Debt Contracts with Short-Term Commitment

Natalia Kovrijnykh*

Department of Economics
University of Chicago

February 13, 2008

Abstract

This paper analyzes the role of short-term commitment by the lender in a dynamic relationship where the borrower cannot be legally forced to make repayments. I show that short-term commitment can decrease social welfare compared to both the full and no-commitment cases considered by most of the literature. I show that the size of investment is positively related to the borrower’s income. In addition, both underinvestment and overinvestment can occur in equilibrium. I also introduce the borrower’s outside option and do comparative statics with respect to it. I show that the social welfare is non-monotonic in the borrower’s outside option. If the borrower’s outside option is interpreted as a measure of competitiveness of the credit market, this implies that an increase in the strength of competition has an ambiguous effect on welfare. Furthermore, numerical results suggest that as the outside option of the borrower increases, the renegotiation-proof equilibria converge to the Markov equilibrium, where the agents’ strategies depend only on the borrower’s liquidity. That is, the welfare gain from using complicated history-dependent strategies instead of simple Markov strategies is small when the borrower’s outside option is high.

1 Introduction

Consider a situation where an agent, the borrower, can operate a stochastic production technology, but does not have access to the necessary capital input. Another agent, the lender, has enough

*I would like to thank Fernando Alvarez, Gadi Barlevy, Marco Bassetto, Jeffrey Campbell, William Fuchs, Veronica Guerrieri, Anil Kashyap, Vijay Krishna, Robert Lucas, Alessandro Pavan, Robert Shimer, Hugo Sonnenschein, Nancy Stokey, Balazs Szentes, Harald Uhlig, and Marcelo Veracierto for helpful discussions and criticism. I am also grateful for comments by seminar participants at the University of Chicago, Federal Reserve Bank of Chicago, University of Minnesota – Carlson School of Management, University of Houston, Texas A&M, Arizona State University, and University of Pennsylvania – Wharton.
capital, but cannot operate the technology himself. The agents enter into repeated relationship, where they jointly generate and share surplus. In such a dynamic relationship, the lack of commitment can cause inefficiencies for the following reason. If the lender expects not to be repaid in the future, he might not invest into the borrower’s technology. Similarly, the borrower might not repay the lender if she expects the lender not to invest in the future. If the parties cannot credibly commit to future decisions, such pessimistic expectations can be self-fulfilling and lead to inefficiencies. The goal of this paper is to analyze how the amount of commitment power affects investment and welfare.

An example of a bilateral relationship described above is the interaction between a foreign investor (the lender) and the host country (the borrower). Indeed, the seminal paper by Thomas and Worrall (1994), among others, analyzes lending contracts in the context of foreign direct investment. In this case, commitment problems are particularly important because international contracts are hard to enforce in general. Another example is a close long-term relationship between a bank and a firm.\footnote{The literature provides many explanations for the existence of such close relationships. For example, a bank might internalize informational problems and serve as a corporate monitor of its client firms – see, e.g., Diamond (1984) and Hoshi, Kashyap, and Scharfstein (1991).}

The lack of commitment on the side of the borrower can arise because, for example, the firm can use accounting tricks to divert cash flows. (See, e.g. Hart and Moore 1998, and Albuquerque and Hopenhayn 2004.)

Most of the studies on dynamic lending with limited commitment on the side of the borrower make one of the two extreme assumptions about the lender’s ability to commit: They either assume that the lender has full commitment power or cannot commit at all. Full commitment means that, at the beginning of the relationship, the lender is able to commit to investment decisions for all future periods and all possible histories. In the no-commitment case, all contracts must be self-enforcing. Interestingly, the common prediction of models with either of these assumptions is that, in optimal subgame perfect equilibria, the socially efficient outcome can be achieved over time if the agents are patient enough. However, the reasons for efficiency are quite different in the two cases. If the lender has full commitment power, efficiency can be achieved because he can commit to efficient investment decisions in the future. The lender is willing to do so for the following reason. The borrower, knowing that the lender will invest efficiently in the future, compensates the lender for past investments by large payments whenever she has enough liquidity. In contrast, if neither parties can commit, autarky becomes an equilibrium. In the optimal contract, any deviation from the equilibrium behavior is punished by entering into autarky forever. If agents are sufficiently...
patient, this punishment is strong enough to induce the agents to make efficient decisions.

Therefore, it is just natural to ask: What happens if the lender does not have full commitment power, but can at least partially commit to future decisions? And perhaps most importantly: How does the introduction of such an intermediate assumption affect social welfare? Indeed, my main objective in this paper is to investigate predictions of a model with a partial, short-term commitment structure, where the lender is able to commit to one-period contracts.

I model short-term commitment in the following way. Before the borrower decides how much to repay to the lender, the lender provides her with a menu specifying the amount of investment as a function of the borrower’s repayment. If the borrower makes a certain amount of repayment, the lender invests the amount that corresponds to the borrower’s payment in the menu. That is, the lender is able to credibly commit to such a menu.\(^2\) This way of modeling short-term commitment explicitly allows new contracts being offered in each period. Indeed, in lender-borrower relationships we often observe parties negotiating the terms of a contract over and over again. Hence, in a model of such a relationship, the explicit assumption of the possibility of signing new contracts seems realistic.

I show that introducing short-term commitment creates a nontrivial incentive problem on the side of the lender that has not been previously analyzed in the literature. When the lender can fully commit, his future incentives are irrelevant. Without commitment, the payoff to the lender who deviates from the optimal contract (i.e., violates its terms), is his autarkic payoff.\(^3\) Hence the lender has incentives to follow the equilibrium contract as long as his payoff exceeds this value. Allowing the lender to commit to one-period contracts increases the lowest payoff that the lender can receive in equilibrium (which can be used as a punishment for a deviation). The reason is that the lender can now credible promise investment to the borrower, and the borrower is willing to pay for it. This allows the lender to generate positive surplus. Therefore, since introducing short-term commitment increases the payoff to the lender from deviating, it makes it harder to discipline the lender in equilibrium.

The above discussion leads to a somewhat surprising prediction that, in environments where short-term contracts can be enforced, the social welfare might be lower than in environments where no contracts can be enforced. While the majority of papers on lending with limited commitment

\(^2\)This essentially means that the lender is able to commit to one-period contracts. It would not be hard to generalize this model so that the commitment is over a finite number of periods.

\(^3\)Even if one requires renegotiation-proofness, as I do in this paper, so that autarky is not considered a credible punishment, the lowest payoff to the lender in renegotiation-proof equilibria is still his autarkic value. See Thomas and Worrall (1994).
focuses on the borrower’s incentives, this paper shows that the possibility of writing short-term contracts makes it crucial to analyze the incentives of the lender.

I characterize renegotiation-proof equilibria in this model. The main consequences of short-term commitment are the following. First, the amount of investment is increasing in the borrower’s liquidity. This result is consistent with empirical findings in the corporate finance literature.\(^4\) Second, both underinvestment and overinvestment can occur in equilibrium. On the one hand, if the borrower does not have enough liquidity to compensate the lender, the lender underinvests. On the other hand, when the borrower’s income is high, the lender overinvests. The reason is that overinvestment increases the next period’s expected income, and hence the next period’s expected investment. Such a contract is attractive to the borrower, and she is willing to pay for it. That is, the lender uses overinvestment as a way of committing himself to high investments in the future. Thus, the short-term commitment causes a positive relationship between the current income and investment, which, in turn, induces overinvestment. Third, investment fluctuates over time. The reason is that the borrower’s production technology is stochastic, and therefore, her liquidity also varies stochastically across different periods. These findings are in contrast to the predictions of most models with either the full- or no-commitment assumptions, where only underinvestment occurs until the time when the maximum level of investment (which is often the efficient amount) is reached.

In a model where there is one lender and one borrower, an important question arises: Would competition among lenders improve efficiency? In order to address this question, I introduce the outside option of the borrower, which is the payoff that she receives after her contractual relationship with the lender comes to an end. This outside option can be interpreted, for example, as the value from an alternative use of the production technology, and perhaps most importantly, the payoff from searching for another lender. Hence one can consider the borrower’s outside option as a measure of competitiveness of the credit market.

One of the most interesting observations is that the social welfare is non-monotonic in the borrower’s outside option: it decreases first and then increases. The reason for this result is that the effect of an increase in the borrower’s outside option on the investment can be negative or positive depending on the level of the borrower’s income. For high income levels, the borrower’s

\(^4\)See Hubbard (1998) for a survey. Many of existing papers explain liquidity sensitivity of investment using models of asymmetric information. This paper provides an alternative explanation - the lack of commitment. Close bank-firm relationships are often suggested to mitigate the information problems and hence the associated liquidity problems, see, e.g., Hoshi, Kashyap, and Scharfstein (1991). However, Fohlin (1998) finds that relationship banking provides no consistent lessening of firms’ liquidity sensitivity.
participation constraint binds, and the borrower with a higher outside option would prefer to reject the lender’s old contract. Therefore, the lender is forced to invest more at high income levels. But this observation has another consequence: it is now more costly for the lender to extract the borrower’s income. Therefore, when the income is low so that the borrower’s participation constraint does not bind and the lender only takes into account his own future gains, he has less incentives to invest. I show that the first, positive effect dominates when the borrower’s outside option is sufficiently high, and the second, negative effect dominates when the outside option is low.

An implication of this result is that the strength of competition in the credit market has an ambiguous effect on social welfare. More precisely, if the competition is weak in the credit market, then a marginal increase in the strength of the competition can actually lead to a decrease in social welfare. However, if the credit market is competitive enough, making the competition even stronger increases efficiency. This finding is in line with empirical predictions that bank competition can have both positive and negative economic effects.\textsuperscript{5} The nonmonotonicity result of my model is in contrast to predictions of similar models with full or no commitment by the lender, which suggest that the social surplus is decreasing in the borrower’s outside option.\textsuperscript{6}

Furthermore, I show with numerical computations that as the outside option of the borrower increases, the set of renegotiation-proof equilibria converges to the Markov equilibrium. (In a Markov equilibrium, the agents’ strategies are allowed to depend only on the current income but not on the history of the play.) That is, the welfare gain from using complicated history-dependent strategies instead of simple Markov strategies is small when the borrower’s outside option is high.

A possible interpretation of this result is the following. In this model, the renegotiation-proof equilibria have the property that the previous history can be summarized by the value to the lender, which can be interpreted as debt. The next period’s debt can be viewed as being part of a contract, indicating to the parties which equilibrium will be played in the next period. Since contracts are complete, the next period’s debt can be conditioned on the next period’s output. In the case of a Markov equilibrium, the current income of the borrower fully determines which strategies are played in a given period. Therefore specifying the level of debt in a contract is irrelevant, as is the possibility of making the level of debt conditional on output. Hence, this paper provides a possible explanation for why we often observe debt contracts that do not specify


state-contingent repayment schedules, even when this possibility appears feasible.

**Related Literature.**— The papers that are most related to mine are Thomas and Worrall (1994), Albuquerque and Hopenhayn (2004), and Kovrijnykh and Szentes (2007). Thomas and Worrall (1994) consider optimal contracts between one lender and one borrower in the context of foreign direct investment. They assume no commitment on either side. Albuquerque and Hopenhayn (2004) study a relationship between a bank and an entrepreneur, where the borrower cannot commit, but the lender has full commitment power. The main prediction of these papers is that while there is underinvestment initially, borrowing constraints cease to bind eventually, and the first-best investment is made from that point on. In Thomas and Worrall, the economy converges to the steady state over time. In contrast, in my model there is no convergence in general, and both underinvestment and overinvestment can occur in equilibrium. In addition, Albuquerque and Hopenhayn predict that the value of the project is decreasing in the outside option of the entrepreneur, in contrast to the prediction of my model that this relationship is non-monotonic.

Kovrijnykh and Szentes (2007) consider a framework similar to mine, with two lenders simultaneously making offers to the borrower. Furthermore, contracts are incomplete: the next-period’s debt cannot be conditioned on the next-period’s income of the borrower. In contrast, in my model only one lender makes offers to the borrower, and contracts are complete in the sense that they can be history-dependent.

Doyle and Van Wijnbergen (1994) study a repeated bargaining model of foreign direct investment, where, similar to my model, negotiation over the output transfer occurs in every period. Bulow and Rogoff (1989) present a “constant recontracting model of sovereign debt” with a finite horizon, where agents engage in Rubinstein bargaining and offers are made in each period. Output is exogenous in both of these papers. The offers only specify how the surplus is shared, and the models make no predictions about the investment process and associated inefficiencies. In contrast, in my model negotiations over both the investment and the output transfers take place.

The following papers on foreign direct investment are also related. Schnitzer (1999) analyzes an infinite horizon model where all investment is made in period zero and is then sunk. She shows that the risk of expropriation may cause underinvestment if the outside option of the investor is too low, and overinvestment if it is too high. Janeba (2002) studies a model where a country cannot commit to a long-run tax policy, and a multinational firm cannot commit to invest in only one country. The author shows that the lack of commitment may induce the firm to invest in

---

7 The authors assume that there are many lenders initially, but they do not play a strategic role, as the borrower writes a long-term contract with only one of them.
a high-cost (but more credible) country. Alternatively, the firm might invest an optimal amount in a low-cost (but low-credibility) country, but hold excess capacity in the high-cost country for strategic purposes. In my model the lender invests only into one country, but similarly to Janeba I am interested in how incentives of the lender are affected by the limited commitment on the side of the borrower. A recent paper by Opp (2007) considers a framework similar to Thomas and Worrall (1994), where the government has access to a relatively inefficient autarkic technology, and the government is more impatient relative to the foreign firm. The author finds that the steady-state payoff to the government is non-monotonic in the autarkic productivity. The steady-state investment, and hence the total steady-state surplus in his model are non-increasing in the autarkic productivity as long as the equilibrium is not the autarky. The autarkic productivity can be interpreted as the outside option of the borrower. That is, Opp’s result is in contrast to the prediction of my model where the total surplus is non-monotone in the borrower’s outside option even while autarky is not an equilibrium.

This work is also related to the following papers on corporate lending. Sigouin (2003) considers a relationship between a risk-neutral creditor and a risk-averse entrepreneur, where assets and capital serve as physical collateral. The author shows that both underinvestment and overinvestment can occur in equilibrium, in line with my predictions. However, in his model overinvestment occurs because it facilitates risk sharing and because capital plays a role of collateral. In contrast, in my model agents are risk neutral and debt is unsecured. The reason why overinvestment occurs is because it increases the next period’s liquidity and thus increases the lender incentives to invest in the next period. Clementi and Hopenhayn (2006) analyze a borrower-lender relationship with asymmetric information where the lender does not observe either the use of funds or the output. The optimal contracts have a feature that the economy eventually converges to one of two absorbing states: A sequence of bad shocks results in the liquidation, while a sequence of good shocks leads to the first best.\footnote{Also see Quadrini (2004) who shows that the liquidation of the firm can arise as an outcome of a renegotiation-proof contract.}

Atkeson (1991) studies optimal sovereign lending contracts with moral hazard between a borrower and overlapping generations of short-lived lenders. In Atkeson’s model the lenders do not observe whether the borrower invests or consumes borrowed funds. To provide incentives to the borrower, the optimal contract must specify a fall in consumption and investment for the lowest realizations of output. My model produces a similar result – investment is increasing in output, and investment is inefficiently small if the output is small.
2 Model

Production and Preferences

There are two risk neutral agents, a lender and a borrower. The time horizon is infinite, time is discrete, and agents discount the future according to the discount factor $\beta \in (0, 1)$.

The borrower can operate a stochastic technology that transforms capital goods into consumption goods. If the amount of capital investment is $K$, then the output in the next period, in terms of consumption goods, is $F(K) = sf(K)$, where $s$ is the realization of a random shock. The function $f$ is strictly increasing, strictly concave, and satisfies the Inada conditions. The shock is distributed according to the continuous cumulative distribution function $G$, which has a strictly positive density $g$ on $[0, 1]$. The goods are perishable and completely depreciate every period.

The lender has enough capital to invest in production in every period. In addition, the lender can instantaneously transform one unit of capital good into one unit of consumption good and vice versa. This means the lender is indifferent between the two goods. Each agent’s goal is to maximize the discounted present value of expected consumption.

Timing and Contracts

The lender can commit to one-period contracts only. A typical contract is a pair $(C, K) \in \mathbb{R}_+^2$, where $C$ is the consumption of the borrower, and $K$ is the investment in the production technology. By offering such a contract, the lender commits to invest $K$ if the borrower makes a repayment of $I - C$ to the lender. More generally, a contract can be a probability mixture of these pairs.

The timing is as follows. At the beginning of a period, the output realization is $I$, which is simultaneously observed by the agents. The lender offers a set of contracts to the borrower. The borrower either accepts a contract from this set or rejects all contracts. If the borrower accepts a contract $(C, K)$, she gives $I - C$ units of consumption good to the lender, and the lender invests $K$ units of capital in the production technology. The borrower consumes $C$, the lender consumes $I - C - K$, and the period ends. If the borrower rejects all contracts, the relationship is automatically terminated and the game ends.

---

9 Having two different goods ensures the borrower cannot invest into the technology herself. This assumption is also used by Thomas and Worrall (1994) and Kovrijnykh and Szentes (2007).

10 Since the borrower is risk neutral, $C$ can always be assumed to be deterministic even though contracts can be random.

11 If the borrower accepts a random contract, then the outcome of the lottery is first observed.

12 The assumption that the game ends if the borrower rejects all offers means that the lender can commit to terminate the relationship. I also consider an extension of this model, which can be found at http://home.uchicago.edu/~nkovrijn/canstay.pdf, where I relax this assumption. Instead I assume that after re-
are zero to the lender and $I + \beta \theta$ to the borrower, where $\theta \geq 0$ is the value of the borrower’s outside option.

To see how this game with short-term commitment is different from a game where the lender has no commitment power, consider the following setup, which is similar to the one analyzed in Thomas and Worrall (1994). At the beginning of each period, first the lender chooses the level of investment. Production takes place and the income is realized and observed by both agents. Then the borrower decides how much to repay to the lender from the current output. The agents consume and the period ends. In this setup, in each period each agent chooses his action to maximize the present value of his utility at that point of time. In particular, the investment choice maximizes the lender’s payoff after the borrower has made the repayment. This is not the case with the short-term commitment structure that I consider. After the borrower makes the repayment $I - C$, it might not be in the lender’s interest ex-post to invest the amount $K$ specified in the contract. The lender does invest $K$ because he committed to it previously. In other words, without commitment the lender chooses a strategy according to his reaction function, while with the short-term commitment the lender commits to a reaction function.

Also notice that, in the model without commitment, nothing links the current output to future investment except for the equilibrium behavior of the agents. This is not the case with the short-term commitment. The reason is that the amount of the borrower’s liquidity limits the repayment that the lender can extract from her, and therefore it affects the level of offered investment. I will show that in equilibrium investment is increasing in the liquidity of the borrower.

**The First-Best Investment.**— Since both agents are risk neutral, the first-best investment maximizes $-K + \beta \int_0^1 s f(K) dG(s)$. The solution, $K^{FB}$, is defined by the following first-order condition: $1 = \beta f'(K^{FB}) \int_0^1 sdG(s)$. Let $S^{FB}$ denote the first-best social surplus if the borrower’s income is zero. That is, $S^{FB} = [-K^{FB} + \beta f(K^{FB}) \int_0^1 sdG(s)]/(1 - \beta)$. When the borrower’s income is $I$, the first-best surplus equals $S^{FB} + I$.

**Equilibrium Concept**

The goal of this paper is to analyze renegotiation-proof equilibria in the game described above. The concept of renegotiation-proofness requires that the Pareto frontier corresponding to optimal subgame perfect equilibria has the property that the off-equilibrium continuation payoffs themselves must lie of the constrained Pareto frontier. The idea behind this equilibrium concept is thatjecting the lender’s offer, the borrower can choose between taking the outside option and staying in the relationship with the current lender. This model is harder to analyze analytically, but the main qualitative results remain the same. For sufficiently high values of the borrower’s outside option the solutions to the two problems coincide.
even if a player deviates from the equilibrium behavior, the agents can renegotiate their implicit contract and can reach a constrained efficient agreement. Renegotiation-proofness is usually defined for repeated games. The game that I consider is not repeated for the following reason. The income $I$ affects the game played in a given period, in particular, it affects the value to the borrower from taking her outside option. In addition, the distribution of income in each period is determined by the investment made in the previous period. However, this game still has a repeated-game-like structure, with the income parametrizing the changing physical environment. This simple structure allows me to generalize the definition of renegotiation-proofness given for repeated games to the considered environment. This extension of the definition is one of the contributions of this paper.\(^{13}\)

Let $\Sigma (I)$ be a set of subgame perfect equilibria for income $I$, and $\Sigma = \{ \Sigma (I) \mid I \geq 0 \}$ be a set of subgame perfect equilibria for all income levels. Denote $\mathcal{P} (I) = \{ p (I) \mid p (I)$ is the payoff profile corresponding to $\sigma (I) \in \Sigma (I) \}$ the corresponding set of payoffs for income $I$. The collection of these sets for all income levels is denoted by $\mathcal{P} = \{ \mathcal{P} (I) \mid I \geq 0 \}$.

**Definition 1** $\mathcal{P}$ is weakly renegotiation-proof if

(a) $\forall I, \forall p \in \mathcal{P} (I)$ $\exists p' \in \mathcal{P} (I)$ such that $p'$ Pareto dominates $p$ (that is, $p' \succeq p$),\(^{14}\) and

(b) $\forall I, \forall p \in \mathcal{P} (I)$, $p$ is generated using continuation payoffs from $\mathcal{P}$.

**Definition 2** $\mathcal{P}$ is renegotiation-proof if $\mathcal{P}$ is weakly renegotiation-proof, and there is no other weakly renegotiation-proof set of payoffs $\mathcal{P}'$, $\mathcal{P}' \neq \mathcal{P}$, that Pareto dominates $\mathcal{P}$, i.e., there is no $\mathcal{P}'$ such that $\forall I, \forall p \in \mathcal{P} (I)$ $\exists p' \in \mathcal{P}' (I)$ such that $p' \succeq p$.

As we will see later, a weakly renegotiation-proof set of payoffs will be a fixed point of an operator, corresponding to a recursive problem. Before setting up this recursive problem, I define Markov equilibrium. I am interested in Markov equilibria for several reasons. First, as I will show below, any Markov equilibrium is weakly renegotiation-proof. This allows me to prove existence of a renegotiation-proof equilibrium. Second, I could prove certain results analytically for Markov equilibria and only show numerically for renegotiation-proof equilibria. And finally, I

---

\(^{13}\)I extend the definition of weak renegotiation-proofness (internal consistency) given in Bernheim and Ray (1989), and Farrell and Maskin (1989). Also see Ray (1994). The additional requirement for renegotiation-proofness extends the corresponding condition given in Van Damme (1991).

\(^{14}\)Some definitions use the notion of weak Pareto dominance here, i.e., they impose $p' \succ p$ ($p'_i > p_i$, $i = 1, 2$) instead of $p' \succeq p$. It can be shown that in this model the two requirements lead to the same results. That is, I am not imposing a stronger requirement by using Pareto dominance instead of weak Pareto dominance.
show with numerical computations that as the outside option of the borrower increases, the set of renegotiation-proof equilibrium payoffs converges to the set of Markov equilibrium payoffs.

**Definition 3** A subgame perfect equilibrium is Markov if in each period the strategies of the players depend only on the current income realization, but not on the previous history of the play.

### 3 Bellman Treatment

In this section, I formulate recursive problems corresponding to Markov and renegotiation-proof equilibria.

**Markov Equilibria**

Fix a Markov equilibrium, and let $L^M(I)$ and $B^M(I)$ denote the values to the lender and to the borrower at income $I$, respectively. The lender solves the following maximization problem:

$$
\hat{L}^M(I) = \max_{C,K} I - C - EK + \beta EL^M(sf(K))
$$

$s.t. \ C + \beta EB^M(sf(K)) \geq I + \beta \theta,$

and

$$
L^M(I) = \max_{I'} \pi \max_{\pi \in [0,1]} \pi \max \left\{ 0, \hat{L}^M(I') \right\} + (1 - \pi) \max \left\{ 0, \hat{L}^M(I'') \right\}
$$

$s.t. \ \pi I' + (1 - \pi) I'' = I.

The choice variables are consumption of the borrower and investment.\(^{15}\) The maximization is subject to the participation constraint of the borrower. It requires that the equilibrium payoff of the borrower at income $I$ is at least as high as her outside option, $I + \beta \theta$. Problem (2) incorporates the possibility that if the value $\hat{L}^M(I)$ is less than zero, then the lender prefers to terminate the relationship (by offering a contract that the borrower would reject) and receive a payoff of zero. Hence at income $I$, the payoff to the lender is at least $\max \left\{ 0, \hat{L}^M(I) \right\}$. The lender can achieve a possibly higher payoff by using lotteries.

The payoff frontier for a given income level $I$ is a singleton: $\mathcal{P}(I) = (B^M(I), L^M(I))$. Hence by the definition of weak renegotiation-proofness, we have the following result:

**Claim 1** Any Markov equilibrium is weakly renegotiation-proof.

Although a Markov equilibrium is weakly renegotiation-proof, it is not renegotiation-proof in general. Furthermore, in a Markov equilibrium the lender is never constrained to deliver more to

\(^{15}\)I allow the use of lotteries, hence the expectations in the continuation values are taken both with respect to the randomness in the contract and the shock.
the borrower than her outside option. That is, the lender always has the full bargaining power. This is not the case in the problem that I will consider below.

Weakly Renegotiation-Proof Equilibria

The problem of finding a set of weakly renegotiation-proof payoffs can be written recursively, with the income and the value to one of the agents as state variables, and the continuation values as control variables. Let $L(I, B)$ denote the value to the lender if the income is $I$ and the value to the borrower is at least $B$. Essentially, the artificial state variable $B$ summarizes the previous history of the play. The maximization problem of the lender is as follows:

$$
L(I, B) = \max_{C,K,B^*(\cdot)} I - C - EK + \beta E L(s f(K), B^*(s f(K)))
$$

s.t. $C + \beta EB^*(s f(K)) \geq B$,

$$
B^*(I') \geq B_{\min}(I') \text{ for all } I',
$$

$$
L(I', B^*(I')) \geq d(I') \text{ for all } I',
$$

where $B_{\min}(I) = \sup \{B | B \in \arg \max_{B \geq I + \beta \theta} L(I, B) \}$ corresponds to the lowest payoff to the borrower on the frontier for income $I$. The value $d(I)$ is the lender’s value from the most profitable deviation when the borrower’s income is $I$. In problem (3), $B \geq I + \beta \theta$ and $L(I, B) \geq d(I)$, for otherwise one of the agents would deviate.

The value to the deviating lender, $d(I)$, is defined by the following two problems:

$$
\hat{d}(I) = \min_{B^d(\cdot)} \left[ \max_{C,K} I - C - EK + \beta EL(s f(K), B^d(s f(K))) \right]
$$

s.t. $C + \beta EB^d(s f(K)) \geq I + \beta \theta$

$$
B^d(I') \geq B_{\min}(I') \text{ for all } I',
$$

$$
L(I', B^d(I')) \geq d(I') \text{ for all } I',
$$

and

$$
d(I) = \max_{I',I'', \pi \in [0,1]} \pi \max \left\{0, \hat{d}(I')\right\} + (1 - \pi) \max \left\{0, \hat{d}(I'')\right\}
$$

s.t. $\pi I' + (1 - \pi) I'' = I$.

Below, I describe the above problems in details, and later establish any weakly renegotiation-proof set of payoffs corresponds to a solution to these problems.

Consider problem (3) first. The choice variable of the lender is an offer that he makes, $(C, K)$. The continuation values $B^*(I')$ are not part of the lender’s strategy. They specify which equilibrium is played in the next period if the income realization is $I'$, and the lender has not deviated. Given $B$, the maximization with respect to $B^*(\cdot)$ ensures that problem (3) generates the constrained Pareto efficient outcome.

The first constraint in problem (3) is the “promise-keeping” constraint, which guarantees that the value that the borrower receives is indeed at least $B$. The second and the third constraints
determine which continuation payoffs can be feasible in equilibrium. The second constraint is the next period’s participation constraint of the borrower. It says that the borrower prefers to play her equilibrium strategy and receive the payoff of $B^* (I')$ to taking her outside option. The value $B_{\text{min}} (I')$ appears on the right-hand side of the constraint instead of $I' + \beta \theta$ for the following reason. For some values of income $I$, if the borrower is only promised her outside option, $B = I + \beta \theta$, the promise-keeping constraint might not bind, and hence the actual lowest payoff that the borrower receives in equilibrium for income $I$ exceeds her outside option: $B_{\text{min}} (I) > I + \beta \theta$. Incorporating $B_{\text{min}} (I')$ into the second constraint guarantees that $B^* (I')$ indeed corresponds to the actual payoff that the borrower receives at income $I'$. The third constraint in problem (3) is the next period’s participation constraint of the lender. It ensures that the lender prefers playing his equilibrium strategy to deviating.

I now turn to the characterization of the function $d$, the value to the lender from the most profitable deviation. Suppose the income of the borrower is $I$. Instead of making an equilibrium offer and delivering a payoff of at least $B$ to the borrower, as in problem (3), the lender can make a different offer. The borrower will accept it as long as it delivers her a payoff of at least $I + \beta \theta$. The agents expect that if the lender deviates, the continuation payoff of the borrower at income $I'$ will be $B^d (I')$. The lender chooses $(C, K)$ to maximize his payoff subject to the participation constraint of the borrower, taking $B^d (\cdot)$ as given. Again, the second and third constraints give restrictions on which $B^d (\cdot)$ can be supported as the equilibrium continuation payoffs, so that neither of the agents deviates. To satisfy renegotiation-proofness, the continuation payoff of the lender for income $I'$ is again given by the value function $L$, that is, the continuation payoffs lie on the same frontier where the equilibrium payoffs lie even after the lender has deviated. The value to the lender is minimized with respect to $B^d (\cdot)$ because the lower the value to the lender from the deviation, the higher welfare it is possible to achieve. That is, the lender who contemplates a deviation expects that, as a punishment, the continuation equilibria will be chosen in the least favorable way to him. Problem (5) is an analog of problem (2).

**Claim 2** Any weakly renegotiation-proof equilibrium corresponds to a solution to the recursive problem described by (3) – (5).

Notice that for a fixed $B$, the value to the lender given by problem (3) depends on $I$ in an additively separable way, and the choice of the lender in this problem only depends on $B$. Hence

---

16If $K$ is deterministic, the second and the third constraints can be written as $B^* (s f (K)) \geq B_{\text{min}} (s f (K))$ and $L (s f (K), B^* (s f (K))) \geq d (s f (K))$, for all $s \in [0, 1]$. The form in which the constraints are written in problem (3) means that they have to be satisfied for each pairs of realizations of the random variables $K$ and $s$. 

13
I can rewrite problem (3) using $B$ as the only state variable. Using this observation, define $N (B) = L (I, B) - I$ to be the value to the lender minus income, when the payoff to the borrower is at least $B$.

I want to rewrite problems (3) and (4) in terms the function $N$ instead of $L$. The lowest payoff to the borrower for income $I$, $B_{\min} (I)$, can be written as $\max \{ I + \beta \theta, B_{0} \}$, where $B_{0} = \sup \{ B \mid B \in \arg \max_{B \geq \beta \theta} N (B) \}$. If the promise-keeping constraint in problem (3) does not bind at $\beta \theta$, then $B_{0}$ would be the largest $B$ at which this constraint does not bind. Then problems (3) and (4) can be written as

$$N (B) = \max_{C, K, B} \{ C - E K + \beta E [s f (K) + N (B^* (s f (K))) \}$$

s.t. $C + \beta E B^* (s f (K)) \geq B,$

$$B^* (I') \geq \max \{ I' + \beta \theta, B_{0} \}, \text{ for all } I',$$

$$I' + N (B^* (I')) \geq d (I') \text{ for all } I'.$$

and

$$\hat{d} (I) = \min_{B_d (\cdot)} \left[ \max_{C, K} \{ C - E K + \beta E [s f (K) + N (B_d (s f (K))) \} \right]$$

s.t. $C + \beta E B_d (s f (K)) \geq I + \beta \theta,$

$$B_d (I') \geq \max \{ I' + \beta \theta, B_{0} \} \text{ for all } I',$$

$$I' + N (B_d (I')) \geq d (I') \text{ for all } I'.$$

Problems (5), (6) and (7) define the operator $T$ in the following way. For a pair of functions $(N, d)$, the value generated by (6) is $T_1 (N, d)$, and the value generated by (5) and (7) is $T_2 (N, d)$. Then $T (N, d) = (T_1 (N, d), T_2 (N, d))$. Any fixed point of the operator $T$ corresponds to a set of weakly renegotiation-proof payoff frontiers. However, not any fixed point corresponds to a set of payoffs which is renegotiation-proof. The reason is that renegotiation-proofness also requires that there is no other fixed point that generates payoff frontiers that are Pareto dominating.

I showed in Claim 1 that any Markov equilibrium is weakly renegotiation-proof. It is easy to verify that a Markov equilibrium corresponds to a fixed point of the operator $T$. Define $N^M (I + \beta \theta) = L^M (I) - I$, the value to the lender minus income in a Markov equilibrium. Then $(N^M, L^M)$ is a fixed point of $T$ such that $B_{\min} (I) = B_{\max} (I) = B^M (I)$ and $I + N (I + \beta \theta) = d (I) = L^M (I)$ for all $I$. Then $B'^* (I') = B^* (I') = B^M (I')$ for all $I'$ and $I + T_1 (N, d) (I + \beta \theta) = T_2 (N, d) (I) = L^M (I)$ for all $I$. Although a Markov equilibrium is weakly renegotiation-proof, it is not renegotiation-proof in general. A fixed point corresponding to a renegotiation-proof frontier satisfies $N (B) \geq N^M (B)$ for all $B$.

In this model there is a payoff frontier for each income level. I rewrote the problem in such a way that this family of frontiers is summarized by two functions, $N$ and $d$. Let $(N, d)$ be a
fixed point of $T$ corresponding to a set of renegotiation-proof payoff frontiers $\mathcal{P}$. I will refer to the function $N$ as the “aggregate” renegotiation-proof frontier, because it comprises in itself all the frontiers for different income levels.

For convenience, define $N^d(I + \beta \theta) = d(I) - I$. Figure 1 shows graphically how the set of payoff frontiers $\mathcal{P}$ can be obtained from the functions $N$ and $N^d$. The functions $N$ and $N^d$ are depicted by black solid and dashed curves, respectively. Recall that the frontier for income $I$ is $\mathcal{P}(I) = \{(B, I + N(B)) \mid B \in [B_{\min}(I), B_{\max}(I)]\}$. The lowest equilibrium payoff to the borrower is $B_0$. Consider, e.g., an income level $I$ such that $I + \beta \theta > B_0$. Then $B_{\min}(I) = I + \beta \theta$ is the lowest payoff to the borrower on the renegotiation-proof frontier for income $I$. The graph shows how to find the highest payoff to the borrower for income $I$, $B_{\max}(I)$. First draw the vertical line through $B_{\min}(I)$. Its intersection with $N^d$ is marked with a white dot. Draw a horizontal line through this intersection until it crosses the curve $N$ to obtain $B_{\max}(I)$: $N^d(I + \beta \theta) = N(B_{\max}(I))$. The two black dots on the curve $N$ correspond to the payoff pairs $(B_{\min}(I), N(B_{\min}(I)))$ and $(B_{\max}(I), N(B_{\max}(I)))$. In order to obtain $\mathcal{P}(I)$, the portion of the $N$-curve in between the two black dots should be shifted vertically by $I$. The grey curves show frontiers for different income levels. Since $L(I, B) = I + N(B)$, these frontiers have the same slope along any vertical line. The set of such frontiers for all income levels gives us the set $\mathcal{P}$.

In most papers, the lowest equilibrium payoff to the lender is his outside option. It is either exogenous or is only influenced by physical variables that are determined in equilibrium, for example, the amount of capital, as in Sigouin (2003). In this paper the lender’s lowest equilibrium payoff is not his outside option, which is zero, but his value from the best deviation while keeping
the relationship in place. This value is endogenously determined and is influenced by, and in turn, influences the whole equilibrium structure.

4 Analysis and Results

I start by describing properties of Markov equilibria in the next subsection. Then I will analyze properties of weakly renegotiation-proof equilibria.

4.1 Properties of Markov Equilibria

Consider problem (1) introduced in Section 3. Since random offers are allowed, the lender’s value function is concave, and any concave function is differentiable everywhere but on at most a countable set. Using concavity and the fact that she shock is continuous, I will use the first-order approach to solve the problem. For simplicity of the exposition, I will assume in most of the analysis that \( \theta \) is low enough so that \( L^M(I) > 0 \) for all \( I \), and hence \( L^M(I) = \hat{L}^M(I) \) for all \( I \).

Let \( \lambda^M \) and \( \gamma^M \) be the Lagrange multipliers on the constraints (1b) and \( C \geq 0 \), respectively. The first-order condition for \( C \) is

\[
1 + \lambda^M + \gamma^M = 0.
\]

The Envelope condition for \( I \) is

\[
L^M(0) = 1 - \lambda^M = \gamma^M \in [0, 1].
\]

That is, \( L^M(I) > 0 \) as long as \( C \geq 0 \) binds. Hence we have

**Lemma 1** The function \( L^M \) is concave and \( L^M \in [0, 1] \).

**Lemma 2** \( B^M(I) = \max \{ I + \beta \theta, B^M(0) \} \).

**Investment.**— The first-order condition with respect to \( K \) is

\[
1 = \beta f'(K) Es \left[ L^{M'}(sf(K)) + \lambda M B^{M'}(sf(K)) \right].
\]  

Let \( K^M(I) \) denote the lender’s choice of investment in problem (1) if only deterministic offers were allowed. Furthermore, define \( \bar{I} = \inf \{ I \mid L^{M'}(I) = 0 \} = \sup \{ I \mid C^M(I) = 0 \} \), the highest income level for which the borrower’s consumption, \( C^M \), is zero. The following proposition characterizes investment in a Markov equilibrium.
Proposition 1  

(i) \( K^M (I) \) \begin{cases} 
= K^M (0) & \text{for } I \leq B^M (0) - \beta \theta \\
\text{strictly increasing} & \text{for } I \in [B^M (0) - \beta \theta, \bar{I}] \\
= K^M (\bar{I}) & \text{for } I \geq \bar{I}. 
\end{cases}

(ii) No random offers are used, that is, investment \( K^M (I) \) is made at \( I \) with probability one.

(iii) \( K^M (0) < K^{FB} \).

(iv) \( K^M (\bar{I}) > K^{FB} \).

The above proposition says that if the participation constraint of the borrower does not bind, then the lowest level of investment is made, which is strictly below the first-best level, \( K^{FB} \). On the other hand, if the borrower consumes, then the highest level of investment is made, which exceeds \( K^{FB} \). Otherwise, the investment strictly increases in the borrower’s income.

What causes inefficiencies in this environment is that the borrower controls the output and can choose not to repay to the lender. If the income is low today and the lender does not have to deliver more than the outside option to the borrower, then he underinvests. This is because the borrower cannot commit to repay to the lender tomorrow for his current investment. Without an up-front compensation, the lender is only willing to invest an amount that makes the surplus from the future relationship worth it. A higher investment can only be induced by a higher up-front payment from the current output. On the other hand, when the borrower’s income is sufficiently high, investment exceeds the first-best level. This increases the expected output, and hence the expected investment in the next period. That is, the lender uses overinvestment as a way of committing himself to high investments in the next period. Such a contract is attractive to the borrower, and she is willing to pay for it.\(^{17}\)

Existence.— First I show that a Markov equilibrium exists, and that it is not autarky. Notice that the function \( B^M (I) \) appears in the constraint of problem (1), and this function is affected by the function \( L^M (I) \) (see the proof of Lemma 2). As a result, the monotonicity property of Blackwell’s sufficient conditions fails to hold. Therefore, the usual dynamic programming techniques cannot be used to show existence and uniqueness of the fixed point. I define an operator, \( T^M \), that maps \( L^M \) into \( T^M (L^M) \), and apply Schauder’s fixed-point theorem to establish the existence of a Markov equilibrium. By Claim 1, any Markov equilibrium is weakly renegotiation-proof. Hence part (i) of the following proposition shows that a weakly renegotiation-proof equilibrium exists. Then it is straightforward to show that a renegotiation-proof equilibrium also exists, see part (iii).

\(^{17}\)If the borrower had access to a storage technology with the rate of return equal \( \eta \), then \( K^M (\bar{I}) > K^{FB} \) would still hold as long as \( \eta < \rho \), where \( \rho = \beta^{-1} - 1 \) is the rate of time preference. If \( \eta = \rho \) then the first-best level of investment is indeterminate, and if \( \eta < \rho \) then \( K^{FB} = \infty \).
Proposition 2  (i) A Markov equilibrium exists.

(ii) If $\theta < S^{FB}/\beta$, then autarky is not a Markov equilibrium.

(iii) A renegotiation-proof equilibrium exists.

Complete Information Case.— Suppose there is no uncertainty. Assume, for example, that $s_t = \int_0^1 s dG(s) \equiv a$ is constant in all periods. It is straightforward to show that parts (i) – (iii) of Proposition 1 still apply. Let $\{K_t\}_{t \geq 1}$ be a sequence of investment levels over time along a Markov equilibrium path if there is no uncertainty. That is, $K_{t+1} = K^M(a f (K_t))$ for $t \geq 1$, and $K_1 = K^M(I_1)$, where $I_1$ is some initial level of income of the borrower. Since $K^M(0) < K^{FB}$, if $I_1$ is low enough, then there is underinvestment initially.

Claim 3 If there is no uncertainty, then $K^M(I) = \max_I K^M(I) = K^{FB}$ and $\exists \tau$ such that for $t \geq \tau$, $K_t = K^{FB}$.

Notice the differences and similarities between the investment decisions in the Markov equilibria with and without uncertainty. In both cases the commitment problem on the side of the borrower causes underinvestment. If the output is deterministic, then over time the borrower accumulates enough liquidity to buy the first-best level of investment in each period. The borrower’s future participation constraints seize to bind and the first-best surplus is achieved from that point on. On the contrary, if the output is uncertain, the liquidity problem is never eliminated.

4.2 Properties of Weakly Renegotiation-Proof Equilibria

In this subsection I will characterize some properties of weakly renegotiation-proof equilibria. Consider problem (6). Let $\lambda$ and $\gamma$ be the Lagrange multipliers on the promise-keeping constraint and $C \geq 0$, respectively. The first-order condition with respect to $C$ is $-1 + \lambda + \gamma = 0$. The Envelope condition with respect to $B$ is $-N'(B) = \lambda = 1 - \gamma$. Since $\gamma \geq 0$, we have $-N'(B) \in [0,1]$. Furthermore, $-N'(B) < 1$ whenever $C \geq 0$ binds. Define $\overline{B} = \inf\{B \mid N'(B) = -1\} = \sup\{B \mid C(B) = 0\}$, the highest value to the borrower such that the borrower’s consumption is zero. The aggregate payoff frontier, $N$, is downward sloping because an increase in $B$ is costly to the lender. The above results are summarized in Lemma 3. Concavity follows because random offers are allowed.

Lemma 3 The function $N$ is concave and $-N' \in [0,1]$.

Continuation Values and Bargaining Power.— The following two claims describe optimal choices of the continuation values $B^*$ and $B^d$. 

18
Claim 4 If the promised value to the borrower is $B$, then the following choice of continuation values $B^*$ is optimal:

$$B^*(I'; B) = \begin{cases} B, & \text{if } B_{\min}(I') \leq B \leq B_{\max}(I'), \\ B_{\max}(I'), & \text{if } B > B_{\max}(I'), \\ B_{\min}(I'), & \text{if } B < B_{\min}(I'). \end{cases}$$

Since $N$ is concave, it is optimal to set $B^*(I')$ equal to $B$ whenever possible. For $I'$ at which the borrower would deviate at $B$, $B^*(I')$ is set to $B_{\min}(I')$. Similarly, for $I'$ at which the lender would deviate at $B$, $B^*(I')$ is set to $B_{\max}(I')$. The value $B_{\min}(I')$ corresponds to the point on the frontier for income $I'$ where the lender extracts all the surplus, or, in other words, has the full bargaining power. Similarly, $B_{\max}(I')$ corresponds to the point on the frontier where the borrower extracts all the surplus. Hence Claim 4 shows how the bargaining power of the agents changes endogenously over time depending on the state of the economy.

Lemma 4 The function $d$ is concave and $d'(I) \in [0, 1]$.

Claim 5 (i) If $I$ and $\theta$ are low enough (so that $d'(I) = 1$), then $B^d(I') = B_{\max}(I')$ for all $I'$.

(ii) If $I$ is high enough (so that $d'(I) = 0$), then $B^d(I') = B_{\min}(I')$ for all $I'$.

The result of Claim 5 is based on the fact that $N(B)$ is decreasing in $B$ and $B + N(B)$ is increasing in $B$. The value to the lender can be written as $\max_K -K + \beta E[sf(K) + N(B^d(sf(K))) + \min\{I, \beta EB^d(sf(K)) - \beta \theta\}$, where the last term is the repayment $R = I - C$. For $I$ low enough the value to the lender equals $\max_K I - K + \beta E[sf(K) + N(B^d(sf(K)))$, which is minimized if $B^d(sf(K))$ is the highest possible for each $s$, so that $N(B^d(sf(K)))$ is the lowest possible for each $s$. On the other hand, if income is very high, the value to the lender is $\max_K -K + \beta E[sf(K) + N(B^d(sf(K))) + B^d(sf(K))] - \beta \theta$, which is minimized if $B^d(sf(K))$ is the lowest possible for each $s$, so that $N(B^d(sf(K))) + B^d(sf(K))$ is the lowest possible for each $s$.

It is worth comparing Claims 4 and 5. For example, for a high enough current income $I$, $B^*(I'; I + \beta \theta) = B_{\max}(I')$ for all $I'$, while $B^d(I'; I) = B_{\min}(I')$. This means that when the income is high, the lender making the equilibrium offer expects that in the next period the borrower will have the full bargaining power. On the other hand, if the lender deviates, he has the full bargaining power in the next period.

Investment. Let $\beta g(s) \mu(sf(K))$ and $\beta g(s) \phi(sf(K))$ be the Lagrange multipliers on the participation constraints of the borrower, $B^*(sf(K)) \geq \max\{sf(K) + \beta \theta, B_0\}$, and the lender,
\[ sf(K) + N(B^*(sf(K))) \geq d(sf(K)). \] Then the first-order condition with respect to \( K \) is
\[
1 = \beta f'(K) Es \left\{ 1 + \phi(sf(K)) \left[ 1 - d'(sf(K)) \right] - \mu(sf(K)) \right\}. \tag{9}
\]

Notice that a binding participation constraint of the borrower \((\mu(sf(K)) > 0)\) decreases the marginal value of income to the lender next period, and hence the lender’s marginal value of investment. On the other hand, a binding participation constraint of the lender \((\phi(sf(K)) > 0)\) increases his marginal value of investment. (By Lemma 4, \( 1 - d'(sf(K)) \geq 0 \).)

The following proposition describes the property of the lender’s investment decision. Let \( K(B) \) denote the lender’s choice of investment in problem (6) if only deterministic offers were allowed.

**Proposition 3**

1. **\( i \)** \( K(B) \)
   - \( = K(B_0) \) for \( B \in [\beta_0,B_0] \),
   - strictly increasing for \( B \in [B_0,\bar{B}] \),
   - \( = K(\bar{B}) \) for \( B \geq \bar{B} \).

2. **\( ii \)** No random offers are used, that is, at \( B \) investment \( K(B) \) is made with probability one.

3. **\( iii \)** \( K(B_0) < K^{FB} \).

4. **\( iv \)** If \( \bar{B} \leq B_{\text{max}}(0) \), then \( K(\bar{B}) = K(B_{\text{max}}(0)) = K^{FB} \).

5. **\( v \)** If \( \bar{B} > B_{\text{max}}(0) \), then \( K(B_{\text{max}}(0)) < K^{FB} \) and \( K(\bar{B}) > K^{FB} \).

Part (i) claims that the investment is increasing in the promised value to the borrower. Since the value to the borrower is positively related to the borrower’s income, investment is also positively related to income. For example, \( I_1 > I_2 \) implies \( K(B_{j}(I_1)) \geq K(B_{j}(I_2)) \), \( j \in \{\text{min, max}\} \).\(^{18}\)

Thus, according to Propositions 1 and 3, the size of investment is positively related to the borrower’s liquidity, in line with empirical predictions of the corporate finance literature (see Hubbard 1998 for a survey).

Part (iii) of Proposition 3 says that the lender underinvests as long as the promised value to the borrower is low enough. When the lender does not have to deliver a high value to the borrower, he underinvests because his own expected marginal value of the next period’s income is less than one, as it is costly for the lender to extract the income from the borrower.

Parts (iv) and (v) show that there can be two possible cases. In the first case (which occurs if at \( B_{\text{max}}(0) \) the borrower consumes) the maximum level of investment is the first best. In the second case (which occurs if at \( B_{\text{max}}(0) \) the borrower does not consume) the maximum investment exceeds \( K^{FB} \). I cannot show analytically for which parameter values which of the two cases can

\(^{18}\)Investment \( K(B_{\text{min}}(I)) \) is made when the borrower’s income is \( I \) and she is promised her outside option.
arise. However, in numerical computations I always obtained the second case, even for very high values of the discount factor $\beta$.\footnote{\textsuperscript{19}In the extension of the model where the borrower can choose to stay in the relationship if she rejects all contracts, the first case occurs if the outside option of the borrower is sufficiently low and the discount factor is sufficiently high.}

The intuition for overinvestment is essentially the same as in the case of a Markov equilibrium (see the discussion following Proposition 1). When the lender has to deliver a high value to the borrower, he uses overinvestment as a way of promising high investment to the borrower in the next period.

Let $\{s_t\}_{t \geq 1}$ be a sequence of random shock realizations. Consider sequences $\{B_t\}_{t \geq 1}$ and $\{K_t\}_{t \geq 1}$, where $K_t = K(B_t)$ and $B_{t+1} = B^*(s_t f(K(B_t)); B_t)$ for all $t \geq 1$. The following lemma establishes some properties of the dynamics of the equilibrium behavior.

**Lemma 5**

(i) If $B_t \in [B_{\min}(0), B_{\max}(0)]$, then $B_{t+1} \geq B_t$.

(ii) If $B_t \geq \min \{B_{\max}(0), \bar{B}\}$, then $B_\tau \geq \min \{B_{\max}(0), \bar{B}\}$ for all $\tau \geq t$.

Proposition 4 characterizes the properties of $\{K_t\}$, the equilibrium investment time series. The two cases in Proposition 4 correspond to the two cases in Proposition 3, parts (iv) and (v).

**Proposition 4**

(i) If $\bar{B} \leq B_{\max}(0)$ then $K_t$ is non-decreasing over time, attaining the maximum level, which is $K^{FB}$, with probability one.

(ii) If $\bar{B} > B_{\max}(0)$, then $K_t$ is always inefficient and fluctuates between underinvestment and overinvestment along any equilibrium path.

Lemma 5 and Proposition 4 lead to the following prediction. As long renegotiation-proof equilibria are different from a Markov equilibrium, we have $B_{\max}(0) > B_{\min}(0)$ and hence $K(B_{\max}(0)) > K(B_{\min}(0))$, by part (i) of Proposition 3. It then follows from Lemma 5 that along an equilibrium path underinvestment is more severe initially. In addition, investment is initially increasing over time, until the point is reached after which $K_t \geq K(B_{\max}(0))$ for all $t$, and investment starts fluctuating (as long as $\bar{B} > B_{\max}(0)$ so that $K(B_{\max}(0)) < K^{FB}$). Hence, although the liquidity problem is relaxed over time, it is not completely eliminated in general. The described behavior of investment is consistent with empirical findings that borrowing constraints are more important for younger firms, and that firm growth is negatively related to firm age. (See, e.g., Evans 1987.)
4.3 Comparative Statics with respect to the Outside Option $\theta$

This subsection analyzes how investment and welfare in a Markov equilibrium are affected by a change in the outside option of the borrower. Although I could not prove the same results analytically for renegotiation-proof equilibria, I show numerically in Section 5 that they do hold. In what follows I use subscripts to denote the dependence of equilibrium value and policy functions, and certain variables, on the parameter $\theta$.

**Lemma 6** $L_0^M (I)$ is decreasing in $\theta$ for all $I$.

The above lemma says that an increase in the borrower’s outside option is costly for the lender, because it makes it harder for him to satisfy the borrower’s participation constraint. The following proposition establishes that the effect of $\theta$ on the investment in Markov equilibria depends the level of the borrower’s income.

**Proposition 5** Let $\theta_2 > \theta_1$.

(i) For $I \in \left[ B_{\theta_j}^M (0) - \beta \theta_j, \bar{I}_{\theta_j} \right]$, $j \in \{1, 2\}$, $K_{\theta_2}^M (I) > K_{\theta_1}^M (I)$.

(ii) For $I \in \left[ 0, B_{\theta_j}^M (0) - \beta \theta_j \right]$, $j \in \{1, 2\}$, $K_{\theta_2}^M (I) < K_{\theta_1}^M (I)$.

Part (i) says that for income levels such that the borrower’s participation constraint binds and her consumption is zero, the investment strictly increases with $\theta$. Part (ii) says that for income levels so low that the borrower’s participation constraint does not bind, the investment strictly decreases with $\theta$. The reason is the following. If the borrower’s participation constraint binds, then the borrower with a higher $\theta$ would prefer to reject the lender’s old contract. Therefore, the lender offers a higher investment level to compensate the borrower for repayments. (Increasing consumption is not optimal for $I < \bar{I}$.) This, in turn, makes high realizations of the borrower’s income less attractive to the lender, as it is now more costly to extract the borrower’s income. Therefore, when the borrower’s participation constraint does not bind and the lender only takes into account his own future gains when he makes the investment decision, he has less incentives to invest. The above result leads us to the following proposition regarding the nonmonotonicity of welfare in the outside option.

**Proposition 6** (i) If $\theta$ is high enough so that $B_{\theta}^M (I) = I + \beta \theta$ for all $I$, then $L_0^M (I) + B_{\theta}^M (I)$ is strictly increasing in $\theta$ for all $I$.

(ii) If $\theta$ is low enough and $\beta$ is high enough, then $L_0^M (I) + B_{\theta}^M (I)$ is strictly decreasing in $\theta$ for $I \in \left[ 0, B_{\theta}^M (0) - \beta \theta \right]$.
Proposition 6 suggests that, in general, the social surplus is non-monotonic in $\theta$: it is decreasing for low values of $\theta$, and increasing for high values of $\theta$. As I described above, an increase in the outside option has two effects on investment: negative for low income realizations and positive for high income realizations. Which effect dominates, in general, depends on how large the threshold level of income is below which the participation constraint of the borrower does not bind, $B^M_\theta (0) - \beta \theta$. When the outside option is low, this threshold income level is high, and income realizations are likely to be smaller than the threshold. Hence, investment decreases on a large range of states, and the negative effect dominates. Since the equilibrium investment is inefficiently small, this implies that social welfare decreases in the outside option. Similarly, if the outside option is high, the threshold is low and the borrower’s income is likely to fall above the threshold. Investment increases on a large range of states, which implies that social welfare increases in the outside option.

An interpretation of the above result is that the strength of competition in the credit market has an ambiguous effect on social welfare. This is consistent with empirical predictions that bank competition can have both positive and negative economic effects. (See the Introduction for the references.)

The result that the value of the project is non-monotonic in the borrower’s outside option is in contrast to the prediction of Albuquerque and Hopenhayn (2004), who assume that the lender has full commitment power. Their model predicts that a lower outside option of the entrepreneur increases the value of the project, because it relaxes the borrower’s participation constraints imposed on the optimal contract. Similarly, the paper by Opp (2007), which assumes no commitment by the lender, suggests that the steady state social surplus is non-increasing in the borrower’s autarkic productivity. My paper shows that such a result does not hold in general once the incentive problem of the lender caused by the short-term commitment has to be taken into account.

5 Numerical Computations

In the numerical computations presented in this section I used the following parameter values. The deterministic part of the production function is $f(K) = 2K^{.75}$, the discount factor $\beta = 1/(1 + \rho)$, where $\rho = .1$, and the distribution of the shock is uniform, $G = U[0, 1]$.

A. Payoff Frontiers and Investment: Short-Term Commitment vs. No Commitment. The dashed curve on panel a of Figure 2 plots the aggregate renegotiation-proof frontier, $N(B)$, for $\theta = 0$. The circle and the asterisk on this frontier correspond to the points $(B_{\max} (0), N(B_{\max} (0)))$ and $(\hat{B}, N(\hat{B}))$, respectively. The part of the curve to the left of the
circle is the renegotiation-proof frontier for zero income. The functions $N$ is downward sloping and concave, with the absolute slope between zero and one, as suggested by Lemma 3. The absolute slope is exactly one to the right of the asterisk. The solid curve is the renegotiation-proof frontier for the model without commitment, similar to Thomas and Worrall (1994). A portion of this frontier coincides with part of the first-best frontier. One can see that less aggregate welfare is generated with short-term commitment relative to the no-commitment case. Moreover, efficiency is achieved in the no-commitment case, but not in the short-term commitment case.

Figure 2. Renegotiation-proof frontiers and investment with short-term commitment vs. no commitment.

Panel b of Figure 2 plots the corresponding investments in the two cases. The shape of the investment policy function $K(B)$ (dashed line) is as described in part (i) of Proposition 3. Since in this case $B_{\text{max}}(0) < \bar{B}$, part (v) of Proposition 3 and part (ii) of Proposition 4 apply. In particular, investment is always inefficient, and both underinvestment and overinvestment occur. The policy function in the no-commitment case exhibits underinvestment for low values of $B$, and it coincides with the first-best level for high values of $B$.

**B. Time Series.** Figure 3 plots time series generated by the renegotiation-proof equilibria following a sequence of random shocks in the case of $\theta = 0$. The initial conditions are $I = 0$ and $B = B_{\text{min}}(0)$. On panel b, the solid (dashed) series is the value to the borrower (lender). The horizontal lines mark $B_{\text{max}}(0)$ and $\bar{B}$. For the first few periods $B_t < B_{\text{max}}(0)$, and underinvestment is most severe. Eventually $B_t \geq B_{\text{max}}(0)$ (for $t \geq 4$), as suggested by part (ii) of Lemma 5. When $B_t \geq \bar{B}$, the maximum investment is made, and it exceeds the first-best level, $K_t = K(\bar{B}) > K^{FB}$, see parts (i) and (iv) of Proposition 3. The borrower consumes only when $B_t \geq \bar{B}$ and $K_t = K(\bar{B})$. The economy exhibits fluctuations between underinvestment and overinvestment, as suggested by part (ii) of Proposition 4. Comparing panels c and d, one can see that the investment is positively
correlated with the borrower’s liquidity. In sharp contrast, in the model without commitment the borrower’s value and the investment are increasing over time, and the investment attains $K_{FB}$ with probability one. (See panel b of Figure 2.) That is, investment is uncorrelated with income in the long run.

Figure 3. Time series.

C. History-Dependence, the Outside Option, and State-Contingent Debt. Figure 4 plots the aggregate renegotiation-proof frontiers, $N(B)$, and the aggregate Markov frontiers, $N^M(B)$, for different values of $\theta$. Again, the portions of the solid curves to the left of the circle are the renegotiation-proof frontiers for zero income. The Markov frontiers for zero income, which are singletons $(B^M(0), L^M(0))$, are the left end points of the dashed curves. One can see that as $\theta$ increases, $N$ becomes closer and closer to $N^M$, and for sufficiently high values of $\theta$ the two curves are visually indistinguishable. In my numerical computations $\|N_\theta(B) - N^M_\theta(B)\|_{\sup}$ is strictly decreasing in $\theta$. The interpretation is the following. Recall that a Markov equilibrium depends only on the current income realization, while renegotiation-proof equilibria depend on the whole previous history of the play. Therefore, my results suggest that as the outside option of the borrower increases, the history-dependence in the renegotiation-proof equilibria matters less and

\[ \|L_\theta(I, I + \beta \theta) - L^M_\theta(I)\|_{\sup}, \quad L_\theta(0, \beta \theta) - L^M_\theta(0), \]
\[ \|L_\theta(I, B_j(I)) + B_j(I) - L^M_\theta(I) - B^M_\theta(I)\|_{\sup}, \quad L_\theta(0, B_j(0)) + B_j(0) - L^M_\theta(0) - B^M_\theta(0), \quad j \in \{\min, \max\}. \]

The following values are also strictly decreasing in $\theta$: $\|L_\theta(I, I + \beta \theta) - L^M_\theta(I)\|_{\sup}$, $L_\theta(0, \beta \theta) - L^M_\theta(0)$, $\|L_\theta(I, B_j(I)) + B_j(I) - L^M_\theta(I) - B^M_\theta(I)\|_{\sup}$, $L_\theta(0, B_j(0)) + B_j(0) - L^M_\theta(0) - B^M_\theta(0)$, $j \in \{\min, \max\}$. The difficulty of proving these results analytically is due to the fact that it is not true, e.g., that $L(I, I + \beta \theta) - L^M(I)$ is decreasing in $\theta$ for all $I$, so the convergence is not pointwise.
less. In fact, especially for high values of $\theta$, the Markov equilibrium does nearly as well in terms of welfare as the renegotiation-proof equilibria.

![Figure 4. The aggregate renegotiation-proof and Markov payoff frontiers. $S^F_B \approx .79$.](image)

Another interpretation of the above result is due to the observation that history-dependence is related to the use of state-contingent debt, as I explain below. The value to the lender can be interpreted as debt. The next period’s debt can be viewed as being part of a contract, indicating to the parties which equilibrium will be played in the next period. Since contracts are complete, the next period’s debt can be conditioned on the next period’s output. In the case of a Markov equilibrium, the current income of the borrower fully determines which equilibrium is played in a given period. That is, there is a one-to-one correspondence between income and debt. Therefore specifying the level of debt in a contract is irrelevant, as is the possibility of making the level of the next period’s debt conditional on the next period’s output. The above result therefore suggests that for sufficiently high values of the borrower’s outside option, the benefit of having state-contingent debt is rather limited. This provides a possible explanation for why we often observe debt contracts that do not specify state-contingent repayment schedules, even when this possibility appears feasible.

**D. Nonmonotonicity of Investment and Social Welfare in $\theta$.** Panel a of Figure 5 plots investment in the renegotiation-proof equilibria for different values of $\theta$. The value $K(I + \theta \beta)$ is the investment made at income $I$ when the borrower’s promised value is her outside option. As one can see, the effect of $\theta$ on investment depends on the income level. (For the Markov equilibrium, it is suggested by Proposition 5.) This results in nonmonotonicity of social welfare in $\theta$, as shown on panel b. Line i plots the social welfare in the Markov equilibrium for zero income, and the shape of this line is as suggested by Proposition 6. Lines ii, iii, and iv correspond to the renegotiation-proof
equilibria. Line ii (iii) plots the social welfare which is generated if the relationship begins with zero income, and all the ex-ante surplus goes to the lender (borrower). Line iv is the social welfare if the relationship starts with a very high income level.

Figure 5. a: Investment in renegotiation-proof equilibria for different $\theta$. b: Social welfare as functions of $\theta$.

6 Final Remarks

The main question addressed in this paper is how the amount of commitment power in a lender-borrower relationship affects the agents’ incentives and social welfare. I show that assuming that the lender has some commitment power as opposed to no commitment power at all makes it more difficult to provide the lender with the incentives to invest into the borrower’s production technology. As a result, lower welfare is generated in general. While the majority of papers on lending with limited commitment focuses on the borrower’s incentives, this paper shows that a realistic assumption of the possibility of writing short-term contracts makes it crucial to analyze the incentives of the lender.

The following results of my model are in contrast to predictions with full- or no-commitment assumptions on the lender’s side. I find that investment is positively related to the borrower’s liquidity, consistent with empirical facts. In addition, both investment and overinvestment can occur in equilibrium. I also show that an increase in the outside option of the borrower, which can be interpreted as a measure of competitiveness of the credit market, affects the lender’s incentives to invest both positively and negatively. Therefore, the social welfare is generally non-monotonic in the strength of competition in the credit market. Numerical results suggest that as the outside option of the borrower increases, the welfare benefit of using history-dependent renegotiation-proof equilibria as opposed to simple Markov equilibria declines.
Appendix: Omitted Proofs

**Proof of Claim 1.** Since for each $I \mathcal{P}(I)$ is a singleton, no payoff in $\mathcal{P}(I)$ Pareto dominates another payoff in $\mathcal{P}(I)$. Furthermore, since equilibrium strategies in each period only depend on $I$, the continuation payoffs used to construct $\mathcal{P}(I)$ for each $I$, automatically lie in $\mathcal{P}$. ■

**Proof of Claim 2.** For any weakly renegotiation-proof set of payoffs in this model, for all $I \mathcal{P}(I)$ can be written in a form \( \{(B, \tilde{L}(I, B)) | B \in [B_1(I), B_2(I)]\} \) for some functions $\tilde{L}$ and $\tilde{d}$, where $B_2(I)$ solves $\tilde{L}(I, B_2(I)) = \tilde{d}(I)$, and $\tilde{L}$ is downward sloping on $[B_1(I), B_2(I)]$ for each $I$. I want to show that $\tilde{L} = L$ and $\tilde{d} = d$, where $L$ and $d$ satisfy (3) – (5). Part (b) of Definition 1 says that the continuation payoffs must lie in $\mathcal{P}$. In problems (3) and (4) it corresponds to the following two conditions. First, the next-period’s participation constraints guarantee that $B^*(I'), B^d(I') \in [B_1(I'), B_2(I')]$. Second, the continuation payoffs of the lender for income $I'$ are given by the same value function, and hence $(B, L(I, B))$ is generated by using the continuation payoffs $(B^*(I'), L(I', B^*(I'))), (B^d(I'), L(I', B^d(I'))) \in \mathcal{P}(I')$ for all $I'$. Part (a) of Definition 1 says that there is no payoff in $\mathcal{P}(I)$ that Pareto dominates $(B, L(I, B))$. I need to verify that if $B$ is the payoff to the borrower at income $I$, then (3) and (4) are defined such $L(I, B)$ is maximized. This is true for the following reasons. In (3) the value is maximized with respect to $B^*(.)$ to deliver the highest possible payoff to the lender following the equilibrium contract. And in (4) the value is minimized with respect to $B^d(.)$, in order to minimize $d(I)$. As a result, the lender’s participation constraints in future periods are as weak as possible, which in turn maximizes $L$ in (3) and minimizes $\tilde{d}$ in (4) and $d (5)$. Hence $\tilde{L} = L$ and $\tilde{d} = d$. ■

**Proof of Lemma 2.** Suppose at $I = 0$ the lender could ignore the participation constraint of the borrower, (1b). Then his maximization problem at $I = 0$ would be $\max_K -K + \beta \int_0^1 L^M(s f(K)) \, dG(s)$. The strict concavity of $f$ and the concavity of $L^M$ imply that this problem has a unique solution, $K_0$. If autarky is not an equilibrium, that is, $L^M$ is not constant zero, then $K_0 > 0$. There are two cases to be considered. Case 1: (1b) does not bind at $I = 0$: $B^M(0) > \beta \theta$. This case happens if $\theta$ is low enough so that $\beta[Esf(K_0) + \beta \theta] > \beta \theta$. Then at $I = 0$ the lender invests $K_0$. As long as the constraint (1b) does not bind, that is, for $I \in [0, B^M(0) - \beta \theta]$, the lender solves the same problem, and hence offers the same contract, $(0, K_0)$. Therefore $B^M(I) = B^M(0)$ for $I \in [0, B^M(0) - \beta \theta]$. The value to the lender on this interval is $L^M(I) = I + L^M(0)$. As the income increases further, the constraint (1b) starts to bind and hence $B^M(I) = I + \beta \theta$ for $I \geq B^M(0) - \beta \theta$. Case 2: (1b) binds for all $I$: $B^M(I) = I + \beta \theta$ for all $I$. This happens if $\theta$ is high enough so that $\beta[Esf(K_0) + \beta \theta] \leq \beta \theta$. (Notice that since with $K_0$ the constraint (1b) is violated, the lender is forced to invest more that $K_0$.) Combining the two cases, we have
\[ B^M(I) = \max \{ I + \beta \theta, B^M(0) \}. \]

**Lemma 7** Let \( \theta < S^{FB}/\beta \). Then \( \exists I \) such that for any \( \delta > 0 \), \( [L^M(I + \delta) - L^M(I)]/\delta > 0 \).

**Proof of Lemma 7.** From Lemma 1, \( [L^M(I + \delta) - L^M(I)]/\delta \geq 0 \) for all \( I \). Suppose that for all \( I \) and any \( \delta > 0 \) \( [L^M(I + \delta) - L^M(I)]/\delta = 0 \). That is, \( L^M(I) = l > 0 \) for all \( I \). Notice that it must be the case that \( (1b) \) binds for all \( I \) (otherwise \( L^M(I) > 0 \) for some \( I \)). Hence \( B^M(I) = I + \beta \theta \) for all \( I \). Suppose that \( I \) large enough so that \( C^M(I) > 0 \). Then the maximization problem of the lender is \( \max K - K + \beta l + \beta Esf(K) - \beta(1 - \beta)\theta \). The solution is clearly \( K^{FB} \). Since \( L^M(I) + B^M(I) - I = l + \beta \theta \) for all \( I \), the generated surplus is the same at all income levels. Hence \( K^M(I) = K^{FB} \) for all \( I \). At \( I = 0 \), the consumption of the borrower is \( C^M(0) = \beta \theta - \beta EB^M(sf(K^{FB})) = \beta(1 - \beta)\theta - \beta Esf(K^{FB}) = (1 - \beta) [\beta \theta - S^{FB}] - K^{FB} < 0 \) because \( \theta < S^{FB}/\beta \), a contradiction. 

**Proof of Proposition 1.** (i) If \( B^M(0) - \beta \theta > 0 \), then \( (1b) \) does not bind at \( I = 0 \). Therefore, for \( I \leq B^M(0) - \beta \theta \) the lender offers the same contract \( (0, K^M(0)) \). For \( I \geq \bar{I} \), \( L^M(I) = 0 \), the marginal product of capital is the same for all \( I \geq \bar{I} \), and hence \( K^M(I) = K^M(\bar{I}) \) for all \( I \). On this interval \( C^M(I) > 0 \), and a one-dollar increase in the borrower’s income leads to a one-dollar increase in her consumption. Let \( I \in [B^M(0) - \beta \theta, \bar{I}] \). Consider a marginal increase in the income of the borrower from \( I \) to \( I + \varepsilon \). Since \( \beta EB^M(sf(K^M(I))) - \beta \theta = I < I + \varepsilon \), and increasing consumption is not optimal, investment must strictly increase.

(ii) Let \( I_1, I_2 \in [B^M(0) - \beta \theta, \bar{I}] \), \( I_1 < I_2 \), and \( I = \pi I_1 + (1 - \pi) I_2 \), where \( \pi \in (0,1) \). Denote \( K_\pi = \pi K^M(I_1) + (1 - \pi) K^M(I_2) \). For \( I \leq \bar{I} \), \( C^M(I) = 0 \), and hence at the optimum the value to the lender from solving problem (1) at income \( I \) is \( I - K^M(I) + \beta EL^M(sf(K^M(I))) \). Since \( L^M(I) \) is concave in \( I \) and \( f \) is strictly concave, \(-K + \beta EL^M(sf(K)) \) is concave in \( K \). Hence the value to the lender from solving problem (1) at income \( I_\pi \) is \( \hat{T}^M L^M(I_\pi) = I_\pi - K^M(I_\pi) + \beta EL^M(sf(K^M(I_\pi))) + (1 - \pi)[I_2 - K^M(I_2) + \beta EL^M(sf(K^M(I_2)))] = \pi \hat{T}^M L^M(I_1) + (1 - \pi) \hat{T}^M L^M(I_2) \). The first inequality follows from the fact that \( K^M(I_\pi) \) is an optimal choice at \( I = I_\pi \). The second inequality follows from concavity of \(-K + \beta EL^M(sf(K)) \). Furthermore, \(-K + \beta EL^M(sf(K)) \) is strictly concave for \( K \) such that \( f(K) \in [B^M(0) - \beta \theta, \bar{I}] \). That is, the second inequality is strict if \( I_1, I_2 \in [B^M(0) - \beta \theta, \bar{I}] \). Hence \( K^M(I) \) will be the actual investment choice made at income \( I \), and no random offers will be made.

(iii) I want to show that \( K^M(0) = \min_I K^M(I) < K^{FB} \). Case 1) At \( I = 0 \), \( (1b) \) holds with a strict inequality. (This happens if \( \theta \) is sufficiently low.) Then \( K^M(0) = \arg \max -K + \beta \theta - \beta Esf(K^{FB}) = \beta(1 - \beta)\theta - \beta Esf(K^{FB}) = (1 - \beta) [\beta \theta - S^{FB}] - K^{FB} < 0 \), a contradiction.
Suppose that \( f \) with respect to \( I \) for all \( L \). Since for all \( I' \), \( K_M (0) = K_{FB} \). Then it must be the case that for all \( I \leq f (K_{FB}) \), \( L_M (0) = L_M (I) \) and \( K_M (I) = K_{FB} \). Therefore, \( L_M (I) = 0 \) for all \( I \), for otherwise the borrower never consumes. Lemma 7 gives a contradiction.

(iv) Denote \( K_M = K_M (I) \). For \( I \geq I' \), \( L_M = 0 \), and hence the first-order condition with respect to \( K \) is \( 1 = \beta f' (K) E s [L_M (sf (K)) + B_M (sf (K))] \). We have \( L_M (I) + B_M (I) = 1 \) for \( I < B_M (0) - \beta \theta \) and \( L_M (I) + B_M (I) \geq 1 \) for \( I \geq B_M (0) - \beta \theta \). Furthermore, since \( L_M (I) \neq 0 \) for all \( I \) (see Lemma 7), we have that \( L_M (I) + B_M (I) > 1 \) for \( I \in [B_M (0) - \beta \theta, I'] \). Hence \( \beta f' (K_M) E s [L_M (sf (K_M)) + B_M (sf (K_M))] \geq \beta f' (K_{FB}) E s = 1 \), and therefore \( K_M \geq K_{FB} \).

Suppose that \( K_M = K_{FB} \). Then it must be the case that \( f (K_M) \leq B_M (0) - \beta \theta \). Then all equilibrium income realizations lie in \( [0, B_M (0) - \beta \theta] \), for which \( C^M (I) = 0 \), a contradiction.

**Proof of Proposition 2.** (i) First, notice that \( 0 \leq L_M (I), B_M (I) \leq I + S_{FB} \). Next, I define the set of possible candidates for \( L_M \). Consider the following set of functions: \( \Gamma = \{ L_g^M \mid L_g^M \in C[0, \infty), L_g^M \geq 0, L_g^M \text{ is concave}, \forall x L_g^M (x) \in [0, 1], L_g^M (x) = 0 \text{ on } [S_{FB}, \infty) \} \). From Lemma 1 we know that \( L_g^M \) is increasing and concave, with slope less than one. If \( I \geq S_{FB} \), then the monopolistic lender clearly unable to offer a contract that specifies \( C = 0 \), because the discounted present value from a contract cannot exceed \( S_{FB} \). Therefore, it follows that \( L_g^M \neq 0 \) on \( [S_{FB}, \infty) \). Hence, if there exists an equilibrium, \( L_g^M \in \Gamma \). Observe that \( \Gamma \) with the supremum norm is a convex compact set. I am going to define a fixed-point operator on \( \Gamma \), where \( \Gamma \) is the set of potential candidates for \( L_M \). For all \( L_g^M \in \Gamma \), let \( K_{0g} = \arg \max_K -K + \beta \int_0^1 L_g^M (sf (K)) dG (s) \).

Since \( g \) is concave, \( K_{0g} \) is well-defined.

**Case 1** If \( \beta E s f (K_{0g}) + \beta^2 \theta > \beta \theta \), that is, \( E s f (K_{0g}) \leq (1 - \beta) \theta \), then (1b) cannot bind for all \( I \). In particular, it does not bind at \( I = 0 \). Then define the value to the borrower at zero income, \( B_g^M (0) \), by the following equation:

\[
B_g^M (0) = \beta G \left( \frac{B_g^M (0) - \beta \theta}{f (K_{0g})} \right) B_g^M (0) + \beta \int_{B_g^M (0) - \beta \theta f (K_{0g})}^{f (K_{0g})} [sf (K_{0g}) + \beta \theta] dG (s). \tag{10}
\]

Since in this case \( B_g^M (0) > \beta \theta \), \( B_g^M (0) \) is well-defined. **Case 2** If \( \beta E s f (K_{0g}) + \beta^2 \theta \leq \beta \theta \), then (1b) binds at all \( I \). In this case define \( B_g^M (0) = \beta \theta \).

Now, define the value to the borrower by \( B_g^M (I) = \max \{ I + \beta \theta, B_g^M (0) \} \). Define the operator
$T^M : \Gamma \rightarrow \Gamma$ as follows.

$$T_0^M L_g^M (I) = \max_{K,L} I - C - K + \beta \int_0^1 L_g^M (sf (K)) dG (s)$$

(s.t. $I - C = \min \left\{ I, \beta \int_0^1 B_g^M (sf (K)) dG (s) - \beta \theta \right\}$, \hspace{1cm} (11)

and

$$T^M L_g^M = \text{conc} \left( \max \left\{ T_0^M L_g^M, 0 \right\} \right),$$

where $\text{conc}$ denotes the concavification of a function. To see that $T^M$ indeed maps into $\Gamma$, one can show that $T^M L_g^M$ is continuous and concave with slope less than one. The concavity of $T^M L_g^M$ and $T^M L_g^M \geq 0$ follows immediately from construction. The continuity of $T^M L_g^M$ and the fact that its slope is less than one follow from Lemma 1. I will next show that (a) The operator $T^M$ has a fixed point, and (b) There is a bijection between fixed points and Markov equilibria.

(a): I apply Schauder’s Fixed-Point Theorem to the operator $T^M$. As I mentioned, the set $\Gamma$ is a convex compact set. It remains to show that the operator $T^M$ is continuous with respect to the supremum norm. This would clearly follow from the continuity of $T^M$. Notice that the functionals $K_g$ and $B_g (0)$ are continuous in $L_g^M$. Therefore, the maximand in (11) and the constraint (12) are both continuous. Thus $T_0^M$ is a continuous operator.

(b): If $L_g^M = L^M$, where $L^M$ is an equilibrium value function of the lender, then $L_g^M$ is obviously a fixed point of $T^M$. If $L_g^M$ is a fixed point of $T^M$, then it follows from the proof of Theorem 9.2 in Stokey and Lucas (1989) that $L^M = L_g^M$, corresponds to an equilibrium.

(ii) Suppose, to the contrary, that autarky is a Markov equilibrium. That is, $L^M (I) = 0$ and $B^M (I) = I + \beta \theta$ for all $I$. To get a contradiction, I show that whenever $I$ is large, in particular, $I \geq \beta E sf (K^{FB}) - \beta (1 - \beta) \theta$, the lender can generate a strictly positive payoff. The maximization problem of the lender can be written as $\min_{K} K + \beta E L^M (sf (K)) + \min \left\{ I, \beta EB^M (sf (K)) - \beta \theta \right\} = \max_{K} -K + \min \left\{ I, \beta E sf (K) - \beta (1 - \beta) \theta \right\}$. The solution is clearly $K^{FB}$, and $C \geq 0$ as long as $I \geq \beta E sf (K^{FB}) - \beta (1 - \beta) \theta$. The payoff to the lender is $-K^{FB} + \beta E sf (K^{FB}) - \beta (1 - \beta) \theta = (1 - \beta) S^{FB} - \beta (1 - \beta) \theta > 0$ as long as $\theta < \beta S^{FB}$. A contradiction.

(iii) By part (i), there exists a fixed point of the operator $T, (N^M, L^M)$, that corresponds to a Markov equilibrium. If there exists no other fixed point $(N, d)$ of $T$, such that $N \geq N^M$, $N \neq N^M$, then using Claim 1 the Markov equilibrium is renegotiation-proof. Suppose there exists a fixed point $(N, d)$ such that $N \geq N^M$, $N \neq N^M$, yet there is no fixed point $(N', d')$ that corresponds to a renegotiation-proof set of payoffs. This is only possible if each weakly renegotiation-proof set
of payoffs is Pareto dominated by another weakly renegotiation-proof set of payoffs. Construct a corresponding sequence of fixed points \( \{(N_k, d_k)\}_{k \geq 1} \), where it must be the case that \( N_{k+1} \geq N_k \) for all \( k \). Since \( N_k \) is bounded from above by the first-best frontier, it converges to some limit \( \bar{N} = \lim_{k \to \infty} N_k \). It then also follows that the corresponding sequence of functions \( d_k \) must converge as well: \( \bar{d} = \lim_{k \to \infty} d_k \). Since the set of constraints is compact, by continuity \((\bar{N}, \bar{d})\) is the fixed point of \( T \). Since the set of payoffs corresponding to \( \bar{N} \) also has the property that no other weakly renegotiation-proof set of payoffs Pareto dominates it, it is renegotiation proof. A contradiction. ■

**Proof of Claim 3.** The first-order condition for investment in a Markov equilibrium without uncertainty is \( 1 = \beta f'(K) a[L^M(af(K)) + \lambda^M B^M(af(K))] \). The maximum investment, \( \bar{K}^M \), is made when \( \lambda^M = 1 \) (\( C^M(I) > 0 \)). Since \( B^M(af(\bar{K}^M)) = 1 \) and \( L^M(af(\bar{K}^M)) = 0 \), the first-order condition becomes \( 1 = \beta f'(\bar{K}^M) a \). Hence \( \bar{K}^M = K^{FB} \). Since eventually the state must be reached such that \( C^M(I) > 0 \), \( \exists \tau \) such that for \( t \geq \tau \), \( K_t = K^{FB} \). ■

**Proof of Claim 4.** Suppose that the promised value to the borrower is \( B \), and a deterministic offer is made.\(^{21}\) Let \( \beta g(s) \mu(sf(K)) \) and \( \beta g(s) \phi(sf(K)) \) be the Lagrange multipliers on the participation constraints of the borrower and the lender. The first-order condition with respect to \( B^*(sf(K)) \) is

\[
-N'(B^*(sf(K))) = \frac{\lambda + \mu(sf(K))}{1 + \phi(sf(K))} = -N'(B) + \frac{\mu(sf(K))}{1 + \phi(sf(K))},
\]

where the second equation uses the Envelope condition. From the first-order condition with respect to \( B^*(I') \), if \( B \in [B_{\min}(I'), B_{\max}(I')] \) so that \( \mu(I') = \phi(I') = 0 \), then \( B^*(I') = B \). If \( B > B_{\max}(I') \), then \( \phi(I') > 0 \), \( \mu(I') = 0 \) and \( B^*(I') = B_{\max}(I') \). If \( B < B_{\min}(I') \), then \( \mu(I') > 0 \), \( \phi(I') = 0 \) and \( B^*(I') = B_{\min}(I') \).

**Proof of Lemma 4.** Consider \( \hat{d}\left(I; \hat{B}(.)\right) = \max_{C, K} \left[ I - C - EK + \beta E[sf(K) + N(\hat{B}(sf(K)))] \right] \) s.t. \( C + \beta E\hat{B}(sf(K)) \geq I + \beta \theta \), where \( \hat{B}(.) \) is some function satisfying \( \hat{B}(I') \in [B_{\min}(I'), B_{\max}(I')] \) for all \( I' \). Let \( \hat{\lambda} \) and \( \hat{\gamma} \) be the Lagrange multipliers on the participation constraint and \( C \geq 0 \), respectively. The first-order condition with respect to \( C \) is \(-1 + \hat{\lambda} + \hat{\gamma} = 0 \). The Envelope condition is \( \partial \hat{d}\left(I; \hat{B}(.)\right) / \partial I = 1 - \hat{\lambda} = \hat{\gamma} \in [0, 1] \). We have \( \hat{d}(I) = \hat{d}\left(I; B^d(.; I)\right) \), where \( B^d(.; I) \) are the continuation values minimizing the value to the deviating lender at income \( I \). I.e., the function \( \hat{d} \) is the lower envelope of the functions \( \hat{d}\left(I; \hat{B}(.)\right) \) for all possible choices of continuation values \( \hat{B}(.) \). Since \( \partial \hat{d}\left(I; \hat{B}(.)\right) / \partial I \in [0, 1] \) hold for all \( I \) and all \( \hat{B}(.) \), it is also true for their lower envelope:

---

\(^{21}\)The formula for the optimal choice of \( B^*(.) \) is the same in the case of a random offer. (See the proof of part (ii) of Proposition 3.)
\[ \hat{d}'(I) \in [0, 1]. \] 
\[ d'(I) \in [0, 1] \text{ automatically follows.} \] 
\[ d \text{ is concave because random offers are allowed.} \]

**Proof of Claim 5.** The value to the deviating lender can be written as 
\[ \hat{d}(I) = \max_K -EK + \beta E[sf(K) + N(B^d(sf(K)))] + \min\{I, \beta EB^d(sf(K)) - \beta \theta\}. \]

(i) Suppose \( I \) and \( \theta \) are sufficiently low (e.g., \( I = \theta = 0 \)), so that the participation constraint of the borrower does not bind, \( \beta EB^d(sf(K)) > I + \beta \theta \). This corresponds to \( d'(I) = 1 \). Then the deviating lender solves \( \max_K I - K + \beta E[sf(K) + N(B^d(sf(K)))] \). This value is minimized if \( N(B^d(I')) \) is the smallest possible for all \( I' \). I.e., \( B^d(I') = B_{\max}(I') \) for all \( I' \).

(ii) Suppose that \( I \) is so high that \( \min\{I, \beta EB^d(sf(K)) - \beta \theta\} = \beta EB^d(sf(K)) - \beta \theta \). This corresponds to \( d'(I) = 0 \). Then the value to the deviating lender can be written as 
\[ \hat{d}(I) = \max_K -K + \beta E[sf(K) + N(B^d(sf(K)))] + B^d(sf(K)) - \beta \theta. \]
Since the deviating lender extracts all the continuation surplus, \( B^d(I') \) must be such that \( N(B^d(I')) + B^d(sf(K)) \) is the lowest for each \( I' \). By Lemma 3, 
\[ N(B_{\min}(I')) + B_{\min}(I') \leq N(B) + B \] for all \( B \in (B_{\min}(I'), B_{\max}(I')] \), with the strict inequality for \( I' \) such that \( |N(B_{\min}(I'))| < 1 \). Thus \( B^d(I') = B_{\min}(I') \) for all \( I' \).

**Proof of Proposition 3.** (i) Suppose that \( B_0 > \beta \theta \). Then the promise-keeping constraint of the borrower does not bind for \( B \leq B_0 \). Hence for all \( B \in [\beta \theta, B_0] \) the optimal contract is the same, \( (C, K) \), where \( C = 0 \) and \( K = K_0 \), where \( K_0 = \arg \max_K -K + \beta E[sf(K) + N(B_{\min}(sf(K)))]. \]
Let \( B \in [B_0, \bar{B}] \) and consider a marginal increase in the promised value to the borrower from \( B \) to \( B' \), where \( B' - B \) is arbitrarily close to zero. Denote \( K = K(B) \) and \( K' = K(B') \). Let \( s_1 \) and \( s_2 \) be such that \( B_{\max}(s_1 f(K)) = B = B_{\min}(s_2 f(K)) \), and \( s'_1 \) and \( s'_2 \) be such that \( B_{\max}(s'_1 f(K)) = B' = B_{\min}(s'_2 f(K)) \). (\( s_1 < s'_1 < s_2 < s'_2 \)) As \( B \) increases to \( B' \), by (13) and Claim 4 some of the constraints \( B^*(sf(K)) \geq B_{\min}(sf(K)) \) are relaxed while some of the constraints \( B^*(sf(K)) \leq B_{\max}(sf(K)) \) are tightened. The change in the marginal product of capital at \( K \) is 
\[ \Delta = \beta f'(K) \left[ \int_{s_1}^{s'_1} s \left\{ 1 + \phi(sf(K)) \left[ 1 - d'(sf(K)) \right] \right\} dG(s) + \int_{s_2}^{s'_2} s \mu(sf(K)) dG(s) \right]. \]
(14) 
Since \( B' < \bar{B}, |N'(B''')| > |N'(B')| \) for \( B'' > B' \) and \( |N'(B'')| < |N'(B')| \) for \( B'' < B' \). Thus the Lagrange multipliers in the above expression are strictly positive, and therefore \( \Delta > 0 \). That is, at \( K \) the right-hand side of (9) is strictly greater than one. Hence \( K' > K \).

Recall that \( \bar{B} \) is the largest \( B \) such that \( C(B) = 0 \). For \( B \geq \bar{B}, |N'(B)| = |N'(\bar{B})| = 1 \) and hence in (14), \( \phi(sf(K)) = 0 \) for \( s \geq s_1 \) and \( \mu(sf(K)) = 0 \) for \( s \geq s_2 \). Therefore, \( \Delta = 0 \) and \( K(B) = K(\bar{B}) \). The optimal contract at \( \bar{B} \) is \((0, K(\bar{B}))\), and the optimal contract at \( B \geq \bar{B} \) is \((B - \bar{B}, K(B))\).

(ii) Suppose that at \( B_1, B_2 \leq \bar{B} \) deterministic investments, \( K_1 \) and \( K_2 \), are made. Since
From Claim 4 and equation (15), the Lagrange multipliers condition for investment, \( f (K_1) + \beta \theta > B_1 \). Let \( B = \pi_1 B_1 + \pi_2 B_2 \in (B_1, B_2) \), \( \pi_i > 0 \), \( \pi_1 + \pi_2 = 1 \), and let \( \pi_1 \) be small enough so that \( B \leq f (K_1) + \beta \theta \). Suppose at \( B \) the lender is best off making an offer that specifies \( K_1 \) with probability \( \pi_1 \) and \( K_2 \) with probability \( \pi_2 = 1 - \pi_1 \). Denote \( K_\pi = \pi_1 K_1 + \pi_2 K_2 \). Then lender's value can be written as \( N (B) = \max N(K_\pi, B^*) - C + \sum_{i=1,2} \pi_i \{ -K_i + \beta E [s f (K_i) + N(B^* (s f (K_i)))] \} \) s.t. \( C + \sum_{i=1,2} \pi_i \beta E B^* (s f (K_i)) \geq B, B^* (s f (K_i)) \in [B_{\min} (s f (K_i)), B_{\max} (s f (K_i))], i = 1, 2, s \in [0, 1] \). (The first-order conditions with respect to \( B^* (s f (K_i)) \) is the same as (13). The Lagrange multipliers on the borrower's and lender's participation constraints are \( \beta \pi_i g (s) \mu (s f (K_i)) \) and \( \beta \pi_i g (s) \phi (s f (K_i)) \).) Since \( f (K_1) + \beta \theta \geq B \), we have \( f (K_2) > f (K_\pi) > f (K_1) \geq B - \beta \theta \), by part (i). Next, for \( I' + \beta \theta \geq B \) we have \( B^* (I') = I' + \beta \theta \), by Claim 4. Hence for \( I' + \beta \theta > B \), \( N (B^* (I')) = N (I' + \beta \theta) \) is concave in \( I' \). Furthermore, \( f \) is strictly concave. That is, the random investment choice is made on the strictly concave part of the objective function. But then a deterministic contract \((0, K_\pi)\) delivers a payoff to the lender which is strictly higher than \( N (B) \). This contradicts optimality of the random offer. I have shown that if at \( B_1 \) and \( B_2 \) a deterministic offer is made, then on \((B_1, f (K (B_1)) + \beta \theta)\) also a deterministic offer is made. Next consider \( B'_1 = f (K (B_1)) + \beta \theta \) and \( B \in (B'_1, B_2) \) such that \( B \leq f (K (B'_1)) + \beta \theta \). (Since \( B'_1 > B_1 \), \( K (B'_1) > K (B_1) \) by part (i).) Proceeding in this way, one shows that no random offers are made on \([B_1, B_2]\), and it is true for any interval where deterministic offers are chosen at the end points of the interval.

(iii) It is enough to show that \( |N' (B_0)| < 1 \). Then from (13), Claim 4 and (9) it follows that \( K(B_0) < K^{FB} \). Suppose that \( |N' (B_0)| = 1 \). Then clearly \( B_0 = \beta \theta \). (If \( B_0 > \beta \theta \), then \( |N' (B_0)| = 0 \).) That is, \( B_{\min} (I) = I + \beta \theta \) for all \( I \). Hence at \( B = \beta \theta \) consumption of the borrower is \( C (\beta \theta) = \beta \theta - \beta E B^M (s f (K^{FB})) = \beta (1 - \beta) \theta - \beta E s f (K^{FB}) = (1 - \beta) [\beta \theta - S^{FB}] - K^{FB} < 0 \) because \( \theta < S^{FB}/\beta \), a contradiction.

(iv) If \( \bar{B} \leq B_{\max} (0) \) then \( K (\bar{B}) = K (B_{\max} (0)) \) by part (i). I want to show that \( K (\bar{B}) = K^{FB} \). From equation (13),

\[
\mu (s f (K) ; B) = |N' (B_{\min} (s f (K)))| - |N' (B)|,
\]

\[
|N' (B_{\max} (s f (K)))| [1 + \phi (s f (K) ; B)] = |N' (B)|.
\]

From Claim 4 and equation (15) the Lagrange multipliers \( \mu (s f (K)) \) that enter into the first-order condition for investment, \( (9) \), are all zero: \( \mu (s f (K)) = |N' (B_{\min} (s f (K)))| - |N' (\bar{B})| = 1 - 1 = 0 \) because \( B_{\min} (s f (K)) \geq \bar{B} \). Hence \( K (\bar{B}) \geq K^{FB} \). Now consider the Lagrange multipliers \( \phi (s f (K)) \) that enter into \( (9) \). From Claim 4 and (16), \( \phi (s f (K)) = |N' (\bar{B})| / |N' (B_{\max} (s f (K)))| - 1 = 0 \) because \( |N' (B_{\max} (0))| = |N' (\bar{B})| = 1 \). Hence \( (9) \) becomes \( 1 - \beta E s f' (K (\bar{B})) \) so that
\( K(\bar{B}) = K^{FB} \).

(v) Suppose \( \bar{B} > B_{\text{max}}(0) \). I first want to show that \( K(B_{\text{max}}(0)) < K^{FB} \). Using Claim 4, at \( B = B_{\text{max}}(0) \), for all \( s \phi(sf(K)) = 0 \) in equation (9). Hence \( K(B_{\text{max}}(0)) \leq K^{FB} \). From (15), \( \mu(I^{'}) = |N'(B_{\text{min}}(I^{'})| - |N'(B_{\text{max}}(0))| \). Consider \( \bar{I} \) such that \( B_{\text{min}}(\bar{I}) = B_{\text{max}}(0) \). For \( I' > \bar{I}, \mu(I^{'}) > 0 \) because \( |N'(B_{\text{max}}(0))| < 1 \). Thus \( K(B_{\text{max}}(0)) < K^{FB} \). Now I will show that \( K(\bar{B}) > K^{FB} \). As I showed in part (i), \( K(\bar{B}) \geq K^{FB} \). Denote \( I_0 = \inf \{ I' | d'(I') < 1 \} \). For \( I' \leq I_0, d'(I') = 1 \) so that \( B_{\text{max}}(I') = B_{\text{max}}(0) \). Hence for \( I' > I_0, \phi(I'[1 - d'(I')] > 0 \) by Lemma 4, equation (16), and \( \bar{B} > B_{\text{max}}(0) = B_{\text{max}}(I_0) \). Therefore, \( K(\bar{B}) > K^{FB} \). □

**Proof of Lemma 5.** (i) Notice that \( B_{\text{max}}(I) \), defined by \( N(B_{\text{max}}(I)) = d(I) - I \), is increasing in \( I \) because \( d(I) - I \) is decreasing in \( I \) ( \( d'(I) \in [0,1] \), see Lemma 4) and \( N \) is decreasing in \( B \) \( (N'(B) \in [-1,0] \), see Lemma 3). Hence \( B_{\text{max}}(I) \geq B_{\text{max}}(0) \) for all \( I \), that is, if \( B \leq B_{\text{max}}(0) \), then \( B \leq B_{\text{max}}(I) \) for all \( I \). Then for \( B \in [B_{\text{min}}(0), B_{\text{max}}(0)] \) we have \( B_{\text{max}}(sf(K)) \geq B \) for all \( s \), and hence by Claim 4 \( B^*(sf(K);B) = B_{\text{max}}(sf(K)) \) is never chosen. Using Claim 4, if \( s \) is such that \( B_{\text{min}}(sf(K)) \leq B, \) then \( B^*(sf(K);B) = B \geq B \). If \( s \) is such that \( B_{\text{min}}(sf(K)) > B, \) then \( B^*(sf(K);B) = B_{\text{min}}(sf(K)) > B \). Thus for \( B \in [B_{\text{min}}(0), B_{\text{max}}(0)], B^*(sf(K);B) \geq B \).

(ii) Suppose \( \bar{B} > B_{\text{max}}(0) \), so that \( \min \{ B_{\text{max}}(0), \bar{B} \} = B_{\text{max}}(0) \). Then \( |N'(B_{\text{max}}(0))| < 1, \) and by Claim 4, if \( B \geq B_{\text{max}}(0) \) then the optimal choice of \( B^* \) must satisfy \( B^*(sf(K);B) \geq B_{\text{max}}(0) \) for all \( s \). Indeed, if \( s \) is such that \( B < B_{\text{min}}(sf(K)) \) then \( B^*(sf(K)) = B_{\text{min}}(sf(K)) > B \geq B_{\text{max}}(0) \). If \( s \) is such that \( B_{\text{min}}(sf(K)) \leq B \leq B_{\text{max}}(sf(K)) \), then \( B^*(sf(K)) = B \geq B_{\text{max}}(0) \). If \( s \) is such that \( B > B_{\text{max}}(sf(K)) \) then \( B^*(sf(K)) = B_{\text{max}}(sf(K)) \geq B_{\text{max}}(0) \) for all \( s \). I have shown that if \( B_t \geq B_{\text{max}}(0), \) then \( B_{\tau} \geq B_{\text{max}}(0) \) for all \( \tau \geq t \). Suppose now that \( \bar{B} \leq B_{\text{max}}(0) \). Then \( |N'(B_{\text{max}}(0))| = |N'(\bar{B})| = 1 \). Therefore if \( B \geq \bar{B}, \) then from (13) \( |N'(B^*(sf(K);B))| = |N'(B)| = 1 \) for all \( s \), and, e.g., both \( B^*(0;B) = B_{\text{max}}(0) \) and \( B^*(0;B) = \bar{B} \) can be optimal choices (with different levels of consumption). Thus \( B^*(sf(K);B) \geq \min \{ B_{\text{max}}(0), \bar{B} \}. □

**Proof of Proposition 4.** (i) Suppose \( \bar{B} \leq B_{\text{max}}(0) \). If \( B_0 < \bar{B}, \) then there is underinvestment initially by part (iii) of Proposition 3. By Lemma 5 \( B_t \) is increasing over time until it reaches \( \bar{B} \). Hence investment is also increasing over time because \( B_t \) is increasing. Furthermore, since on \( [B_0, \bar{B}] K(B) \) is strictly increasing by part (i) of Proposition 3, \( B_t \) reaches \( \bar{B} \) with probability one. For \( B \geq \bar{B}, K(B) = K^{FB} \) by parts (i) and (iv) of Proposition 3.

(ii) Suppose \( \bar{B} > B_{\text{max}}(0) \). Then \( K(B_{\text{max}}(0)) < K^{FB} \) and \( K(\bar{B}) > K^{FB} \) by part (v) of Proposition 3. Consider \( B_t < \bar{B} \) so that \( K_t < K^{FB} \). Since the borrower must consume, with probability one \( \exists s \) such that \( B_s > \bar{B} \) and \( K_s = K(\bar{B}) > K^{FB} \). Since \( K(B) \) is strictly increasing
on $[B_0, \tilde{B}]$, following a sequence of bad shocks (which happens with probability one, because the shocks are i.i.d.) $\exists s'$ such that $B_s < \tilde{B}$ so that $K_s < K^{FB}$. That is, a sequence of good shocks leads to overinvestment, and a sequence of bad shocks leads to underinvestment. ■

Proof of Lemma 6. The proof is by induction. Consider a fixed point $L^M_\theta$ corresponding to $\theta$, and increase $\theta$ marginally to $\theta'$. Apply the operator $T^M_\theta$ to $L^M_\theta$. Denote $L^M_{(0)} = L^M_\theta$, and $L^M_{(k)} = T^M_{\theta'}\left(L^M_{(k-1)}\right)$, $k \geq 1$. Denote $K_{0;(k)} = \arg \max_K -K + \beta E L^M_{(k-1)}(sf(K))$. We have $K_{0;(1)} = K^\theta_0$.

Suppose first that $\theta$ is sufficiently small so that the borrower’s participation constraint, (1b), does not bind at $I = 0$. Then $B^M_{(1)}(0)$ defined by $B^M_{(1)}(0) = \beta E \max \left\{ sf(K^\theta_0) + \beta \theta', B^M_{(1)}(0) \right\}$ satisfies $B^M_{(1)}(0) > B^M_{(0)}(0) = B^M_\eta(0)$ and hence $B^M_{(1)}(I) > B^M_{(0)}(I)$ for all $I$. If $\theta$ is high enough so that (1b) binds at $I = 0$, then it will also bind at $\theta'$, and hence $B^M_{(1)}(I) = I + \beta \theta' > I + \beta \theta = B^M_{(0)}(I)$ for all $I$. Furthermore, a marginal increase in $\theta$ increases the right-hand side of (1b) by $\beta$, while the left-hand side increases by at most $\beta^2$. Thus $L^M_{(1)}(I) \leq L^M_{(0)}(I)$, with the strict inequality for $I$ such that (1b) binds. Furthermore, $L^M_{(0)}(I) - L^M_{(1)}(I)$ is increasing in $I$, because the borrower’s participation constraint, (1b), binds more for higher income levels (by concavity of the value function, $\lambda^M(I)$ is increasing in $I$). As a result, $L^M_{(1)}(I) \leq L^M_{(0)}(I)$ for all $I$, with the strict inequality for some $I$. I want to show that if $L^M_{(k)}(I) \leq L^M_{(k-1)}(I)$ and $L^M_{(k)}(I) \leq L^M_{(k-1)}(I)$, then $L^M_{(k+1)}(I) \leq L^M_{(k)}(I)$ and $L^M_{(k+1)}(I) \leq L^M_{(k)}(I)$. If $B^M_{(k-1)}(0) = \beta \theta'$, then on each iteration $k \geq 2$ the constraint set is unchanged. Hence $L^M_{(k)}(I) \leq L^M_{(k-1)}(I)$ implies $L^M_{(k+1)}(I) \leq L^M_{(k)}(I)$ for all $I$, and we are done. Suppose that $B^M_{(k-1)}(0) > \beta \theta'$. The inequality $L^M_{(k)}(I) \leq L^M_{(k-1)}(I)$ implies that $K_{0;(k+1)} \leq K_{0;(k)}$ and hence $B^M_{(k+1)}(0) \leq B^M_{(k)}(0)$ so that $B^M_{(k+1)}(I) \leq B^M_{(k)}(I)$ for all $I$. Thus the constraint (1b) becomes tighter (the left-hand side decreases while the right-hand side stays the same). Therefore $L^M_{(k+1)} \leq L^M_{(k)}$. In addition, $L^M_{(k)}(I) - L^M_{(k+1)}(I)$ is increasing in $I$. The reason is that (1b) binds more for higher income levels, $L^M_{k-1}(I) - L^M_{k}(I)$ is increasing in $I$, and $K^M(I)$ is increasing in $I$ by Claim 1. Applying induction, we have a decreasing sequence of functions, \(\left\{ L^M_{(k)} \right\}_{k \geq 0}\), with $L^M_{(0)} = L^M_\theta$. Since $L^M_{(k)} \geq 0$, the sequence has a limit, $L^M_*$. By continuity this limit is a fixed point for $\theta'$: $L^M_* = L^M_{\theta'}$. Therefore $L^M_{\theta'} \leq L^M_\theta$. ■

Proof of Proposition 5. (i) Since $I > B^M_{\theta_j}(0) - \beta \theta_j$, $j = 1, 2$, (1b) binds in equilibrium. I showed in the proof of Lemma 6 that an increase in $\theta$ increases the right-hand side of (1b) by more than the left-hand side. Hence at the old investment choice (1b) is violated: $\beta EB^M_{\theta_1} (sf(K^M_{\theta_1}(I))) < I + \beta \theta_2$. Since $I < \bar{I}_{\theta_j}$, $j = 1, 2$, the borrower’s consumption is zero, and the lender increases investment to make the borrower accept an offer. Thus $K^M_{\theta_2}(I) > K^M_{\theta_1}(I)$.

(ii) From the induction in the proof of Lemma 6, $\left\{ K_{0;(k)} \right\}_{k \geq 0}$ is a decreasing sequence (with $K_{0;(k+1)} > K_{0;(k)}$ at least for some $k$), and hence $K^\theta_0$ is strictly decreasing in $\theta$. Suppose $\theta$ is such
that \( B^M_\theta (0) > \beta \theta \). Then for \( I \in [0, B^M_\theta (0) - \beta \theta) \), \( K^M_\theta (I) = K_{0, \theta} \), implying \( K^M_{\theta^2} (I) < K^M_\theta (I) \).

**Proof of Proposition 6.**  (i) Suppose that \( B^M_\theta (I) = I + \beta \theta \) for all \( I \). Then \( dB^M_\theta (I) / d\theta = \beta \). I want to show that in this case \(-dL^M_\theta (I) / d\theta < \beta \). It would then follow that \( d \left[ L^M_\theta (I) + B^M_\theta \right] / d\theta > 0 \) for all \( I \). From the induction algorithm in the proof of part (i) of Proposition 5, we have that \( L^M_{(k)} (I) - L^M_{(k+1)} (I) \) is increasing in \( I \) for all \( k \geq 1 \). Hence, \(-dL^M_\theta (I) / d\theta \) is increasing in \( I \). The constraint (1b) binding at all \( I \) implies that it can be written as \( C + \beta E s f (K) \geq I + \beta (1 - \beta) \theta \).

Using the Envelope condition \( \lambda^M = 1 - L^M_{\theta^2} (I) \),

\[
-dL^M_\theta (I) / d\theta = -\beta EdL^M_\theta \left( s f (K^M (I)) \right) / d\theta + [1 - L^M_{\theta^2} (I)] \beta (1 - \beta)
\]

\[
\leq -\beta EdL^M_\theta \left( s f (K^M (I)) \right) / d\theta + [1 - L^M_{\theta^2} (I)] \beta (1 - \beta),
\]

where the inequality uses the fact that \(-dL^M_\theta (I) / d\theta \) is increasing in \( I \). Integrating (17), obtain

\[
-EdL^M_\theta \left( s f (K^M (I)) \right) / d\theta \leq -\beta EdL^M_\theta \left( s f (K^M (I)) \right) / d\theta + \beta (1 - \beta) [1 - EL^M_{\theta^2} (s f (K^M (I)))].
\]

Rearranging terms yields

\[
-EdL^M_\theta \left( s f (K^M (I)) \right) / d\theta \leq \beta [1 - EL^M_{\theta^2} (s f (K^M (I)))].
\]

Using this inequality and (17), obtain

\[
-dL^M_\theta (I) / d\theta \leq \beta^2 [1 - EL^M_{\theta^2} (s f (K^M (I))) + [1 - L^M_{\theta^2} (I)] \beta (1 - \beta) < \beta^2 + [1 - L^M_{\theta^2} (I)] \beta (1 - \beta) \leq \beta,
\]

where the first inequality uses \( EL^M_{\theta^2} (s f (K^M (I))) \) \( (0, 1) \), and the second inequality uses \( L^M_{\theta^2} (I) \in [0, 1] \). Hence \(-dL^M_\theta (I) / d\theta < \beta \) for all \( I \).

(ii) Denote \( z = [B^M_\theta (0) - \beta \theta] / f (K_{0, \theta}) \). Then the expression for \( B^M_\theta (0) \) becomes

\[
(1/\beta - G(z)) \left( z f (K_{0, \theta}) + \beta \theta \right) = \int_z^1 [s f (K_{0, \theta}) + \beta \theta] dG(s) = f (K_{0, \theta}) \int_z^1 s dG(s) + \beta \theta \left( 1 - G(z) \right).
\]

Rearranging terms, we have

\[
\frac{1 - \beta}{2} \theta = \int_z^1 s dG(s),
\]

Using integration by parts, the above expression simplifies to

\[
\frac{1 - \beta}{2} \theta = 1 - \int_z^1 G(s) ds.
\]

Differentiating (19) with respect to \( \beta \),

\[
[1/\beta - G(z)] dz / d\beta = z / \beta^2 - \theta d[(1/\beta - \beta) / f (K_{0, \theta})] / d\beta,
\]

where \( d[(1/\beta - \beta) / f (K_{0, \theta})] / d\beta = -1 / f (K_{0, \theta}) - (1 - \beta) / f^2 (K_{0, \theta}) df (K_{0, \theta}) / d\beta, \)

and \( z / \beta^2 > 0 \). It can be shown that \( df (K_{0, \theta}) / d\beta > 0 \), but I do not prove it here. However, assuming that \( \theta \) is low enough and \( \beta \) high enough is sufficient to guarantee that \( dz / d\beta > 0 \).

Differentiating (19) with respect to \( \theta \), obtain

\[
dz / d\theta = -[(1 - \beta) / 1/\beta - G(z)] \{1 / f (K_{0, \theta}) - \theta / f^2 (K_{0, \theta}) df (K_{0, \theta}) / d\theta \} < 0,
\]

because \( df (K_{0, \theta}) / d\theta < 0 \) and \( \beta < 1 \). Furthermore, for \( \theta = 0 \), (18) simplifies to

\[
1/\beta - 1 = \int_z^1 s / z dG(s) - [1 - G(z)] = \int_z^1 s / z - 1 dG(s).
\]

From this equation, as \( \beta \to 1 \) we have \( z \to 1 \). This shows that for \( \beta \) high enough and \( \theta \) low enough, \( z \) is close to 1.
Recall that investment $K_{0,\theta}$ is made on $[0, B^M_{\theta}(0) - \beta \theta]$, and $K_{0,\theta}$ is decreasing in $\theta$. Then for $z$ sufficiently close to 1 (so that $G(z)$ is sufficiently close to 1), given that $I \in [0, B^M_{\theta}(0) - \beta \theta]$ today, the probability of the next-period’s output realization being below $B^M_{\theta}(0) - \beta \theta$ is very high, and hence the probability of investment being equal to $K_{0,\theta}$ in the next period is very high. Since the shocks are i.i.d., this argument applies to all periods. This argument shows that for some parameter values (in particular, for sufficiently low $\theta$) the social surplus for $I \in [0, B^M_{\theta}(0) - \beta \theta]$ can be strictly decreasing in $\theta$. (In fact, computations show that for $\beta$ high enough and $\theta$ low enough $B^M_{\theta}(I) + F^M_{\theta}(I)$ is strictly decreasing in $\theta$ for all $I$.) ■

References


