \), where \( w \) denotes the wage in the economy, and given the demand function from (2), it charges a price \( \frac{w}{p_{\varphi}} \). Then, the quantity sold domestically, the revenues and profits from domestic sales of a firm with productivity \( \varphi \) are, respectively,

\[
q_d(\varphi) = R P^{\sigma-1} \left( \frac{\eta w}{p_{\varphi}} \right)^{-\sigma}, \quad r_d(\varphi) = R P^{\sigma-1} \eta^{-\sigma} \left( \frac{w}{p_{\varphi}} \right)^{1-\sigma}, \quad \pi_d(\varphi) = \frac{r_d(\varphi)}{\sigma} - w f. \tag{3}
\]

Foreign demand for domestic variety \( v \) is given by \( A p_{\text{exp}}(v)^{-\sigma} \), where \( A \) is exogenously fixed and \( p_{\text{exp}}(v) \) is the price charged by an exporter. A firm which decides to export must pay a fixed cost \( f_{\text{exp}} \) to access the foreign market.\footnote{Introducing per-unit trade costs would not affect our results, so we chose to leave them out to simplify notation.} Also, we assume that it receives an ad-valorem export subsidy \( s - 1 > 0 \), calculated over export revenues, so that an exporter charging price \( p_{\text{exp}} \) gets

For more detailed derivations see Appendix.
for each unit sold abroad. Thus, exporters maximize

\[ \pi_{\text{exp}}(\varphi) = sp_{\text{exp}}^{1-\sigma} - (w/\varphi)Ap_{\text{exp}}^{\sigma} - wf_{\text{exp}}, \]

and charge price \( p_{\text{exp}}(\varphi) = \frac{w}{sp_{\text{exp}}} \). The quantity exported, and the revenues and profits from exporting are, respectively,

\[ q_{\text{exp}}(\varphi) = A\left( \frac{w}{p_{\text{exp}}\varphi} \right)^{-\sigma}, \quad r_{\text{exp}}(\varphi) = As^{\sigma}\left( \frac{w}{p_{\text{exp}}\varphi} \right)^{1-\sigma}, \quad \pi_{\text{exp}}(\varphi) = \frac{r_{\text{exp}}(\varphi)}{\sigma} - wf_{\text{exp}}, \quad (4) \]

Since profits of domestic producers and exporters are increasing in \( \varphi \), we can define two productivity cutoffs, \( x \) and \( y \), for domestic producers and exporters, respectively, so that only firms with productivity above \( x \) produce for the domestic market, and only firms with productivity above \( y \) export. The conditions for these cutoffs are derived from equalizing profits from each option to zero,

\[ RP^{\sigma-1}\varphi^{-\sigma} \left( \frac{px}{w} \right)^{\sigma-1} = \sigma w f, \quad (5) \]

(\text{EXP}) condition \[ As^{\sigma}w^{1-\sigma}(py)^{\sigma-1} = \sigma w f_{\text{exp}}. \quad (6) \]

Using the cutoffs \( x \) and \( y \), the price index can now be expressed as

\[ P^{1-\sigma} = p_{z}^{1-\sigma} + \int_{x}^{\infty} (p(\varphi))^{1-\sigma} M \frac{dG(\varphi)}{1-G(x)} = 1 + \theta M \left( \frac{px}{\eta w} \right)^{\sigma-1}, \]

where

\[ \theta = \frac{\beta}{\beta - (\sigma - 1)}. \]

We consider only equilibria with \( y > x \), i.e., there are some firms that do not export, which is consistent with the empirical evidence. Specifically, firms with \( \varphi \in [b, x) \) exit without production, firms with \( \varphi \in [x, y) \) produce only for the domestic market, and firms with \( \varphi \in [y, \infty) \) produce for both home and foreign markets. Thus, if \( M_e \) is the mass of entrants in the economy and \( M \) is the mass of active firms in the economy, then in steady state \((1-G(x))M_e = \delta M \). This simply states that new successful entrants exactly replace exiting firms. In addition, the mass of exporters is \( M_{\text{exp}} = m_{\text{exp}}M \), where \( m_{\text{exp}} = \frac{1-G(y)}{1-G(x)} \) is the share of exporters among the whole population of active firms in the economy.

Following Melitz (2003), define

\[ \tilde{\varphi}(x)^{\sigma-1} = \int_{x}^{\infty} \varphi^{\sigma-1} \mu(\varphi) d\varphi, \quad \text{where} \quad \mu(\varphi) = \frac{g(\varphi)}{1-G(x)} = \beta \frac{x^{\beta}}{\varphi^{\beta+1}}. \]

\(^5\text{Note that to model an export tax} \tau, \text{it is enough to assume that} \ s < 1, \text{so that} \ \tau = 1 - s. \text{All derivations are the same for any value of} \ s.\)

\(^6\text{We chose to look at this particular type of subsidy, since as was empirically estimated by Das, Roberts and Tybout (2003), such policy is far more effective at stimulating exports than policies that subsidize exporters’ costs of entering foreign markets.}\)
Then one can show that the expected profit from entering is given by
\[
\bar{\pi} = \pi_d(\tilde{\varphi}(x)) + m_{\text{exp}}\pi_{\text{exp}}(\tilde{\varphi}(y)) = \omega f(\theta - 1) + \omega m_{\text{exp}}f_{\text{exp}}(\theta - 1).
\]
(7)

The free entry condition, \( \frac{\pi}{\theta} (1 - G(x)) = \omega f_e \), can then be written as
\[
(\text{FE}) \text{ condition} \quad (\theta - 1) x^{-\beta} \left[ f + m_{\text{exp}}f_{\text{exp}} \right] = \frac{\delta f_e}{b^\gamma}.
\]
(8)

Now let us derive the trade balance condition. Total export revenues for the country (at world prices) are
\[
V = \int_y^{r_{\text{exp}}(\varphi)} s M(\varphi) d\varphi,
\]
whereas the foreign international value of imports is \( R_m \equiv p_{z} = R \rho^{\sigma-1} \). Trade balance is simply
\[ V = R_m \].
To characterize this further, we need to derive an expression for \( R \). To do so, note that total expenditures are \( R = w + T \), where \( T \) are lump sum transfers defined as
\[
T = -(s - 1) V - (1/\eta - 1) R_d,
\]
where \( R_d \) are total expenditures on domestic goods and equals \( \eta \int_x r_d(\varphi) M(\varphi) d\varphi \). Using (5) and (6), and some simplification, we obtain
\[
R = w - w \sigma \theta M \left[ \frac{s - 1}{s} f_{\text{exp}}m_{\text{exp}} + (1 - \eta) f \right].
\]
Simplifying this expression and using (6), the trade balance condition can then be rewritten as
\[
(\text{TB}) \text{ condition} \quad 1 = \sigma \theta M \left[ f_{\text{exp}}m_{\text{exp}} \left( 1 + \frac{1}{s} \theta M \left( \frac{p_{\text{exp}}}{\eta w} \right)^{\sigma - 1} \right) \right] + (1 - \eta) f.\]
(9)

We also need to derive the formula for the mass of firms in the economy. Note that the total revenue obtained by domestic producers, \( M \sigma (\bar{\pi} + w (f + m_{\text{exp}}f_{\text{exp}})) \), must be equal to \( w \).\(^8\) Using (7) then we can write this as
\[
(\text{M}) \text{ condition} \quad M = \frac{1}{\sigma \theta (f + m_{\text{exp}}f_{\text{exp}})} = \frac{(\theta - 1) b^\beta}{\sigma \theta \delta f_e} x^{-\beta},
\]
(10)

where the last equality follows from the (FE) condition.

Now we have our equilibrium system of equations (6), (8), (9), and (10) with four unknown variables, \( x, y, w, \) and \( M \).

We are interested in exploring how different policies affect welfare, which is captured by the

\(^7\)The free entry condition equalizes the costs of entry to the present value of average profits multiplied by the probability of successful entry.

\(^8\)To prove this, note that the total expenditure at home is \( V + R_d = R_m + R_d = w + T = w - (s - 1) V - \frac{1 - \eta}{\eta} R_d \), where \( R_d \) and \( R_m \) are expenditures on the domestic and foreign goods, respectively, and \( V \) is the value of exports. This implies \( s V + R_d/\eta = w \). Note that while consumers pay only \( p(\varphi) \), a domestic producer with productivity \( \varphi \) receives \( p(\varphi) \). However, while foreign consumers pay \( p_{\text{exp}}(\varphi) \), exporters receive \( sp_{\text{exp}}(\varphi) \). Thus, the total revenues of domestic firms are \( s V + R_d/\eta \), which equals \( w \) from above.
utility of the representative consumer. To obtain a useful expression for this utility, we first introduce some new definitions. First, let \( Q_d \) be the total quantity consumed of domestic goods,

\[
Q_d = M \int_x q(\varphi) \mu(\varphi) d\varphi = f(\sigma - 1) \frac{\beta}{\beta - \sigma} M x, \tag{11}
\]

and let \( Q^C = z + Q_d \) be the total quantity consumed of the imported and domestic goods. Similarly, let \( Q_{exp} \) be the total quantity of goods exported,

\[
Q_{exp} = M \int_y q_{exp}(\varphi) \mu(\varphi) d\varphi = f_{exp} (\sigma - 1) \frac{\beta}{\beta - \sigma} M_{exp} y \tag{12}
\]

and let \( Q^P = Q_{exp} + Q_d \) be the total quantity produced for both the domestic and foreign markets. Then utility per capita can be expressed as

\[
U = Q^P Q^C \left[ \left( \frac{z}{Q^C} \right)^\rho + M^{1-\rho} \left( \frac{Q_d M(f_x q^0(\varphi) \mu(\varphi) d\varphi)^{1/\mu}}{Q^C} \right)^\rho \right]^{1/\rho}.
\]

The first component in the product above is the productivity index in the economy measured as the total output per worker. This definition of productivity differs from that in Melitz (2003), who aims to capture “measured” productivity. In particular, he adds value added across firms and divides this sum by the industry level price, whereas we sum up value added across firms dividing by the price, or \((pq)/p = q\). It is important to note here that we are simply adding physical units of different goods to arrive at a concept of aggregate quantities consumed and produced. We comment on this further below.

The second component is the trade-adjusted terms of trade (TOT) index, which tells us the ratio of consumption to production in an open economy. This is, of course, affected by the usual TOT expression, which here would be \( P_{exp}/p_z \), with \( P_{exp} = V/Q_{exp} \), where \( V \) is the value of exports. But the impact of this relative price on utility depends on the importance of trade. Our TOT index captures the precise way in which both \( P_{exp}/p_z \) and the importance of trade affect utility.\(^9\)

The final component of the utility function is the variety index:

\[
\text{Variety index} = \left[ \left( \frac{z}{Q^C} \right)^\rho + M^{1-\rho} \left( \frac{Q_d M(f_x q^0(\varphi) \mu(\varphi) d\varphi)^{1/\mu}}{Q^C} \right)^\rho \right]^{1/\rho}.
\]

To better understand the latter components, it is useful to imagine for a moment that there was no heterogeneity across domestic varieties, so that \( q(\varphi) = q \) and \( q_{exp}(\varphi) = q_{exp} \) for all \( \varphi \). Then \( Q^C = z + Q_d = z + M q \), and the variety index would be equal to

\[
\left[ \left( \frac{z}{z + M q} \right)^\rho + M^{1-\rho} \left( \frac{M q}{z + M q} \right)^\rho \right]^{1/\rho}.
\]

The variety index of imports is one, since there is no variety, and the variety index of domestic\(^9\) Simple algebra reveals that, given trade balance, our TOT index is equal to \( \frac{P_{exp}}{P_z} \left[ \left( \frac{Q_{exp}}{Q_d + Q_{exp}} \right) / \left( \frac{z}{Q_d + z} \right) \right] \). This expression captures the way in which the volume of exports and imports affects the impact of the relative price \( P_{exp}/p_z \) on utility.
goods is $M^{1-\rho} = M^{\frac{1}{\rho}}$. We have two groups of goods, foreign and domestic, and we use the share of each group in the total quantity consumed as its weight in the variety index. With heterogeneity, we have an additional term in the formula for the variety index,

$$\frac{M(\int_x q^\rho(\varphi)\mu(\varphi)\,d\varphi)^{1/\rho}}{Q_d} = \frac{\beta - \sigma \theta^{1/\rho}}{\beta} < 1.$$  

This term captures what we could call "curvature," which measures the way in which heterogeneity across firms affects utility. It is easy to show that this curvature term rises (meaning less cost of heterogeneity) if the dispersion of the productivity distribution falls, i.e., if $\beta$ rises. And it converges to 1 as $\beta \to \infty$. In other words, for any value of $\sigma$, this term becomes closer to 1 as firms differ less, and it equals to 1 if firms are identical. Moreover, if the elasticity of substitution $\sigma$ rises, then the curvature term rises as well and becomes closer to 1, which reflects the fact that with a higher elasticity of substitution differences between varieties and their prices matter less to consumers.

We can now comment a bit more on our productivity index. Consider a closed economy. Utility in equilibrium is simply $(\rho \tilde{\varphi}(x))M^{1/(\sigma-1)}$. Following Melitz (2003), one could think of $\rho \tilde{\varphi}(x) = \theta^{1/(\sigma-1)}\rho x$ as measuring productivity, and $M^{1/(\sigma-1)}$ as measuring variety, so that utility would be the product of a Productivity Index and a Variety Index. Our approach is slightly different. We measure productivity simply by the quantum $Q^\rho$, which is equal to $(\frac{1}{\beta})\left(\frac{\beta}{\beta - \sigma}\right)\rho x$. In turn, our variety index includes the curvature term discussed above and equals $\left(\frac{\beta - \sigma \theta^{1/\rho}}{\beta - \sigma}\right)M^{1/(\sigma-1)}$. Thus, the difference between our decomposition and that of Melitz is that we put the curvature term in the variety index, whereas he puts it in the productivity index.\(^\text{10}\) Of course, both decompositions are valid, but ours is more useful given our goals in this paper.

To sum up, there are three channels through which trade policy affects welfare in our "small" economy:

$$U = (\text{Productivity Index}) \ast (\text{TOT index}) \ast (\text{Variety Index}).$$  

(13)

3 The First Best Allocation

Now let us look at the social planner’s choice of the optimal policy. In particular, as shown in Appendix, the social planner chooses an allocation that maximizes $(\int_x q^\rho(\varphi)\mu(\varphi)\,d\varphi)^{1/\rho}$ subject to full employment and balanced trade, and that has no goods produced for $\varphi < x$ and exports only goods with $\varphi > y$. Note that if there exists a solution to this problem, it is unique. Moreover, we can prove the following proposition:

**Proposition 1** The first best outcome can be achieved through implementing the following policies:

- a consumption subsidy $\eta = \rho$;
- an export tax $\tau = 1 - \rho$ (or a negative subsidy, $s = \rho < 1$);
- an import tariff $t = \frac{1}{\rho} > 1$.

Moreover, when only one of these instruments is used, utility is decreasing as the policy deviates more from its optimal level.

\(^\text{10}\) Let $C \equiv \frac{\beta - \sigma \theta^{1/\rho}}{\beta - \sigma}$, then $\theta^{1/(\sigma-1)}\rho x \ast M^{1/(\sigma-1)} = (\frac{1}{\beta})\left(\frac{\beta}{\beta - \sigma}\right)\rho x \ast C \ast M^{1/(\sigma-1)}$.  

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The intuition behind these results is the following. First, look at the consumption subsidy. There is a domestic distortion created by the mark-up charged on domestic sales: domestic goods are sold at a price above the opportunity cost, whereas imported goods are sold at a price equal to the opportunity cost, so in the equilibrium there is too little consumption of domestic relative to foreign varieties. This distortion is neutralized with the consumption subsidy, which allows consumers to pay a price equal to the producer’s marginal cost. Another way to neutralize this distortion is to set an import tariff, which makes consumers pay the same "mark-up" for the imported variety as the one they pay for domestic varieties. Finally, a tax on exports makes exporting less attractive, so resources are reallocated toward domestic production and the quantity of each consumed variety rises.

To compare these policies with each other, note that it can be shown that while the "real" values, namely, cutoffs $x$ and $y$, masses of domestic producers and exporters, $M$ and $M_x$, and the quantity of the imported variety $z$, are the same in each case, the "nominal" values, namely, wage rate $w$, total revenues $R$, and price index $P$, can differ. In particular, letting "cs" denote the consumption subsidy case, "exp" denote the export tax case, and "m" denote the import tariff case, it can be shown that

$$w^{cs} = \frac{1}{\rho} w^{exp} = w^m, \quad R^{cs} = R^{exp} = \frac{1}{\rho} R^m, \quad P^{cs} = P^{exp} = \frac{1}{\rho} P^m.$$ 

First, note that the export tax leads to a lower wage compared with the consumption subsidy, but the price index and total revenues are the same. The intuition is that the export tax reduces the demand for labor, since exporting is not that attractive anymore and fewer resources are needed for exporting, and as a result the wage is lower in this case. However, price indices are the same since the price of the imported variety is the same, and the price of any domestic variety is low in both cases either because of the consumption subsidy, or because of lower wage in the export tax case. Total consumer income is the same since in one case the revenues from the export tax compensate for the low labor payments, and in the other case the higher wage compensates for losses due to financing of the consumption subsidy.

Second, the wages are the same in the case of a consumption subsidy and an import tariff, whereas the price index and revenues are higher in the latter case. The price index is higher with an import tariff because consumers have to pay a mark-up on both domestic and imported varieties. Revenues are higher thanks to the revenues from the import tariff, but the extra income exactly compensates for the higher prices.

In addition, Proposition 1 leads to the following straightforward conclusion:

**Corollary 1** In the presence of the optimal consumption subsidy, any trade policy results in welfare losses.

4 The Effects of Export Subsidies

In this section, we assume that the government has in place the optimal consumption subsidy (i.e., $\eta = \rho$) and explore how export subsidies affect the three components of the utility function in (13). Note that from Corollary 1, an export subsidy worsens the equilibrium outcome compared to the case with no export subsidy. Moreover, we prove the following result:

**Proposition 2** An introduction of an export subsidy decreases welfare: the higher is the subsidy, the lower is the welfare level.
To understand better why the increasing export subsidy causes a welfare reduction, we consider the three components of utility in (13). Before analyzing them, we first look at the effect of the export subsidy on the basic variables in the economy (Section 4.1). In Section 4.2 we show that the productivity index rises. In spite of this, welfare falls with the export subsidy because the other two components in (13) together fall and this more than compensates for the productivity increase. In Section 4.3 we analyze the behavior of the second and third components in more detail.

4.1 The equilibrium effects of export subsidies

First, note that from the (FE) condition, the cutoffs for domestic producers and exporters, \(x\) and \(y\), always move in the opposite directions in response to changes in the export subsidy \(s\). Moreover, as we prove in the Appendix, as \(s\) rises, \(y\) must fall, so \(x\) increases. In turn, from (10), \(M\) falls,

\[
M_e = \frac{\delta M}{1-G(x)} = \frac{\theta-1}{\sigma \theta f_e} \text{ remains constant, and } M_{exp} = M \frac{1-G(y)}{1-G(x)} = \frac{(\theta-1)\beta L_y y^{-\beta}}{\sigma \theta f_e} \text{ rises. Finally, it can be shown that the wage increases with } x, \text{ so that it increases with } s \text{ as well. The next Proposition formally states these results:
}

**Proposition 3** As the export subsidy increases, the productivity cutoff for domestic producers rises, the productivity cutoff for exporters falls, the wage rises, the mass of entrants remains unchanged, the mass of domestic producers falls, and the mass of exporters increases.

The only result that merits some additional comment here is that export subsidy does not affect the mass of entrants. In the case of the Pareto distribution, the same result can be reproduced in the framework used by Melitz (2003): a decline in transportation costs does not affect the mass of entrants.

4.2 The Effect of the Export Subsidy on Productivity

Equations (11) and (12) together with conditions (M) and (FE) allow us to write the productivity index as

\[
Q^P = \frac{\beta}{\sigma \theta (\beta - \sigma)} \frac{\delta f_e}{(\theta-1)\beta} \left[ f + f_{exp} \left( \frac{x}{y} \right)^{\beta-1} \right] x^{\beta+1}.
\]

Since \(\beta > \sigma > 1\), and \(x\) and \(\frac{x}{y}\) rise with \(s\), the productivity index rises as well. This leads to the following proposition:

**Proposition 4** The productivity index is an increasing function of the export subsidy.\(^{11}\)

This is precisely the effect of the export subsidy on productivity that we mentioned in the Introduction. An export subsidy raises the expected profits from exporting, leading to more entry, more severe competition, and a reallocation of labor from less to more productive firms, thus increasing productivity.

\(^{11}\)Note that the productivity index also rises with the export subsidy in the absence of the consumption subsidy.
4.3 The Effect of the Export Subsidy on the TOT and Variety Indices

Using Corollary 1 and Proposition 4 together, it is clear that the TOT and variety indices together must fall with $s$. Now we want to look at the behavior of these indices separately. It can be shown numerically that depending on the parameters, each index can rise or fall with $s$. Thus, it is impossible to make unambiguous predictions about the behavior of these two indices in general.

The intuition behind these results is the following. First, let us look at the TOT index. The export subsidy affects the terms of trade through two channels. The first is the intensive margin, i.e., the export subsidy allows the original exporters to increase the quantity they sell abroad, and this leads to the standard negative effect on the terms of trade. The second channel is along the extensive margin, as the export subsidy allows more firms to become exporters. As a result, the average productivity of exporters declines and this improves the TOT.

Now consider the variety index. Since the higher export subsidy results in the exit of the least efficient producers, the mass of domestic varieties falls. However, the amount of the imported variety rises. Thus, when the foreign demand is very small ($A$ is small) and, as a result, the economy imports too little of the foreign variety, an increase in the imported variety can more than compensate for welfare losses due to a fall in the domestic variety, as consumers value an increase in the quantity of the imported variety a lot. As a result, the variety index can fall.

5 Conclusion

Recent trade models with heterogenous firms suggest that export subsidies can indeed increase productivity by inducing a reallocation of labor from less to more productive firms. We have shown in this paper that, with an appropriate measure of productivity, this positive effect is in fact present, but is dominated by the negative effects of the export subsidy on the country’s terms of trade and variety. The main message that arises from our results is that an exclusive focus on productivity can be counterproductive: a broader analysis is necessary.

We have also shown that policy-makers have several options available to improve welfare in the economy we have considered, which is affected by a mark-up distortion: either a consumption subsidy equal to the size of mark-up, or an export tax or an import tariff at the right amounts can lead to the first best allocation in the economy.

References


\(^{12}\) The details can be found in the Appendix.

\(^{13}\) It can be shown that the price set by the original exporters $p_{exp} (\varphi) = \frac{w}{\varphi}$, Since $w/s$ falls with $s$, then this price falls as $s$ rises.

6 Appendix

6.1 Derivation of (TB) condition (formula (8) in the paper).

The trade balance condition can be written as \( p_z = \int_y \frac{r_{\exp}(\phi)}{s} M \mu(\phi) \, d\phi \), or

\[
RP^{\sigma-1} = \frac{1}{s} w \sigma M f_{\exp m_{\exp}} \text{, or } R = \left( \frac{1}{s} w \sigma M f_{\exp m_{\exp}} \right) P^{1-\sigma},
\]

where \( P^{1-\sigma} = 1 + \theta M \left( \frac{px}{\eta w} \right)^{\sigma-1} \) and \( R = w + T \), where \( T \) is

\[
T = - (s - 1) \int_y \frac{r_{\exp}(\phi)}{s} M \mu(\phi) \, d\phi - (1 - \eta) \int_x r_d(\phi) M \mu(\phi) \, d\phi, \text{ so that}
\]

\[
R = w - w \sigma M \left[ \frac{s - 1}{s} f_{\exp m_{\exp}} + (1 - \eta) f \right] = \left( \frac{1}{s} w \sigma M f_{\exp m_{\exp}} \right) \left[ 1 + \theta M \left( \frac{px}{\eta w} \right)^{\sigma-1} \right], \text{ or}
\]

\[
1 = \sigma \theta M \left[ \frac{s - 1}{s} f_{\exp m_{\exp}} + (1 - \eta) f \right] + \left( \frac{1}{s} \sigma \theta M f_{\exp m_{\exp}} \right) \left( 1 + \theta M \left( \frac{px}{\eta w} \right)^{\sigma-1} \right)
= \sigma \theta M \left[ f_{\exp m_{\exp}} + (1 - \eta) f \right] + \left( \frac{1}{s} \sigma \theta M f_{\exp m_{\exp}} \right) \theta M \left( \frac{px}{\eta w} \right)^{\sigma-1}
= \sigma \theta M \left[ f_{\exp m_{\exp}} \left( 1 + \frac{1}{s} \theta M \left( \frac{px}{\eta w} \right)^{\sigma-1} \right) + (1 - \eta) f \right].
\]

6.2 Curvature term in the variety index.

\[
Curvature = \frac{M(f_x q^\phi(\phi) \mu(\phi) d\phi)^{1/\rho}}{Q_d} \frac{M \theta^{1/\rho} f(\sigma - 1) x}{f(\sigma - 1) \frac{1}{\beta - (\sigma - 1)}} = \beta - \sigma \left( \frac{\beta}{\beta - (\sigma - 1)} \right)^{\frac{\sigma - 1}{\beta}}.
\]

Note that in the case of the Pareto distribution, \( G(\phi) = 1 - \left( \frac{b}{\phi} \right)^\beta \), \( \phi > b \), \( E(\phi) = \frac{b}{(\beta - 1)(\beta - 1)} \), and \( Var(\phi) = \frac{b^2}{(\beta - 2)(\beta - 1)^2} \). As a result, if \( \beta \) rises, the mean and dispersion fall, and if \( \beta \to \infty \), then \( E(\phi) \to b \), and \( Var(\phi) \to 0 \). In other words, an increase in \( \beta \) reduces heterogeneity among firms, and if \( \beta \to \infty \), all firms are identical. What happens with the curvature term in both cases? It rises with rising \( \beta \), since its derivative with respect to \( \beta \) is positive:

\[
\frac{\sigma}{\beta^2} \left( 1 + \frac{\sigma - 1}{\beta - (\sigma - 1)} \right)^{\frac{\sigma}{\beta^2} - 1} + \left( 1 - \frac{\sigma}{\beta} \right)^{\frac{\sigma}{\beta} - 1} = \left( 1 + \frac{\sigma - 1}{\beta - (\sigma - 1)} \right)^{\frac{\sigma}{\beta^2} - 1} \frac{\sigma}{\beta^2} \frac{1}{\beta - (\sigma - 1)} > 0,
\]

and it converges to 1 as \( \beta \to \infty \):

\[
\frac{\beta - \sigma}{\beta} \left( \frac{\beta}{\beta - (\sigma - 1)} \right)^{\frac{\sigma}{\beta^2}} = \left( 1 - \frac{\sigma}{\beta} \right)^{\frac{\sigma}{\beta^2} - 1} \to 1.
\]
Moreover, if $\sigma$ rises, the curvature falls \textbf{(given that $\sigma < \beta$)}:

$$\frac{\beta - \sigma}{\beta} \left( \frac{\beta}{\beta - (\sigma - 1)} \right)^{\frac{\sigma}{\beta - (\sigma - 1)}} = \left( 1 - \frac{1}{\beta - (\sigma - 1)} \right) \left( \frac{\beta}{\beta - (\sigma - 1)} \right)^{\frac{1}{\beta - (\sigma - 1)}},$$

where the first component falls as well as the second component. To see this, note that

$$\ln \left[ \left( \frac{\beta}{\beta - (\sigma - 1)} \right)^{\frac{1}{\beta - (\sigma - 1)}} \right] = \frac{1}{(\sigma - 1)} \ln \left( \frac{\beta}{\beta - (\sigma - 1)} \right)$$

falls with $\sigma$.

What happens if $\sigma \to \infty$? In our model, we have a restriction $\beta > \sigma$. Thus, $\sigma$ is always bounded from above. And if $\sigma \to \infty$, it means that $\beta \to \infty$, so that the curvature term $\to 1$.

### 6.3 Proof of Proposition 1

#### 6.3.1 Social Planner’ Problem and Its Solution.

Let $q(\varphi)$ be the quantity consumed of a good with productivity index $\varphi$ and let $Q(\varphi)$ be the quantity produced. It can be easily checked that if all varieties $v \in \Omega$ are produced, then it must be that $q(v)$ with $v \in \Omega$ maximizes utility $\int_{v \in \Omega} q(v)^{\rho} dv$ s.t. $\int_{v \in \Omega} [q(v)/\varphi(v)] dv = K$. This leads to the F.O.C. of $q(v)/q(v') = [\varphi(v)/\varphi(v')]^{\rho}$. On the other hand, if all varieties $v \in \Omega$ are exported, then it must be that $Q(v) - q(v)$ with $v \in \Omega$ maximizes export revenue $\int_{v \in \Omega} a(Q(v) - q(v))^{\rho} dv$ s.t. $\int_{v \in \Omega} [(Q(v) - q(v))/\varphi(v)] dv = J$. This leads to F.O.C. $Q(v)/Q(v') = [\varphi(v)/\varphi(v')]^{\rho}$. Combining both results, we also obtain $Q(v)/Q(v') = [\varphi(v)/\varphi(v')]^{\rho}$. Thus, an allocation in the social planner’s problem would necessarily have $q(\varphi) = \phi \varphi^\sigma$ and $Q(\varphi) - q(\varphi) = \alpha \varphi^\sigma$, with $\alpha, \phi > 0$ (for the appropriate levels of $\varphi$). Moreover, if a variety $v$ with $\varphi(v)$ is consumed (exported), then all varieties with $\varphi > \varphi(v)$ must be consumed (exported). Thus, we will look for an allocation that maximizes welfare, has no goods produced for $\varphi < x$, exports only for goods with $\varphi > y$, subject to full employment and balanced trade. In particular, the social planner maximizes

$$\max_{x,y,z,M} \left\{ z^\rho + \int_x^\infty q(\varphi)^{\rho} M \mu(\varphi) d\varphi \right\} \quad \text{s.t.}$$

$$\int_x^\infty (f + q(\varphi)/\varphi) M \mu(\varphi) d\varphi + \int_y^\infty \left( f_{\text{exp}} + \frac{Q(\varphi) - q(\varphi)}{\varphi} \right) M \mu(\varphi) d\varphi + \frac{M \delta f_0}{1 - G(x)} = 1,$$

$$\int_y^\infty p_{\text{exp}} (Q(\varphi) - q(\varphi)) M \mu(\varphi) d\varphi - p_z z = p_z z.$$

In addition, export revenue is $(Q - q) p_{\text{exp}}$. But $Q - q = A p_{\text{exp}}^{-\sigma}$ implies $a (Q - q)^{-1/\sigma} = p_{\text{exp}}$, where $a = A^{1/\sigma}$. Hence, export revenues are $a (Q - q)^{-1/\sigma} = a (Q - q)^{\rho}$. Also, recall that $p_z = 1$ and we assume the Pareto productivity distribution, $G(\varphi) = 1 - \left( \frac{b}{\varphi} \right)^{\beta}$. Thus, the social planner problem can be rewritten as

$$\max_{x,y,z,M,\alpha,\phi} \left\{ z^\rho + M \phi^\rho x^{\sigma-1} \right\}$$
\[ Mf + M\theta \phi x^{\sigma - 1} + Mm_{\text{exp}}f_{\text{exp}} + Mm_{\text{exp}}\theta \alpha y^{\sigma - 1} + \frac{M\delta f_e}{b^\beta} x^\beta = 1, \]
\[ Mm_{\text{exp}}a\alpha^\rho y^{\sigma - 1} = z, \]

where \( m_{\text{exp}} = \frac{1-G(y)}{1-G(x)} = \left( \frac{x}{y} \right)^\beta \). The Lagrangian is then:

\[
\mathcal{L} = z^\rho + M\phi^\rho \theta x^{\sigma - 1} - \lambda \left( Mf + M\theta \phi x^{\sigma - 1} + Mm_{\text{exp}}f_{\text{exp}} + Mm_{\text{exp}}\theta \alpha y^{\sigma - 1} + \frac{M\delta f_e}{b^\beta} x^\beta - 1 \right) + \zeta \left( Mm_{\text{exp}}a\alpha^\rho y^{\sigma - 1} - z \right). \tag{14}
\]

This must be maximized with respect to \( z, x, y, \alpha, M \). Letting \( h(v) = \frac{g(v)}{1-G(v)} \), then:

\[
(z): \quad \frac{\partial \mathcal{L}}{\partial z} = \rho z^{\sigma - 1} - \zeta = 0, \tag{15}
\]
\[(x): \quad \frac{\partial \mathcal{L}}{\partial x} = M\phi^\rho \theta (\sigma - 1) x^{\sigma - 2} - \lambda M (\sigma - 1) \theta \phi x^{\sigma - 2} - \lambda M m_{\text{exp}}f_{\text{exp}}h(x) - \lambda M m_{\text{exp}}h(x) \theta \alpha y^{\sigma - 1} - \lambda M \delta f_e \frac{h(x)}{1-G(x)} + \zeta M m_{\text{exp}}h(x) a\alpha^\rho y^{\sigma - 1} = 0, \tag{16}
\]
\[(y): \quad \frac{\partial \mathcal{L}}{\partial y} = \lambda M f_{\text{exp}}m_{\text{exp}}h(y) + \lambda M m_{\text{exp}}h(y) \alpha \theta y^{\sigma - 1} - \lambda M m_{\text{exp}}\alpha \theta (\sigma - 1) y^{\sigma - 2} - \zeta M m_{\text{exp}}h(y) a\alpha^\rho y^{\sigma - 1} + \zeta M m_{\text{exp}}a\alpha^\rho (\sigma - 1) y^{\sigma - 2} = 0, \tag{17}
\]
\[(\phi): \quad \frac{\partial \mathcal{L}}{\partial \phi} = M \rho \phi^\rho - \lambda M \theta \phi x^{\sigma - 1} = 0, \tag{18}
\]
\[(\alpha): \quad \frac{\partial \mathcal{L}}{\partial \alpha} = -\lambda M m_{\text{exp}}\theta y^{\sigma - 1} + \zeta M m_{\text{exp}}a\alpha^\rho y^{\sigma - 1} = 0, \tag{19}
\]
\[(M): \quad \frac{\partial \mathcal{L}}{\partial M} = \phi^\rho \theta x^{\sigma - 1} - \lambda f - \lambda M \phi x^{\sigma - 1} - \lambda f_{\text{exp}}m_{\text{exp}} - \lambda m_{\text{exp}}\alpha \theta y^{\sigma - 1} - \lambda \frac{\delta f_e}{1-G(x)} + \zeta m_{\text{exp}}a\alpha^\rho y^{\sigma - 1} = 0. \tag{20}
\]

Note that \( \theta (\sigma - 1)v^{\sigma - 2} = (\theta - 1)h(v)v^{\sigma - 1} \), hence, we have 8 equations with 8 unknown variables:

\[
(z): \quad \rho z^{\sigma - 1} - \zeta = 0,
\]
\[(x): \quad \phi (\theta - 1) x^{\sigma - 1} (\phi^\rho - \lambda) - \lambda m_{\text{exp}}f_{\text{exp}} - \lambda m_{\text{exp}}\theta \alpha y^{\sigma - 1} - \lambda \delta f_e \frac{1}{1-G(x)} + \zeta m_{\text{exp}}a\alpha^\rho y^{\sigma - 1} = 0,
\]
\[(y): \quad \lambda f_{\text{exp}} + \lambda \alpha \theta y^{\sigma - 1} - \lambda \alpha (\theta - 1) y^{\sigma - 1} - \zeta a\alpha^\rho y^{\sigma - 1} + \zeta a\alpha^\rho (\theta - 1) y^{\sigma - 1} = 0,
\]
\[(\phi): \quad \rho \phi^\rho - \lambda = 0,
\]
\[(\alpha): \quad \zeta a\alpha^\rho - \lambda = 0,
\]
\[(M): \quad \phi^\rho \theta x^{\sigma - 1} - \lambda f - \lambda \phi x^{\sigma - 1} - \lambda f_{\text{exp}}m_{\text{exp}} - \lambda m_{\text{exp}}\alpha \theta y^{\sigma - 1} - \lambda \frac{\delta f_e}{1-G(x)} + \zeta m_{\text{exp}}a\alpha^\rho y^{\sigma - 1} = 0,
\]
\[(FE): \quad 1 = Mf + M\theta \phi x^{\sigma - 1} + Mf_{\text{exp}}m_{\text{exp}} + Mm_{\text{exp}}\alpha \theta y^{\sigma - 1} + \frac{M\delta f_e}{1-G(x)},
\]
\[(TB): \quad z = Mm_{\text{exp}}a\alpha^\rho y^{\sigma - 1}.
\]
Moreover, subtracting (M) from (x) gives:
\[
\left(\phi (\theta - 1) x^{\sigma-1} (\phi^\rho - 1 - \lambda) - \lambda m_{\text{exp}} f_{\text{exp}} - \lambda m_{\text{exp}} \theta \alpha y^{\sigma-1} - \lambda \delta f_e \frac{1}{1 - G(x)} + \zeta m_{\text{exp}} \alpha \rho \theta y^{\sigma-1}\right)
- \left(\phi^\rho \theta x^{\sigma-1} - \lambda f - \lambda \theta \phi x^{\sigma-1} - \lambda f_{\text{exp}} m_{\text{exp}} - \lambda m_{\text{exp}} \alpha \theta y^{\sigma-1} - \lambda \frac{\delta f_e}{1 - G(x)} + \zeta m_{\text{exp}} \alpha \rho \theta y^{\sigma-1}\right)
= -\phi^\rho x^{\sigma-1} + \lambda \phi x^{\sigma-1} + \lambda f = 0, \quad \text{or} \quad \phi x^{\sigma-1} - \phi^\rho x^{\sigma-1} = \lambda f.
\]

But from (\phi), \( \lambda = \rho \phi^\rho - 1 \). Thus, (x) condition is \( \phi x^{\sigma-1} = \sigma \rho f \).

Similarly, using (\alpha) in (y) gives (y) condition: \( \alpha y^{\sigma-1} = \sigma \rho f_{\text{exp}} \).

In addition, using new equations (x) and (y) and equations (\phi) and (\alpha), we derive a new (M) condition:
\[
\frac{\delta f_e}{1 - G(x)} = \theta (\sigma \rho f) \left(\frac{1}{\rho} - 1\right) - f - f_{\text{exp}} m_{\text{exp}} + \theta m_{\text{exp}} (\sigma \rho f_{\text{exp}}) \left(\frac{1}{\rho} - 1\right)
= \theta f - f - f_{\text{exp}} m_{\text{exp}} + \theta m_{\text{exp}} f_{\text{exp}} = (\theta - 1) (f + f_{\text{exp}}),
\]
and a new full employment (FE) condition:
\[
1 = M \left[ f + \theta \phi x^{\sigma-1} + f_{\text{exp}} m_{\text{exp}} + m_{\text{exp}} \alpha \theta y^{\sigma-1} + \frac{\delta f_e}{1 - G(x)} \right] = M[f + \theta \rho f]
+ f_{\text{exp}} m_{\text{exp}} + m_{\text{exp}} \theta \rho f_{\text{exp}} + (\theta - 1) (f + f_{\text{exp}}) = \sigma \theta M (f + m_{\text{exp}} f_{\text{exp}}).
\]

Thus, we have the following system of F.O.C.s in the social planner’s problem:

\[
\begin{align*}
(z) & : \quad \rho z^{\rho - 1} = \zeta, \quad (21) \\
(x) & : \quad \phi x^{\rho - 1} = \sigma \rho f, \quad (22) \\
(y) & : \quad \alpha y^{\rho - 1} = \sigma \rho f_{\text{exp}}, \quad (23) \\
(\phi) & : \quad \rho \phi^{\rho - 1} = \lambda, \quad (24) \\
(\alpha) & : \quad \zeta \alpha \rho \phi^{\rho - 1} = \lambda, \quad (25) \\
(M) & : \quad \frac{\delta f_e}{1 - G(x)} = (\theta - 1) (f + m_{\text{exp}} f_{\text{exp}}), \quad (26) \\
(FE) & : \quad 1 = \sigma \theta M (f + m_{\text{exp}} f_{\text{exp}}), \quad (27) \\
(TB) & : \quad z = M m_{\text{exp}} \alpha \rho \theta y^{\rho - 1}. \quad (28)
\end{align*}
\]

**Uniqueness of the Solution** To prove the uniqueness of the solution of the system above, first, note that it can be easily shown that if there are 2 solutions, \((x_1, y_1, z_1, \phi_1, \alpha_1, \lambda_1, \zeta_1, M_1)\) and \((x_2, y_2, z_2, \phi_2, \alpha_2, \lambda_2, \zeta_2, M_2)\), and both solutions have at least one common component (for example, \( \zeta_1 = \zeta_2 \)), then these solutions coincide. We will prove that there is a unique \( x \), which solves the system. Then, if the solution exists, it is unique. To do this, we will exclude all other variables from the system above till we have one equation with one unknown variable, which has a unique solution.

**Step 1.** Let’s exclude \( M \). From (TB), \( M = z / (m_{\text{exp}} \alpha \rho \theta y^{\rho - 1}) \). \( M \) is used only in (FE), which
together with \((y)\) and \((\alpha)\) can be written as:

\[(FE): \quad 1 = \sigma \frac{(f + m_{\text{exp}}f_{\text{exp}})}{m_{\text{exp}}a\alpha^\rho y^\sigma - 1} = \sigma \frac{(f + m_{\text{exp}}f_{\text{exp}})}{m_{\text{exp}}a\alpha^\rho - 1} \frac{f_{\text{exp}}}{f_{\text{exp}}} = \frac{z (f + m_{\text{exp}}f_{\text{exp}})}{m_{\text{exp}}\lambda / \zeta} = \frac{z (f + m_{\text{exp}}f_{\text{exp}})}{m_{\text{exp}}f_{\text{exp}}}.
\]

Thus, we now have 7 unknown variables and 7 equations.

**Step 2.** Let’s exclude \(\lambda\) and \(\zeta\). From \((z)\) and \((\phi)\): \(\zeta = \rho z^\rho - 1\) and \(\lambda = \rho \phi^\rho - 1\). Thus, we have

\[(x) : \quad \phi x^{\phi - 1} = (\sigma - 1) f,\]
\[(y) : \quad \alpha y^{\alpha - 1} = (\sigma - 1) f_{\text{exp}},\]
\[(\alpha) : \quad z^{\alpha - 1} a\alpha x^{\rho - 1} = \phi^\rho - 1,\]
\[(M) : \quad \frac{\delta f_e}{1 - G(x)} = (\theta - 1) (f + m_{\text{exp}}f_{\text{exp}}),\]
\[(FE) : \quad 1 = \left(\frac{z}{\phi}\right)^{\rho - 1} \frac{z (f + m_{\text{exp}}f_{\text{exp}})}{m_{\text{exp}}f_{\text{exp}}}.
\]

**Step 3.** Let’s exclude \(z\). From \((\alpha)\): \(z = \frac{\phi}{\alpha} (a\rho)_{\frac{1}{\rho - 1}} = \frac{\phi}{\alpha} (a\rho)^{-\sigma}\), so we have now

\[(x) : \quad \phi x^{\phi - 1} = (\sigma - 1) f,\]
\[(y) : \quad \alpha y^{\alpha - 1} = (\sigma - 1) f_{\text{exp}},\]
\[(M) : \quad \frac{\delta f_e}{1 - G(x)} = (\theta - 1) (f + m_{\text{exp}}f_{\text{exp}}),\]
\[(FE) : \quad 1 = \left(\frac{1}{\alpha} (a\rho)^{\sigma}\right)^{\rho - 1} \frac{\phi}{\alpha} (a\rho)^\sigma \frac{(f + m_{\text{exp}}f_{\text{exp}})}{m_{\text{exp}}f_{\text{exp}}} = \alpha^{1 - \rho} (a\rho)^{-\sigma \rho} \frac{\phi}{\alpha} (f + m_{\text{exp}}f_{\text{exp}}).
\]

**Step 4.** Let’s exclude \(\phi\) and \(\alpha\). From \((x)\) and \((y)\): \(\phi = \frac{(\sigma - 1) f}{x^{\rho - 1}}\) and \(\alpha = \frac{(\sigma - 1) f_{\text{exp}}}{y^{\rho - 1}}\). Thus,

\[(M) : \quad x = \left(\frac{b^\beta (\theta - 1)}{\delta f_e} (f + m_{\text{exp}}f_{\text{exp}})^{1/\beta}\right),\]
\[(FE) : \quad 1 = (\sigma - 1) f_{\text{exp}}^{1 - \rho} (a\rho)^{-\sigma \rho} \frac{f}{f_{\text{exp}}} y^{(\sigma - 1)(1 - \rho)} (\frac{y}{x})^{\sigma - 1} \frac{(f + m_{\text{exp}}f_{\text{exp}})}{m_{\text{exp}}f_{\text{exp}}}.
\]

**Final Step.** Note that

\[y^{(\sigma - 1)(1 - \rho)} (\frac{y}{x})^{\sigma - 1} = x^{-\rho} \left(\frac{x}{y}\right)^{\rho} y^{\sigma - 1} = \left(b^\beta (\theta - 1) \frac{\delta f_e}{f_{\text{exp}}} (f + m_{\text{exp}}f_{\text{exp}})^{-\frac{\sigma}{\rho}} (m_{\text{exp}})^{\frac{\rho - \sigma}{\rho}}
\]

and equation \((FE)\) can be rewritten as

\[(m_{\text{exp}})^{-\frac{\sigma - 1}{\rho}} \left(\frac{f}{m_{\text{exp}}} + f_{\text{exp}}\right)^{1 - \frac{\sigma}{\rho}} \text{ = Some exogenously given constant.}
\]

Note that since \(\beta > \sigma > 1 > \rho\), then the left-hand side of equation above is a decreasing function of \(m_{\text{exp}}\). Thus, the equation above has a unique solution \(m_{\text{exp}}\). But from \((M)\), it follows, that \(x\) is also unique! Therefore, we proved that if the solution of the system of F.O.C.s exists, it is unique.
**Sufficiency of F.O.C.s**  Let us rewrite the Lagrangian as

\[
\mathcal{L} = U + H, \quad \text{where } U = z^\phi + M \phi \theta x^\sigma - 1 \quad \text{and}
\]

\[
H = -\lambda \left( M f + M \phi x^\sigma - 1 + M m_{\exp} f_{\exp} + M m_{\exp} \theta \alpha y^\sigma - 1 + \frac{M \delta f_x}{b^\beta} x^\beta - 1 \right) + \zeta (M m_{\exp} a \alpha \theta y^\sigma - 1 - z). 
\]

To prove the sufficiency of the first order conditions of the social planner’s problem described above, we need to show that for any vector \( \bar{\chi} \) such that

\[
\bar{\chi} \neq \bar{0} \quad \text{and} \quad \nabla \bar{h}(\text{solution}) \bar{\chi} = \bar{0},
\]

where \( \nabla \bar{h}(\text{solution}) \) is a matrix of the first derivatives of the vector of restrictions in our problem,

\[
\bar{h} = \left( M f + M \phi x^\sigma - 1 + M m_{\exp} f_{\exp} + M m_{\exp} \theta \alpha y^\sigma - 1 + \frac{M \delta f_x}{b^\beta} x^\beta - 1 \right),
\]

with respect to \( \xi' = (z, x, y, \phi, \alpha, M) \), evaluated at the solution point \( \xi^* \), we have

\[
\bar{\chi}' \cdot \mathcal{L}_{\xi \xi}(\xi^*) \bar{\chi} < 0,
\]

where \( \mathcal{L}_{\xi \xi}(\xi^*) \) is the matrix of second derivatives of the Lagrangian with respect to \( \xi' = (x, y, z, \phi, \alpha, M) \), evaluated at the solution point \( \xi^* \). (See, for example, Theorem 3.3.2, p. 214 in G. Giorgi, A. Guerraggio and J. Thierfelder, "Mathematics of Optimization: Smooth and Nonsmooth Case", Elsevier B.V., 2004, which states that if there exist such \( \lambda \) and \( \zeta \), for which the conditions above are satisfied, then the solution we found is a point of global maximum of the objective function subject to our restrictions. And we found such \( \lambda \) and \( \zeta \) already.) In order to do this, we can show that the matrix of the second derivatives can be written as (see the proof below):

\[
\mathcal{L}_{\xi \xi}(\xi^*) = \begin{pmatrix}
  a_{11} & 0 & 0 & 0 & 0 & 0 \\
  0 & a_{22} & 0 & 0 & 0 & 0 \\
  0 & 0 & a_{33} & 0 & 0 & 0 \\
  0 & 0 & 0 & a_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & a_{55} & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

where \( a_{ii} < 0 \) for any \( i \neq 6 \). Thus, \( \bar{\chi}' \cdot \mathcal{L}_{\xi \xi}(\xi^*) \bar{\chi} = \sum_{i=1}^5 a_{ii} \chi_i^2 \leq 0 \), so we need to show that \( \bar{\chi}' \cdot \mathcal{L}_{\xi \xi}(\xi^*) \bar{\chi} \neq 0 \). Note that the only way \( \bar{\chi}' \cdot \mathcal{L}_{\xi \xi}(\xi^*) \bar{\chi} = 0 \) is if \( \bar{\chi}' = (0, 0, 0, 0, 0, \chi_6) \) and \( \chi_6 \neq 0 \). However, in this case (29) is violated since \( \nabla \bar{h}(\text{solution}) \bar{\chi} = \bar{0} \) implies

\[
\left[ \frac{\partial}{\partial M} \left( M f + M \phi x^\sigma - 1 + M m_{\exp} f_{\exp} + M m_{\exp} \theta \alpha y^\sigma - 1 + \frac{M \delta f_x}{b^\beta} x^\beta - 1 \right) \right] \chi_6 = 0,
\]

\[
\left[ \frac{\partial}{\partial M} \left( M m_{\exp} a \alpha \theta y^\sigma - 1 - z \right) \right] \chi_6 = 0, \quad \text{or}
\]

\[
\left( f + \phi x^\sigma - 1 + m_{\exp} f_{\exp} + m_{\exp} \theta \alpha y^\sigma - 1 + \frac{\delta f_x}{b^\beta} x^\beta \right) \chi_6 = 0, \quad \text{and} \quad (m_{\exp} a \alpha \theta y^\sigma - 1) \chi_6 = 0,
\]

which is clearly impossible, since in the second equation above, \( m_{\exp} a \alpha \theta y^\sigma - 1 \neq 0 \) and \( \chi_6 \neq 0 \). Therefore, \( \bar{\chi}' \) cannot be equal to \( (0, 0, 0, 0, 0, \chi_6) \), and we proved that \( \bar{\chi}' \cdot \mathcal{L}_{\xi \xi}(\xi^*) \bar{\chi} < 0 \).

**The derivation of** \( \mathcal{L}_{\xi \xi}(\xi^*) \). Let denote the elements of \( \mathcal{L}_{\xi \xi}(\xi^*) \) by \( [a_{ij}]_{i,j=1,\ldots,6} \).
**Diagonal elements.** First, note that

\[ a_{11} = \frac{\partial^2 \mathcal{L}}{\partial z^2} = \rho (\rho - 1) z^{\rho - 2} < 0. \]

What we do in cases \( i = 2, 3 \) is we take the derivatives and use the property that \( \frac{\partial (x^n)}{\partial x} = \frac{n}{x} (x^n) \). Then we use the corresponding condition to simplify the expression for the derivative and compare it with 0. For example, since \( \frac{\partial (m_{\exp} h (x))}{\partial x} = \frac{\partial (\beta x^{\beta-1} / y^\beta)}{\partial x} = \frac{\delta-1}{x} m_{\exp} h (x) \) and \( \frac{\partial (h (x) / 1-G(x))}{\partial x} = \frac{\beta-1}{x} h (x) \),

\[
a_{22} = \frac{\partial^2 \mathcal{L}}{\partial x^2} = M \phi^\rho \theta (\sigma - 1) (\sigma - 2) x^{\sigma-3} - \lambda \phi \theta (\sigma - 1) (\sigma - 2) x^{\sigma-3} \\
- \frac{\beta-1}{x} \left( \lambda M m_{\exp} f_{\exp} h (x) - \lambda M m_{\exp} h (x) \theta \alpha y^{\sigma-1} \lambda M \delta f_e \frac{h (x)}{1-G (x)} + \zeta M m_{\exp} h (x) a \alpha^\rho \theta y^{\sigma-1} \right). \\
\]

We can use condition (x) to rewrite it as

\[
a_{22} = M \phi^\rho \theta (\sigma - 1) (\sigma - 2) x^{\sigma-3} - \lambda \phi \theta (\sigma - 1) (\sigma - 2) x^{\sigma-3} - \frac{\beta-1}{x} \left[ \phi^\rho \theta (\sigma - 1) x^{\sigma-2} - \lambda \phi \theta (\sigma - 1) x^{\sigma-2} \right] \\
= M \phi \theta (\sigma - 1) x^{\sigma-3} \left[ \phi^{\rho-1} - \lambda \right] \left( \sigma - 1 - \beta \right). \\
\]

Note that since \( \rho \phi^{\rho-1} = \lambda \) and \( \rho < 1 \), then \( \phi^{\rho-1} - \lambda > 0 \), while \( (\sigma - 1) - \beta < 0 \). Thus, \( a_{22} < 0 \).

Similarly, it can be shown that

\[ a_{33} = \frac{\partial^2 \mathcal{L}}{\partial y^2} = - (\sigma - 1) M \lambda f_{\exp} m_{\exp} h (y) \frac{1}{y} < 0, \]

\[ a_{44} = \frac{\partial^2 \mathcal{L}}{\partial \phi^2} = (\rho - 1) \phi^{\rho-2} M \rho \theta x^{\sigma-1} < 0, \]

\[ a_{55} = \frac{\partial^2 \mathcal{L}}{\partial \alpha^2} = (\rho - 1) \zeta M m_{\exp} a \rho \phi^{\rho-2} y^{\sigma-1} < 0, \]

\[ a_{66} = \frac{\partial^2 \mathcal{L}}{\partial M^2} = \frac{\partial}{\partial M} \left( \frac{\partial \mathcal{L}}{\partial M} \right) = 0. \]

**Off-Diagonal elements.** To derive the off diagonal elements, we use Young’s theorem. As a result,

\[ a_{12} = a_{21} = \frac{\partial^2 \mathcal{L}}{\partial z \partial x} = \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial z} \right) = \frac{\partial}{\partial x} (\rho z^{\rho-1} - \zeta) = 0, \]

and similarly,

\[ a_{13} = a_{31} = a_{14} = a_{41} = a_{15} = a_{51} = a_{16} = a_{61} = 0. \]

\[ a_{23} = a_{32} = \frac{\partial^2 \mathcal{L}}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial y} \right) = \frac{\beta}{x} \left( \frac{\partial \mathcal{L}}{\partial y} \right) = 0, \]
since \( \frac{\partial m_{\text{exp}}}{\partial x} = \beta x^{\beta-1} y^{-\beta} = \frac{\beta}{x} m_{\text{exp}} \). Similarly,

\[
\begin{align*}
a_{24} &= a_{42} = \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial \phi} \right) = \frac{\sigma - 1}{x} \left( \frac{\partial \mathcal{L}}{\partial \phi} \right) = 0, \\
a_{25} &= a_{52} = \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial \alpha} \right) = \beta \left( \frac{\partial \mathcal{L}}{\partial \alpha} \right) = 0, \\
a_{26} &= a_{62} = \frac{\partial^2 \mathcal{L}}{\partial M \partial x} = \frac{\partial}{\partial M} \left( \frac{\partial \mathcal{L}}{\partial x} \right) = \frac{1}{M} \left( \frac{\partial \mathcal{L}}{\partial x} \right) = 0.
\end{align*}
\]

Using the same logic, it can be shown that all off-diagonal elements of the matrix are zeros.

6.3.2 The Consumption Subsidy

The Optimal Value of Consumption Subsidy. In order to derive the equilibrium conditions in the presence of the consumption subsidy \( \eta \), it is enough to rewrite the equilibrium conditions derived in Section 2 by setting \( s = 1 \):

\[
\begin{align*}
\text{(EXP) condition} & \quad A \left( \frac{py}{w} \right)^{\sigma - 1} = \sigma w f_{\text{exp}}, \\
\text{(FE) condition} & \quad (\theta - 1) x^{-\beta} [f + m_{\text{exp}} f_{\text{exp}}] = \frac{\delta f_e}{b^\beta}, \\
\text{(TB) condition} & \quad 1 = \sigma \theta M \left[ f_{\text{exp}} m_{\text{exp}} \left( 1 + \theta M \left( \frac{px}{\eta w} \right)^{\sigma - 1} \right) + (1 - \eta) f \right], \\
\text{(M) condition} & \quad M = \frac{1}{\sigma \theta (f + m_{\text{exp}} f_{\text{exp}})} = \frac{(\theta - 1) b^\beta}{\sigma \theta \delta f_e} x^{-\beta}.
\end{align*}
\]

Now we are ready to prove that a consumption subsidy equal to the size of the mark-up (i.e., \( \eta = \rho \)) results in the maximal level of welfare.

Proof. PART 1. First, we prove that when \( \eta \) rises, \( y \) and \( w \) must fall and \( x \) must rise. Note that

\[
\begin{align*}
\text{(EXP):} \quad w^\sigma = \frac{A \rho^{\sigma - 1}}{\sigma f_{\text{exp}}} y^{\sigma - 1},
\end{align*}
\]

thus, \( y \) and \( w \) must move in the same direction. Moreover, from the (FE) condition, \( x \) and \( y \) move in the opposite direction. Then,

\[
\begin{align*}
\text{(TB) + (M)} & \Rightarrow 1 = \frac{1}{(f + m_{\text{exp}} f_{\text{exp}})} \left[ f_{\text{exp}} m_{\text{exp}} \left( 1 + \theta M \left( \frac{px}{\eta w} \right)^{\sigma - 1} \right) + (1 - \eta) f \right], \\
1 & = 1 + \frac{1}{(f + m_{\text{exp}} f_{\text{exp}})} \left[ f_{\text{exp}} m_{\text{exp}} \left( \frac{px}{\sigma f_{\text{exp}} (f + m_{\text{exp}} f_{\text{exp}})} \left( \frac{px}{\eta w} \right)^{\sigma - 1} \right) - \eta f \right], \\
& = \frac{f_{\text{exp}} m_{\text{exp}}}{\sigma (f + m_{\text{exp}} f_{\text{exp}})} \left( \frac{px}{\eta w} \right)^{\sigma - 1} = \eta f.
\end{align*}
\]

Finally,

\[
\begin{align*}
\left( \frac{\rho}{w} \right)^{1-\sigma} \left( 1 + \frac{f}{f_{\text{exp}}} \left( \frac{y}{x} \right)^\beta \right) = \frac{1}{\sigma \eta f} \left( \frac{\eta}{x} \right)^{1-\sigma},
\end{align*}
\]

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i.e., if \( \eta \) rises, the left-hand side of equation above should fall. Now, assume that if \( \eta \) rises, then \( y \) rises and \( x \) falls. Then

from (34): \[
    w = \left( \frac{A \sigma^{-1}}{\sigma f_{exp}} y^{\sigma-1} \right)^{1/\sigma} \text{ must rise;}
\]

from (36): \[
    w = \left( \frac{1}{\sigma \eta f} \left( \frac{\rho \eta}{\eta} \right)^{\sigma-1} \left( 1 + f \frac{\rho \eta}{f_{exp}} \left( \frac{\rho \eta}{\eta} \right)^{\beta} \right) \right)^{1/(\sigma-1)} \text{ must fall.}
\]

Our assumption led to the contradiction, thus, as \( \eta \) rises, \( y \) and \( w \) must fall and \( x \) must increase.

**PART 2.** From the (TB) condition, \( z = R P^{\sigma-1} = \sigma f_{exp} w \theta M_{exp} \), and

\[
    \int_{x}^{\infty} q(\varphi) \mu(\varphi) d\varphi = M [f(\sigma - 1)]^{\rho} x^{\beta - \rho(\sigma - 1)} \int_{x}^{\infty} \varphi^{\rho - \beta - 1} d\varphi = M \theta [f(\sigma - 1) x]^{\rho},
\]

so that

\[
    U^\rho = z^\rho + \int_{x}^{\infty} q(\varphi) \mu(\varphi) d\varphi = (\sigma f_{exp} w \theta M_{exp})^\rho + M \theta [f(\sigma - 1) x]^{\rho}.
\]

We want to show that

\[
    U^\rho = M x^\rho \left[ \left( \frac{w}{x} \sigma f_{exp} \theta \left( \frac{x}{y} \right)^{\beta} \right)^{\rho} M^{\rho-1} + \theta [f(\sigma - 1)]^{\rho} \right] \tag{37}
\]

is decreasing as \( \eta \) increases. To do this, we will show that

\[
    U^\rho = d(\eta) h(\eta),
\]

where \( d(\eta) = M x^\rho \) falls faster than \( h(\eta) = \left( \frac{w}{x} \sigma f_{exp} \theta \left( \frac{x}{y} \right)^{\beta} \right)^{\rho} M^{\rho-1} + \theta [f(\sigma - 1)]^{\rho} \) rises, if \( \eta > \rho \), and vice versa if \( \eta < \rho \). In other words, the utility is maximal at \( \eta = \rho \).

First, we can use the fact that \( x \) rises as \( \eta \) rises to rewrite both functions in terms of \( x \) and \( y(x) \), where \( y(x) \) is a decreasing function of \( x \), and look at the behavior of \( d(x) \) and \( h(x) \) as functions of \( x \). Then we can compare the elasticities of these two functions and show that \( \varepsilon_d < 0 < \varepsilon_h \). Moreover, if \( \eta > \rho \) (\( \eta < \rho \)), then \( |\varepsilon_d| > |\varepsilon_h| \), so that \( d(x) \) falls faster (slower) than \( h(x) \) rises, and \( U = d(x) h(x) \) falls (rises) as a result.

**Step 1.**

\[
    d(x) = M x^\rho = \left[ \frac{(\theta - 1) b^\beta}{\sigma \theta f_{exp}} \right] (x)^{\rho - \beta}.
\]

Since \( \rho < 1 < \beta \), \( d(x) \) is decreasing in \( x \) and

\[
    \varepsilon_d = \rho - \beta < 0. \tag{38}
\]

**Step 2.** \[
    h(x) = \left( \frac{w}{y} \right)^{\rho} (\sigma f_{exp} \theta)^{\rho} \left( \frac{x}{y} \right)^{(\beta - 1)\rho} M^{\rho-1} + \theta [f(\sigma - 1)]^{\rho}.
\]

Note that from the (EXP)
and (M) conditions,

\[ \frac{w}{y} = \left( \frac{A \rho^{\sigma-1}}{\sigma f_{\text{exp}}} \right)^{1/\sigma} \frac{1}{y^{1/\sigma}} \quad \text{and} \quad M = \frac{(\theta - 1) b^\beta}{\sigma \theta \delta f_e} x^{-\beta}. \]

Thus,

\[ h(x) = \gamma + \kappa(x), \quad \text{where} \]

\[ \gamma = \theta [f(\sigma - 1)]^\rho \quad \text{is exogenously defined constant,} \]

\[ \kappa(x) = \left( \frac{A \rho^{\sigma-1}}{\sigma f_{\text{exp}}} \right)^{1/\sigma} \frac{1}{y^{1/\sigma}} \left( \frac{\sigma f_{\text{exp}} \theta}{y} \right)^{\rho} \left( \frac{x}{y} \right)^{(\beta-1)\rho} \left( \frac{(\theta - 1) b^\beta}{\sigma \theta \delta f_e} x^{-\beta} \right)^{\rho-1} \]

\[ = \text{(exogenously defined constant)} [x^{\beta-\rho} y^{-(\beta-\rho)\rho}]. \]

Since \( \beta > 1 > \rho \) and \( y \) falls as \( x \) rises, then \( \kappa(x) \), and in turn \( h(x) \), is increasing with \( x \).

\[ \varepsilon_h = \frac{h'(x)}{h(x)} = \frac{\kappa'(x)}{\gamma + \kappa(x)} = \frac{\kappa'(x)}{\kappa(x)} \frac{\kappa(x)}{\gamma + \kappa(x)} = \varepsilon_\kappa \frac{\kappa(x)}{\gamma + \kappa(x)}. \]

To calculate \( \varepsilon_\kappa \), we use two properties, \( \varepsilon_{a(x) b(x)} = \varepsilon_{a(x)} + \varepsilon_{b(x)} \) and \( \varepsilon_{a(b(x))} = \varepsilon_{a(b)} \varepsilon_{b(x)} \). Then,

\[ \varepsilon_\kappa(x) = \varepsilon_{x^{\beta-\rho}} + \varepsilon_{y^{-\frac{\rho}{\beta-\rho}-(\beta-\rho)\rho}} = (\beta - \rho) - \rho (\beta - \rho) \varepsilon_y(x). \]

From the (FE) condition,

\[ \varepsilon_y(x) = -\frac{f}{f_{\text{exp}}} \left( \frac{y}{x} \right)^{\beta}, \quad \text{so that} \quad \varepsilon_\kappa(x) = (\beta - \rho) \left( 1 + \rho \frac{f}{f_{\text{exp}}} \left( \frac{y}{x} \right)^{\beta} \right), \quad \text{and} \]

\[ \varepsilon_h = (\beta - \rho) \left( 1 + \rho \frac{f}{f_{\text{exp}}} \left( \frac{y}{x} \right)^{\beta} \right) \frac{\kappa(x)}{\gamma + \kappa(x)} > 0. \]

**Step 3.** Now we can compare the absolute values of elasticities from (38) and (39):

\[ |\varepsilon_a| = \beta - \rho \quad \text{versus} \quad |\varepsilon_h| = (\beta - \rho) \left( 1 + \rho \frac{f}{f_{\text{exp}}} \left( \frac{y}{x} \right)^{\beta} \right) \frac{\kappa(x)}{\gamma + \kappa(x)}, \quad \text{or} \]

\[ 1 \quad \text{versus} \quad \left( 1 + \rho \frac{f}{f_{\text{exp}}} \left( \frac{y}{x} \right)^{\beta} \right) \frac{\kappa(x)}{\gamma + \kappa(x)}, \quad \text{or} \quad \gamma \frac{\kappa(x)}{\kappa(x)} \quad \text{versus} \quad \rho \frac{f}{f_{\text{exp}}} \left( \frac{y}{x} \right)^{\beta}, \quad \text{or} \]

\[ \frac{1}{\rho} \frac{f}{f_{\text{exp}}} \left( \frac{x^\beta y}{\kappa(x)} \right) > 1. \]

To compare the left-hand side with 1, we plug the expressions for \( \kappa(x) \) and \( \gamma \):

\[ \frac{1}{\rho} \frac{f}{f_{\text{exp}}} \left( \frac{x^\beta y}{\kappa(x)} \right) = \frac{1}{\rho} \frac{f}{f_{\text{exp}}} \left( \frac{x^\beta y}{\kappa(x)} \right) \frac{\theta [f(\sigma - 1)]^\rho}{\rho f} \left( \frac{w}{y} \right)^{\rho} \left( \frac{\sigma f_{\text{exp}} \theta}{y} \right)^{\rho} \left( \frac{x}{y} \right)^{(\beta-1)\rho} M^{\rho-1} \]

\[ = \rho^{\rho-1} \left( \frac{f}{f_{\text{exp}}} \right)^{1-\rho} \left( \frac{x}{y} \right)^{(1-\rho)\beta} \theta^{1-\rho} \left( \frac{x}{w} \right)^{\rho} M^{1-\rho}. \]
Note that from (35) and (M) condition,
\[
\frac{f_{\text{exp}m_{\text{exp}}}}{\sigma (f + m_{\text{exp}}f_{\text{exp}})} \left( \frac{\rho x}{\eta w} \right)^{\sigma - 1} = \eta f \quad \text{and} \quad M = \frac{1}{\sigma \theta (f + m_{\text{exp}}f_{\text{exp}})} = \frac{\eta f}{\theta f_{\text{exp}m_{\text{exp}}}},
\]
and (41) can be rewritten as
\[
\rho^{\rho - 1} \left( \frac{f_{\text{exp}}}{f} \right)^{1 - \rho} \left( \frac{x}{y} \right)^{\beta(1 - \rho)} \left( \frac{\eta f}{\theta f_{\text{exp}m_{\text{exp}}}} \right)^{1 - \rho} \left( \frac{y}{w} \right)^{\rho} \left( \frac{\rho}{\eta} \right)^{\rho - 1} = \left( \frac{\rho}{\eta} \right)^{\rho - 1} = \frac{\eta}{\rho}.
\]
Thus, the comparison in (40) results in \( \frac{1}{\rho} \left( \frac{z}{y} \right)^{\frac{\beta}{\kappa(x)}} = \frac{2}{\rho} \) versus 1, and we proved our results. \( \blacksquare \)

**First Best Allocation and Consumption Subsidy.** As shown before, the market equilibrium with a consumption subsidy \( \eta = \rho \) satisfies:

\begin{align*}
(M1) : \quad R P^{\sigma - 1} w^{-\sigma} x^{\sigma - 1} & = \sigma \rho f, \\
(M2) : \quad A w^{-\sigma} \rho^{\gamma} y^{\gamma - 1} & = \sigma \rho f_{\text{exp}}, \\
(M3) : \quad \rho & = \int_{y} f_{\text{exp}}(\varphi) M \mu(\varphi) d\varphi = A w^{1 - \sigma} \theta M_{\text{exp}}(\rho y)\gamma - 1 \\
(M4) : \quad \frac{\delta f_{e}}{1 - G(x)} & = (f + m_{\text{exp}}f_{\text{exp}})(\theta - 1), \\
(M5) : \quad M \sigma \theta (f + m_{\text{exp}}f_{\text{exp}}) & = 1,
\end{align*}

where \( (P/w)^{1 - \sigma} = (p_{z}/w)^{1 - \sigma} + \rho^{\sigma - 1} M \theta x^{\sigma - 1}, \) i.e., we have 5 equations with 5 unknown variables, \( R, w, x, y, M. \) If we have a market equilibrium and an optimal allocation \( (z_{0}, x_{0}, y_{0}, \alpha, \phi, M_{0}, \lambda, \zeta), \) which satisfies the system of equations (21)-(28). Is it the case that \( (x_{0}, y_{0}, z_{0}, M_{0}) = (x_{M}, y_{M}, z_{M}, M_{M})? \)

The first indication that this is the case is that equations (M) and (FE) in the optimal allocation coincide with equations (M4) and (M5) in the market equilibrium (denoted by ME below).

One way to complete the answer (assuming the solutions are unique) is to postulate \( (x_{0}, y_{0}, z_{0}, M_{0}) = (x_{M}, y_{M}, z_{M}, M_{M}) \) and then see if there exist \( (z, \alpha, \phi, \lambda, \zeta) \) such that these together with \( (x_{M}, y_{M}, M_{M}) \) satisfy 8 equations for an optimum allocation. This is exactly the case if:
\[
\phi = R P^{\sigma - 1} w^{-\sigma}, \quad \alpha = A w^{-\sigma} \rho^{\sigma}, \quad \zeta = \rho^{\rho - 1} P^{-\rho}, \quad \lambda = R^{\rho} P^{-\rho} \rho w.
\]
Then,

\[ z : \rho z^{\theta-1} = \rho R^{\theta-1} P^{-\rho} \text{ or } z = R^{\sigma-1} P^{-\sigma}, \]  
(formula for import demand in ME)

\[ x : RP^{\sigma-1} w^{-\sigma} x^{\sigma-1} = \sigma \rho f \text{ or } RP^{\sigma-1} \eta^{-\sigma} \left( \frac{\rho x}{w} \right)^{\sigma-1} = \sigma w f, \]  
(zero profit condition for domestic producers in ME, M1)

\[ y : Aw^{\sigma} \rho^{\sigma} y^{\sigma-1} = \sigma \rho f_{\text{exp}} \text{ or } Aw^{1-\sigma} (\rho y)^{\sigma-1} = \sigma w f_{\text{exp}}, \]  
(zero profit condition for exporters in ME, M2)

\[ \phi : \rho (RP^{\sigma-1} w^{-\sigma})^{\rho-1} = R^{\rho-1} P^{-\rho} PW \text{ is an identity.} \]

\[ \alpha : \rho R^{\rho-1} P^{-\rho} A^{1/\alpha} \rho (Aw^{\sigma} \rho^{\sigma})^{\rho-1} = R^{\rho-1} P^{-\rho} PW \text{ is an identity.} \]

\[ M : \frac{\delta f_{e}}{1 - G(x)} = (\theta - 1) (f + m_{\text{exp}} f_{\text{exp}}), \]  
(free entry condition in ME, M4)

\[ (FE) : 1 = \sigma \theta M (f + m_{\text{exp}} f_{\text{exp}}) \text{ or } M = \frac{1}{\sigma \theta (f + f_{\text{exp}} m_{\text{exp}})}, \]  
(the expression for the mass of active firms in ME, M5)

\[ (TB) : z = M m_{\text{exp}} A^{1/\sigma} (Aw^{\sigma} \rho^{\sigma})^{\rho} \theta y^{\sigma-1} \text{ or } p_{z} z = Aw^{1-\sigma} \theta M f_{\text{exp}} (\rho y)^{\sigma-1} = \int_{y} f_{\text{exp}} (\varphi) M \mu (\varphi) d \varphi. \]  
trade balance condition in ME, M3)

Thus, we prove that 2 sets of conditions coincide.

6.3.3 The Export Tax

The Optimal Value of Export Tax. We show that the optimal value of \( s \) is \( s < 1 \). In other words, the optimal "export subsidy" is actually an export tax: an exporter who charges price \( p_{\text{exp}} \) has to pay \((1 - s) p_{\text{exp}}\) as the tax.

The market equilibrium conditions are the same as those in Section 2 with \( \eta = 1 \):

\[ (\text{EXP}) \text{ condition } As^{\sigma} w^{1-\sigma} (\rho y)^{\sigma-1} = \sigma w f_{\text{exp}}, \]  
(42)

\[ (\text{FE}) \text{ condition } (\theta - 1) x^{-\beta} [f + m_{\text{exp}} f_{\text{exp}}] = \frac{\delta f_{e}}{b^{\beta}}, \]  
(43)

\[ (\text{M}) \text{ condition } M = \frac{1}{\sigma \theta (f + f_{\text{exp}} m_{\text{exp}})} = \frac{(\theta - 1) b^{\beta}}{\sigma \theta \delta f_{e} x^{-\beta}}, \]  
(44)

\[ (\text{TB}) \text{ condition } 1 = \sigma \theta M f_{\text{exp}} m_{\text{exp}} \left( 1 + \frac{1}{s} M \left( \frac{\rho x}{w} \right)^{\sigma-1} \right). \]  
(45)

Now we are ready to prove that \textbf{an export tax} \( \tau = 1 - \rho \), \textbf{which is in our notation equivalent to the export "subsidy"} \( s = \rho \) \textbf{results in the maximal level of welfare.}

**Proof.** Step 1. First, note that when \( s \) rises, \( y \) must fall and \( x \) must rise. The proof is the same as in Section 6.3.2, with equations (34), (35), and (36) rewritten as

\[ (\text{EXP}): \frac{w^{\sigma}}{y^{\rho-1}} = \frac{A \rho^{\sigma-1}}{s f_{\text{exp}}} s^{\sigma}, \]  
(46)
\[
\frac{1}{s \sigma (f + m_{\exp f_{\exp}})} \left( \frac{px}{w} \right)^{\sigma - 1} = f, \quad (47)
\]

\[
\left( \frac{w}{x} \right)^{\sigma - 1} \left( 1 + \frac{f}{f_{\exp}} \left( \frac{y}{x} \right)^{\beta} \right) = \frac{1}{s \rho^{\sigma - 1}} \frac{1}{\sigma f}. \quad (48)
\]

i.e., if \( s \) rises, the left-hand side of equation above should fall. Now, assume that if \( s \) rises, then \( y \) rises and \( x \) falls. Then

from (46): \[
w = \left( \frac{A \rho^{\sigma - 1}}{\sigma f_{\exp}} s^\sigma y^{\sigma - 1} \right)^{1/\sigma} \text{ must rise;}
\]

from (48): \[
w = \left( \frac{1}{s \rho^{\sigma - 1}} \frac{1}{\sigma f} \left( \frac{x^{\sigma - 1}}{f_{\exp} \left( \frac{y}{x} \right)^{\beta}} \right) \right)^{1/(\sigma - 1)} \text{ must fall.}
\]

To prove that the wage is an increasing function of \( x \), and as a result, it increases with \( s \) as well, note that from (46),

\[
\frac{w}{s} = \left( \frac{A \rho^{\sigma - 1}}{\sigma f_{\exp}} \right)^{1/\sigma} y^\rho. \quad (49)
\]

Then, from (48) and the (FE) condition,

\[
\left( \frac{w}{x} \right)^{\sigma - 1} \left( \frac{\delta f_e}{(\theta - 1) b^\beta} \right) \frac{y^\beta}{f_{\exp}} = \frac{1}{s \rho^{\sigma - 1}} \frac{1}{\sigma f}, \text{ or } \frac{w}{x} = \left( \frac{\theta - 1}{\sigma f \delta f_e} \rho^{\sigma - 1} \right)^{\frac{1}{\sigma - 1}} \frac{1}{s^{1-\sigma} y^{\frac{\beta}{1-\sigma}}}. \quad (50)
\]

Dividing (50) by (49), we get

\[
\frac{s}{x} = \Delta \left[ s^{\frac{1}{1-\sigma}} y^{\frac{\beta}{1-\sigma} - \rho} \right],
\]

where \( \Delta = \left( \frac{(\theta - 1)^{b^\beta} f_{\exp}}{\sigma f \delta f_e} \rho^{\sigma - 1} \right)^{\frac{1}{\sigma - 1}} / \left( \frac{A \rho^{\sigma - 1}}{\sigma f_{\exp}} \right)^{1/\sigma} \) is a constant. Therefore,

\[
s^{1 + \frac{1}{\sigma - 1}} = s^\rho = \Delta x y^{\frac{\beta}{1-\sigma} - \rho}, \text{ and } s = \Delta^\rho x^\rho y^{-\frac{\beta}{1-\sigma} - \rho^2}.
\]

If we plug the expression above into (49), we get

\[
w = \left( \frac{A \rho^{\sigma - 1}}{\sigma f_{\exp}} \right)^{1/\sigma} \Delta^\rho x^\rho y^{-\frac{\beta}{1-\sigma} - \rho^2},
\]

where \( \left( \frac{A \rho^{\sigma - 1}}{\sigma f_{\exp}} \right)^{1/\sigma} \Delta^\rho \) is defined exogenously. Note that \( \beta > \sigma \) and \( \rho < 1 \), so that \( \rho - \frac{\beta}{1-\sigma} - \rho^2 < 0 \). Thus, since \( x \) rises and \( y \) falls as \( s \) rises, then wage \( w \) must rise with \( s \).

**Step 2.** Now we are ready to derive the optimal export subsidy. As in Section 6.3.2, using the (TB) condition, we can rewrite the utility function as

\[
U^\rho = M x^\rho \left[ \left( \frac{w}{s x^\sigma f_{\exp} \theta \left( \frac{x}{y} \right)^{\beta}} \right)^{\rho} M^{\rho - 1} + \theta [f (\sigma - 1)]^\rho \right]. \quad (51)
\]
We will show that
\[ U \equiv d(x) h(x), \]
where \( d(x) \equiv M \alpha \) falls faster than \( h(x) \equiv \left( \frac{w}{x} \sigma f_{\exp} \left( \frac{x}{y} \right)^{\beta} \right)^{\rho} M^{\rho-1} + \theta [f(\sigma - 1)]^{\rho} \) rises, as \( x \) (and \( s \)) increases. Similarly to the case of the consumption subsidy, we can show that
\[ \varepsilon_d = \rho - \beta < 0, \quad \text{and} \]
\[ \varepsilon_h = (\beta - \rho) \left( 1 + \rho \frac{f}{f_x} \frac{y}{x} \right) \frac{\kappa(x)}{\gamma + \kappa(x)} > 0, \quad \text{where} \]
\( \gamma = \theta [f(\sigma - 1)]^{\rho} \) is exogenously defined constant,
\[ \kappa(x) = \left( \frac{w}{sy} \right)^{\rho} (\sigma f_{\exp} \theta)^{\rho} \left( \frac{x}{y} \right)^{(\beta-1)\rho} M^{\rho-1} \]
\[ = \left( \frac{A \rho^{\rho-1}}{\sigma f_{\exp}} \right)^{1/\beta} \left( \frac{1}{y^{1/\beta}} \right)^{\rho} (\sigma f_{\exp} \theta)^{\rho} \left( \frac{x}{y} \right)^{(\beta-1)\rho} \left( \frac{(\theta - 1) b^\beta}{\sigma \theta \delta f_x} x^{-\beta} \right)^{\rho-1} \]
\[ = \left( \text{exogenously defined constant} \right) [x^{\beta-\rho} y^{-(\beta-\rho)\rho}]. \]

Now we can compare the absolute values of elasticities from (52) and (53):
\[ |\varepsilon_d| = \beta - \rho \quad \text{versus} \quad |\varepsilon_h| = (\beta - \rho) \left( 1 + \rho \frac{f}{f_x} \frac{y}{x} \right) \frac{\kappa(x)}{\gamma + \kappa(x)}, \quad \text{or} \]
\[ \frac{1}{\rho \frac{f}{f_x} \frac{y}{\kappa(x)}} \frac{\gamma}{\kappa(x)} \quad \text{versus} \quad 1. \quad (54) \]

To compare the left-hand side with 1, we plug the expressions for \( \kappa(x) \) and \( \gamma \):
\[ \frac{1}{\rho \frac{f}{f_x} \frac{y}{\kappa(x)}} \frac{\gamma}{\kappa(x)} = \frac{1}{\rho \frac{f}{f_x} \frac{y}{\kappa(x)}} \frac{\theta [f(\sigma - 1)]^{\rho}}{\left( \frac{w}{sy} \right)^{\rho} (\sigma f_{\exp} \theta)^{\rho} \left( \frac{x}{y} \right)^{(\beta-1)\rho} M^{\rho-1} \]
\[ = \rho^{\rho-1} \left( \frac{f}{f_x} \right)^{1-\rho} \left( \frac{x}{y} \right)^{\beta(1-\rho)} \left( \frac{s x}{w} \right)^{\rho} M^{1-\rho}. \quad (55) \]

Note that from (47) and (44), \( M = \frac{\rho^{1-\sigma} f \left( \frac{x}{w} \right)^{\sigma-1} s}{\theta f_x m_{\exp}} \), and (55) can be rewritten as
\[ \rho^{\rho-1} \left( \frac{f_{\exp}}{f} \right)^{1-\rho} \left( \frac{x}{y} \right)^{\beta(1-\rho)} \left( \frac{s x}{w} \right)^{\rho} \left( \frac{\rho^{1-\sigma} f \left( \frac{x}{w} \right)^{\sigma-1} s}{\theta f_{\exp} m_{\exp}} \right)^{1-\rho} \]
\[ = \rho^{\rho-1} \left( \frac{x}{y} \right)^{\beta(1-\rho)} \left( \frac{s x}{w} \right)^{\rho} \left( \frac{\rho^{1-\sigma} f \left( \frac{x}{w} \right)^{\sigma-1} s}{\theta f_{\exp} m_{\exp}} \right)^{1-\rho} \]
\[ = \frac{s}{\rho}. \]

Thus, the comparison in (54) results in \( \frac{s}{\rho} \gg 1 \), and we proved our results.
**First Best Allocation and Export Tax**  Similarly to the case of the consumption subsidy, it can be shown straightforwardly that the market equilibrium conditions in the case of the optimal export tax coincide with system of equations (21)-(28), if

\[ \phi = R P^{\sigma-1} (\frac{p}{w})^\sigma, \quad \alpha = A (\frac{p}{w})^\rho, \quad \lambda = R^{\rho-1} P^{-\rho} w, \quad \zeta = \rho R^{\rho-1} P^{-\rho}. \]

### 6.3.4 The Import Tariff

**The Optimal Value of Import Tariff.** Let us derive the equilibrium conditions in the presence of the import tariff \( t \). Note that these derivations are very similar to those in Section 2 with \( \eta = 1 \) and \( s = 1 \). As a result, we have

\begin{align*}
(\text{EXP}) \text{ condition} & \quad Aw^{1-\sigma} (\rho y)^{\sigma-1} = \sigma w f_{\text{exp}}, \quad \text{and} \quad (56) \\
(\text{FE}) \text{ condition} & \quad (\theta - 1) x^{-\beta} [f + m_{\text{exp}} f_{\text{exp}}] = \frac{\delta f_e}{b^\sigma}. \quad (57)
\end{align*}

We need to derive the trade balance condition. The expenditures in the economy is \( R = w + T \), where \( T \) is a lump sum transfer defined as

\[ T = \frac{(t - 1)^2}{1} [(p z) z] = (t - 1) \left[ R P^{\sigma-1} t^{-\sigma} \right], \]

since \( p_z = 1 \) and the demand for the foreign variety is \( z = R P^{\sigma-1} (t p_z)^{-\sigma} = [R P^{\sigma-1} t^{-\sigma}] \). The trade balance condition implies that

\[ \int_y r_{\text{exp}} (r) M \mu (r) d r = p_z z = R P^{\sigma-1} t^{-\sigma}, \]

where \( P^{1-\sigma} = t^{1-\sigma} + \theta M \left( \frac{p_{z}}{w} \right)^{\sigma-1} \). Then, using some further simplification, the trade balance condition can be rewritten as

\[ Aw^{1-\sigma} \theta M_{\text{exp}} (\rho y)^{\sigma-1} = w \sigma f_{\text{exp}} \theta M_{\text{exp}} = R P^{\sigma-1} t^{-\sigma} = \frac{w t^{-\sigma}}{P^{1-\sigma [1 - (t - 1) [P^{\sigma-1} t^{-\sigma}]]}}, \]

or \( (\text{TB}) \text{ condition:} \quad \sigma \theta M f_{\text{exp}} m_{\text{exp}} = \frac{1}{1 + t^\sigma \theta M \left( \frac{p_{z}}{w} \right)^{\sigma-1}}. \quad (58) \]

Finally,

\[ (\text{M}) \text{ condition} \quad M = \frac{1}{\sigma \theta (f + m_{\text{exp}} f_{\text{exp}})} = \frac{(\theta - 1) b^\beta}{\sigma \theta \delta f_e} x^{-\beta}. \quad (59) \]

Now we are ready to prove that an import tariff \( t = \frac{1}{\rho} \) maximizes welfare.

**Proof.** Step 1. First, we prove that when \( t \) rises, \( x \) falls, while \( y \) and \( w \) rise. From the (FE) condition, \( x \) and \( y \) must move in the opposite direction. Let us rewrite the (EXP) and (TB) conditions and then use the result to show that \( y \) rises and \( x \) falls as the import tariff \( t \) increases.
Assume that $y$ falls. Then $x$ rises and from

$$(\text{EXP}): \frac{w}{y^{\sigma-1}} = \frac{A\rho^{-1}}{\sigma f_{\exp}},$$

(60)

$w$ must fall. On the other hand, from

$$(\text{TB}) + (\text{M}) \Rightarrow \frac{1}{\sigma \theta (f + m_{\exp} f_{\exp})} f_{\exp} m_{\exp} = \frac{1}{1 + t \theta M \left(\frac{\rho x}{w}\right)^{\sigma-1}},$$

or

$$\frac{f}{m_{\exp} f_{\exp}} = \left(\frac{\rho x}{w}\right)^{\sigma-1},$$

(61)

which is equivalent to

$$w^{1-\sigma} = (\theta M)^{-1} \left(\frac{\rho x}{w}\right)^{1-\sigma} t^{-\sigma} \frac{f}{m_{\exp} f_{\exp}}.$$  

(62)

Note that $M = \left(\frac{(\theta - 1) \rho x}{\sigma \theta M} - \beta\right)$ falls, $(\rho x)^{1-\sigma}$ falls, $1/m_{\exp} = (y/x)^{\beta}$ falls, and $t^{-\sigma}$ falls. Thus, $w^{1-\sigma}$ must fall, so that $w$ falls, which contradicts to the previous conclusion about $w$. Thus, we proved that $y$ cannot fall with an increase in $t$, and as $t$ rises, $y$ and $w$ rise as well, and $x$ falls.

**Step 2.** Now we are ready to derive the optimal export subsidy. From the (TB) condition, $z = R P^{\sigma-1} t^{-\sigma} = \sigma f_{\exp} w \theta M_{\exp}$, and

$$\int_{x}^{\infty} q(\varphi) \rho M \mu(\varphi) d\varphi = M \left[f (\sigma - 1)\right]^\rho x^{\beta - \rho (\sigma - 1)} \beta \int_{x}^{\infty} \varphi^{\sigma - \beta - 1} d\varphi = \theta M [f (\sigma - 1) x]^\rho,$$

we can rewrite the utility function as

$$U^\rho = (\sigma f_{\exp} w \theta M_{\exp})^\rho + \theta M [f (\sigma - 1) x]^\rho.$$

We want to look at the behavior of

$$U^\rho = (m_{\exp} x)^\rho \left[(\frac{w}{x} \sigma f_{\exp} \theta M)^\rho + \theta M [f (m_{\exp})^{-1} (\sigma - 1)]^\rho\right]$$

(63)

as $t$ increases. Note that $dx/dt < 0$. Therefore, we can look at the expression above as a function of $x$ and consider its behavior as $x$ falls. To do this, we will show that

$$U^\rho = d(x) h(x),$$

where as $x$ rises, $d(x) = (m_{\exp} x)^\rho$ falls faster than $h(x) = \left[(\frac{w}{x} \sigma f_{\exp} \theta M)^\rho + \theta M [f (m_{\exp})^{-1} (\sigma - 1)]^\rho\right]$ rises, if $t < \frac{1}{\rho}$, and the opposite happens, if $t > \frac{1}{\rho}$. In other words, $U(x)$ falls with $x$, if $t < \frac{1}{\rho}$, and it rises with $x$, if $t > \frac{1}{\rho}$. Then

$$\frac{dU}{dt} = \frac{dU}{dx} \frac{dx}{dt} = \begin{cases} > 0, & \text{if } t < \frac{1}{\rho}, \\ < 0, & \text{if } t > \frac{1}{\rho}. \end{cases}$$

(64)
and the utility reaches its maximum, when \( t = \frac{1}{\beta} \).

We can compare the elasticities of \( d(x) \) and \( h(x) \) and show that \( \varepsilon_d > 0 > \varepsilon_h \). Thus, the behavior of the utility function depends on the comparison of absolute terms \( |\varepsilon_d| \) and \( |\varepsilon_h| \) : if \( |\varepsilon_d| < |\varepsilon_h| \) (\( |\varepsilon_d| > |\varepsilon_h| \)), i.e., \( d(x) \) rises slower (faster) than \( h(x) \) falls, then \( U = d(x)h(x) \) falls (rises) as a result. To calculate elasticities, we will use the following properties:

\[
\varepsilon_{a(x)b(x)} = \varepsilon_{a(x)} + \varepsilon_{b(x)}, \quad \varepsilon_{(a(x))} = \delta\varepsilon_{a(x)}, \quad \varepsilon_{a(b(x))} = \varepsilon_a\varepsilon_{b(x)},
\]

\[
\varepsilon_{a(x) + b(x)} = \frac{a(x)}{a(x) + b(x)}\varepsilon_{a(x)} + \frac{b(x)}{a(x) + b(x)}\varepsilon_{b(x)}.
\]

\( d(x) : \)

\[
d(x) = (m_{\exp}x)^\rho = (x)^{\rho + \beta} y^{-\beta}.
\]

\( d(x) \) is decreasing in \( x \) (y falls as \( x \) rises) and

\[
\varepsilon_d = \rho \left[ 1 + \beta - \beta\varepsilon_{b(x)} \right], \quad \text{(64)}
\]

where from the (FE) condition, \( \varepsilon_y(x) = -\frac{f}{\exp} (\frac{y}{x})^\beta = -\frac{f}{\exp m_{\exp}} \), so that

\[
\varepsilon_d = \rho \left[ 1 + \beta - \frac{f}{\exp m_{\exp}} \right] > 0. \quad \text{(65)}
\]

\( h(x) : \)

\[
h(x) = \left( \frac{w}{x} \sigma f_{\exp} \theta M \right)^\rho + M \theta \left[ f (m_{\exp})^{-1} (\sigma - 1) \right]^\rho.
\]

Note that from the (EXP) condition,

\[
w = \left( \frac{A \rho^{\sigma - 1}}{\sigma f_{\exp}} \right)^{1/\sigma} y^{1 - \frac{1}{\sigma}} = \left( \frac{A \rho^{\sigma - 1}}{\sigma f_{\exp}} \right)^{1/\sigma} y^\rho,
\]

and from the (M) condition, \( M = \frac{(\theta - 1)\beta}{\sigma \theta f_{\exp}} x^{-\beta} \). Let us denote

\[
\left( \frac{w}{x} \sigma f_{\exp} \theta M \right)^\rho \equiv a(x) \quad \text{and} \quad M \theta \left[ f (m_{\exp})^{-1} (\sigma - 1) \right]^\rho \equiv b(x).
\]

Thus,

\[
\varepsilon_{h(x)} = \frac{a(x)}{a(x) + b(x)}\varepsilon_{a(x)} + \frac{b(x)}{a(x) + b(x)}\varepsilon_{b(x)}, \quad \text{where}
\]

\[
\varepsilon_{a(x)} = \rho \left( -\rho \frac{f}{\exp m_{\exp}} - 1 - \beta \right) < 0, \quad \varepsilon_{b(x)} = -\beta + \rho \left( -\beta - \beta \frac{f}{\exp m_{\exp}} \right) < 0,
\]

and \( \varepsilon_{h(x)} < 0 \). Now we can compare the absolute values of elasticities from above:

\[
|\varepsilon_d| = \rho \left[ 1 + \beta - \beta \frac{f}{\exp m_{\exp}} \right] \text{versus}
\]

\[
|\varepsilon_h| = \frac{a(x)}{a(x) + b(x)}\rho \left( \rho \frac{f}{\exp m_{\exp}} + 1 + \beta \right) + \frac{b(x)}{a(x) + b(x)} \left( \beta + \rho \left( \beta + \beta \frac{f}{\exp m_{\exp}} \right) \right), \text{or}
\]

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\[ \rho + \rho \beta + \rho \alpha \frac{f}{f_{\text{exp}m_{\text{exp}}}} \text{ versus } \frac{a(x)}{a(x) + b(x)} \left( \rho^2 \frac{f}{f_{\text{exp}m_{\text{exp}}} + \rho} + \rho \beta \frac{b(x)}{a(x) + b(x)} \right) \frac{f_{\text{exp}m_{\text{exp}}}}{f_{\text{exp}m_{\text{exp}}}}, \]

or

\[ \left( \rho + \rho \beta \frac{f}{f_{\text{exp}m_{\text{exp}}}} \right) \left( \frac{a(x)}{b(x)} + 1 \right) \text{ versus } \frac{a(x)}{b(x)} \left( \rho^2 \frac{f}{f_{\text{exp}m_{\text{exp}}} + \rho} + \beta + \rho \frac{f}{f_{\text{exp}m_{\text{exp}}}} \right). \]  

(66)

Note that from (61),

\[ a(x) - b(x) = w \exp \left( \sigma m_{\text{exp}} \right) \frac{1}{\rho \theta (f + m_{\text{exp}}f_{\text{exp}})} \left( \frac{x}{w} \right)^{\sigma - 1}, \quad \alpha = A \left( \frac{\sigma}{w} \right)^{\alpha}, \quad \lambda = R^{\alpha - 1} P^{-\rho} w, \quad \zeta = R^{\alpha - 1} P^{-\rho}. \]

### 6.4 Proof of Propositions 2 and 3

The market equilibrium conditions for the case of the export subsidy in the presence of the consumption subsidy are derived in Section 3. Hereafter, we assume that the government has in place the optimal consumption subsidy (i.e., \( \eta = \rho \)) and explore how export subsidies affect the three components of the utility function by first proving Proposition 2 and then proving Proposition 3.

**Proof.** The only difference between the equilibrium conditions in the cases with and without consumption subsidy is the (TB) condition (compare (8) and (45)). Thus, the proofs for Propositions 2 and 3 are the same as those in Section 6.3.3 with the only difference in equations (47) and (48), which now can be written as:

\[ \frac{1}{s} \frac{f_{\text{exp}m_{\text{exp}}}}{\sigma (f + m_{\text{exp}}f_{\text{exp}})} \left( \frac{x}{w} \right)^{\sigma - 1} \rho f, \]

and

\[ \left( \frac{w}{x} \right)^{\sigma - 1} \left( 1 + \frac{f}{f_{\text{exp}} \left( \frac{w}{x} \right)^{\rho}} \right) = \frac{1}{s \sigma \rho f}, \]

so that

\[ M = \frac{1}{\sigma \theta (f + m_{\text{exp}}f_{\text{exp}})} = \rho f \left( \frac{w}{x} \right)^{\sigma - 1} s. \]
and the comparison of $\varepsilon_h(x)$ and $\varepsilon_d(x)$ results in the comparison $s \geq 1$. ■

6.5 Quantitative Exercise for Three Components of Utility Function

We know already that $[\text{(TOT index) } * \text{(Variety index)]}$ falls as the export subsidy $s$ falls. Now we want to look at the behavior of these indices separately. We will show below that anything possible, i.e., there are 3 cases: (Which case happens depends on the parameters.)

1. TOT index falls, Variety index falls;
2. TOT index falls, Variety index rises;
3. TOT index rises, Variety index falls;

Thus, it is not possible to make unambiguous predictions about the behavior of these two indices in general case (without knowing the values of the parameters.) The difference between cases below is in the values of $A/f$, $\sigma$, and the highest value of $s$.

1. TOT index falls, Variety index falls

Let’s set the parameters: $\delta = 0.1$, $\beta = 4.1$, $\sigma = 3.8$, $b = 1$, $\frac{f_{\exp}}{d} = 15$, $\frac{f_e}{d} = 6$, $\frac{A}{d} = 100$, $\frac{L}{d} = 10,000$. Then $\theta = \frac{\beta}{\beta - (\sigma - 1)} = 3.1$, $\rho = \frac{\sigma - 1}{\sigma} = 0.74$. We will vary $s$ between 0.5 and 10.8. Then $\beta > \sigma$, $y > x > b$, $M_e > 0$, $wage > 0$, and the figure is\textsuperscript{13}

\[\text{TOT index} \quad \text{Variety index}\]

(2) TOT index falls, Variety index rises

Let’s set the parameters: $\delta = 0.1$, $\beta = 4.1$, $\sigma = 4$, $b = 1$, $\frac{f_{\exp}}{d} = 15$, $\frac{f_e}{d} = 6$, $\frac{A}{d} = 10$, $\frac{L}{d} = 10,000$. Then $\theta = 3.1$, $\rho = 0.74$. We will vary $s$ between 0.5 and 2. Then $\beta > \sigma$, $y > x > b$, $M_e > 0$, $wage > 0$, and the figure is

\[\text{TOT index} \quad \text{Variety index}\]

\textsuperscript{13}Note that in this case Variety index first increases and then starts to fall. We did not include the "falling" part into the figure.
(3) TOT index rises, Variety index falls

Let’s set the parameters: $\delta = 0.1, \beta = 4.1, \sigma = 3, b = 1, \frac{f_{\exp}}{f} = 15, \frac{f_e}{f} = 6, \frac{A}{f} = 10,000, \frac{L}{f} = 10,000$. Then $\theta = 1.95, \rho = 0.667$. We will vary $s$ between 0.5 and 7.1. Then $\beta > \sigma, y > x > \hat{b}, M_c > 0, wage > 0$, and the figure is