Polarization under Incomplete Markets and Endogenous Labor Productivity

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Abstract

We explore the accumulation of assets in the presence of limited insurance against idiosyncratic shocks, borrowing constraints and endogenous labor productivity due to the so-called “nutrition curve”. We show that in such an environment, any stationary equilibrium is characterized by a polarized distribution of wealth. That is, there are only extremely rich and extremely poor agents.

Keywords: Idiosyncratic shocks, incomplete markets, labor supply.
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1 Introduction

According to Mookherjee and Ray (2002), there are three main approaches to the study of economic inequality and its dynamics: 1) inequality tends to die out over time as it is the result of bad shocks and the available investment technologies are convex (Becker and Tomes 1979, Champernowne 1953, Loury 1981), 2) inequality in the long run maps -and possibly amplifies- initial conditions because investment technologies are non convex (Majumdar and Mitra 1982, Banerjee and Newman 1993, Galor and Zeira 1993, Ray and Streufert 1993, among others), and 3) inequality is an intrinsic feature of market economies in which there are frictions such as missing markets, credit rationing, or borrowing constraints that prevent poor households from having access to some investment technologies (Ljungqvist 1993, Piketty 1997, Mookherjee and Ray 2002).\footnote{It is fair to say that even if there are no frictions, inequality may persist due to endogenous differences in marginal propensities to save (Chatterjee 1994, Ventura 1998, Caselli and Ventura 2000, among others).}

We contribute to this literature by showing that loose constraints and polarization in the distribution of wealth, i.e., an extreme form of inequality, may go together in any long run equilibrium.

In this paper we study the long run distribution of wealth corresponding to an economy where agents are subject to idiosyncratic shocks of unemployment, and where all markets are competitive but insurance markets are incomplete. We depart from the standard incomplete markets setup (e.g. Aiyagari 1994, Huggett 1993, 1997) in that labor supply is endogenous, and in that labor productivity of an agent depends on her consumption. The assumption of endogenous labor has been widely used in many studies (see for instance, Kydland and Prescott (1982) and the vast literature about real business cycles following that paper), and its effects under incomplete markets economies have been previously studied, among others, by Ríos-Rull (1994), Díaz-Gimenez (1998), Obiols-Homs (2003), and Marcet, Obiols-Homs and Weil (2007) (MOW from now on). The interest in economics in the linkage between nutrition intakes and productivity of labor goes back, at least, to Leibenstein (1957) and Mazumdar (1959).\footnote{Also, Mirlees (1975), Stiglitz (1976, 1982), and Dasgupta and Ray (1986), among many others, formalized and extended the efficiency wages hypothesis, and pointed to the so-called “nutrition curve” as an explanation for involuntary unemployment.}

Therefore, we do not...
perceive these two assumptions as implying a large deviation from existing
theory or empirical evidence.

Using the model economy outlined above we show that in equilibrium the
stationary distribution of wealth is polarized. That is, in the long run there
are only extremely rich and extremely poor agents, with no middle class in
between, hence, there is no mobility between the two extremes. In addition,
in our model the equilibrium return to assets is equal to the rate of
time preference, and there are many distributions of wealth compatible with
equilibrium. These results stand in sharp contrast with previous results in
incomplete markets models: In the related incomplete markets models the
stationary distribution of wealth puts positive mass in all of the support
of the distribution, and thus, there is a lot of mobility. Furthermore, the
equilibrium prices are such that the return to assets is smaller than the rate
of time preference, and they uniquely determine the long run distribution of
wealth.

The intuition for our results can be described as follows. As shown in MOW,
when leisure is a normal good a sufficiently rich agent chooses not to work
because of a wealth effect on her labor supply. The assumption that the
efficiency units of labor supplied by agents increases with their consump-
tion (although it is bounded above) implies that at low consumption levels,
labor productivity is also low, and therefore, agents prefer not to work,
and in stead, enjoy their leisure. This means that in our model, both, rich
agents that can afford a high consumption level, and poor agents that barely
consume, do not face uncertainty because they choose not to work irrespec-
tively of their occupational status. Then, if the return of assets was larger
(or smaller) than the rate of time preference, asset holdings of all agents
would be, in the long run, sufficiently large (or sufficiently small) so that
non of them would work. At those asset levels the assets market cannot
clear. Thus, the only possible equilibria are such that the return of assets is
equal to the rate of time preference. With those prices, however, the assets
of agents that receive a sufficiently long sequence of good (or bad) idiosyn-
cratic shocks converge to the levels where they choose not to work. This is
why the distribution of wealth is polarized in the long run. Furthermore, it
is possible to construct a continuum of equilibrium distributions simply by
putting more/less mass of agents beyond the critical levels of assets where
they choose not to work. In other words, in our model the Monotone Mixing
Condition sufficient for uniqueness in Theorem 2 in Hopenhayn and Prescott
(1992), is not satisfied.

There is an additional connection with existing work that is worth emphasiz-
The first is that the policy prescriptions leading to avoid inequality in our paper are dramatically different from those obtained in other well-known models such as Banerjee and Newman (1993), and Matsuyama (2004). In these two papers there are frictions in the capital/financial markets that prevent poor households (or countries) to have access to the best investment projects, and so, inequality in the long run equilibrium can be seen as an amplified image of initial inequality. In Banerjee and Newman (1993) this inequality could be alleviated by allowing a larger amount of borrowing. In addition to this policy, in Matsuyama (2004) polarization could be avoided by simply closing down international financial markets, so that all economies would reach the same steady state. Instead, our results suggest that perfect competition together with the development of credit markets, will not reduce inequality nor favor growth and development. Rather, what is needed to reduce inefficiencies is what standard competitive theory a la Arrow-Debreu dictates: complete insurance markets. Hence some government intervention should be designed if polarization is to be avoided. This finding is reminiscent to the efficient provision of unemployment insurance problem with imperfect information studied in Atkeson and Lucas (1995). These authors impose an artificial constraint to prevent “excessive” trading of future consumption for current consumption (de facto a borrowing constraint), which prevents inequality to explode unboundedly. Likewise, polarization could be avoided in our economy by directly introducing a tight borrowing limit, but also, by implementing taxes and consumption subsidies such that the nutrition curve looses its bite.

The empirical evidence supporting A5 can be found, for instance, in Strauss (1986).

The paper continues as follows: section 2 introduces the assumptions that are sustained throughout the analysis, section 3 states the main results of the paper about the stationary distribution of wealth, and section 4 discusses them. The Appendix contains all proofs.

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4See also Marcet and Obiols-Homs (2007) about the effect of too loose borrowing constraints on welfare in a related model.

5Strauss (1986) contains many other interesting references.


2 The model

2.1 Assumptions

Time is discrete and goes on forever. The economy consists of a continuum of ex ante identical consumers, indexed by \( i \), uniformly distributed in the unit interval. Consumers have instantaneous preferences \( U(c, l) \) defined over consumption and leisure, as formalized in assumptions A1-A3.

**A1**: \( U(c, l) = u(c) + v(l) : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R} \), is continuous and differentiable.

**A2**: \( u(c) \) is increasing and concave. There are \( 0 \leq c < \bar{c} \leq +\infty \), such that \( \lim_{c \rightarrow 0} u'(c) = 1/\bar{c} \), and such that \( \lim_{c \rightarrow \bar{c}} u'(c) = 0 \).

**A3**: \( v(l) \) is strictly increasing and strictly concave, with \( \lim_{l \rightarrow 0} v'(l) = +\infty \).

Notice that in A2 we do not necessarily require an Inada condition at zero consumption (this case is obtained with \( \bar{c} = 0 \)). Also, the case of \( \bar{c} < \infty \) allows for satiation in consumption. These possibilities will play a central role in several results of the paper. To the extent that \( \bar{c} \) can be made arbitrarily small, and that \( \bar{c} \) can be made arbitrarily large, assumption A2 does not seem too restrictive. A3 is a standard assumption in models with endogenous labor supply decisions. We introduce leisure in the preferences mainly for technical reasons: as shown in MOW, with endogenous leisure/labor supply there may be a stationary distribution of wealth even if the return to savings equals the rate of time preference.

An agent’s endowment has two parts. The first part consists of one unit of time, and of \( \theta \) units of the consumption good (non random, and the same for all agents). We think of \( \theta \) as fruits that land produces spontaneously and that can be consumed at no cost. The second part is an idiosyncratic endowment of labor productivity (like employment shocks, or health shocks). We therefore model this idiosyncratic state as a random variable \( s_t \) that takes the values 0 and 1. Formally,

**A4**: \( s^i_t \in S \equiv \{1, 0\} \) with \( \sum_{s'} \pi_{s'|s} = 1 \) and \( \pi_{s'|s} > 0 \) for all \( s, s' \in S \), and is independent across \( i \).

The assumptions introduced up to now are similar to those in other models with idiosyncratic shocks, such as Aiyagari (1994), Huggett (1993, 1997), and Krusell and Smith (1998), perhaps with the only exception of assuming endogenous labor (which follows MOW, and Pijoan-Mas 2006, among others). We now depart from the standard setup by assuming that the productivity of the time spent at work depends on the level of consumption of
the agent. That is, if a consumer spends \((1 - l_t)\) units of her time at work, then her labor supply in efficiency units is given by \(P(c_t)(1 - l_t)\) in the “good” or productive state, and zero otherwise. The function \(P(c)\) satisfies:

\[\textbf{A5:} \quad i) \quad P(c) : R_+ \rightarrow [0, 1], \text{ is continuous, differentiable, strictly increasing and concave, with } P'(c) < 1 \text{ for all } c \geq 0; \quad ii) \quad P(0) = 0, \quad \text{and } \lim_{c \to \infty} P(c) = 1.\]

The concavity assumption in part i) simplifies the derivation of the results, which, as we discuss later, would also follow under the assumption of convex-concave \(P(c)\) similar to that in Dasgupta and Ray (1986). The normalization of \(P(0) = 0\) seems appropriate if we think of time periods as weeks or longer spells. That productivity is constant in the limit is assumed in an effort to keep the model close to the usual case for large consumption.\(^6\)

We assume that output in period \(t\) is given by a linear function of aggregate labor in efficiency units, such that one unit of efficient labor is worth one unit of output. To complete the model, we assume that consumers can save in a riskless bond (for simplicity, we assume that this is the only available asset in the economy). Our interpretation is that there is a central authority that gives credit balances at a cost \(q\) and that accepts saving accounts offering a return \((1 - q)/q\). That is, to obtain a credit balance of one unit of consumption goods in the next period, the consumers must pay \(q\) units of goods in the current period. Finally, bond holdings are subject to a borrowing limit such that consumers face the constraint \(b_t \geq B\).

### 2.2 An agent’s problem

The problem of an agent that discounts expected future utility with a factor \(\beta \in (0, 1)\) (which is the same for all agents), and that faces a constant price \(q\) for bonds, can be written formally as follows:

\[
\max_{\{c_t, l_t, b_t\}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(l_t)\}
\]

s. to

\[
c_t + q b_t \leq \theta + s_t P(c_t)(1 - l_t) + b_{t-1},
\]

\[
c_t, l_t \geq 0, \quad l_t \leq 1, \quad b_t \geq B,
\]

and the usual transversality condition.

\(^6\)Again, the literature has explored slightly different specifications for the nutrition curve in which \(P(c) = 0\) for \(c \in [0, c]\), with \(P(c)\) displaying an inverted U-shape for \(c > c\). These specifications complicate the analysis without changing the predictions of our model.
Remark 1: \( q_t > 0 \), otherwise the consumer would be willing to issue an infinite amount of debt.

The first order conditions necessary for optimality that a solution to the problem in (2) must satisfy can be written as follows (the conditions correspond, respectively, to \( c_t \), \( l_t \), and \( b_t \)):

\[
\begin{align*}
    u'(c_t) &\leq \gamma_t (1 - s_t P'(c_t)(1 - l_t)), \text{ and } c_t \geq 0, \\
    v'(l_t) &\leq \gamma_t s_t P(c_t) + \eta_t, \text{ and } l_t \geq 0, \\
    q_t \gamma_t &\geq \beta E_t[\gamma_{t+1}], \text{ and } b_t \geq B,
\end{align*}
\]

and with respect to the multipliers associated to the budget and time constraints in period \( t \) (denoted, respectively, \( \gamma_t \) and \( \eta_t \)):

\[
\begin{align*}
    c_t + q_t b_t &\leq \theta + b_{t-1} + s_t P(c_t)(1 - l_t), \text{ with } \gamma_t \geq 0, \\
    l_t &\leq 1, \text{ with } \eta_t \geq 0.
\end{align*}
\]

Complementary slackness implies that, for each of the above pairs of inequalities, at least one of them holds as an equation. Equation (2) shows that marginal utility of consumption is modified by the factor \((1 - s_t P'(c_t)(1 - l_t))\), which is the marginal income that it costs to consume a marginal unit of consumption; Equation (3) is the usual condition that, for an interior solution for leisure, the marginal rate of substitution between consumption and leisure must be equal to the wage \( P(c_t) \). Finally, Equation (4) is the familiar intertemporal condition for models with savings.

Before we continue with the analysis it is instructive to study the effects of the nutrition curve on the allocation of consumption and leisure. We summarize these effects in the following lemma:

**Lemma 1:** Assume A1-A5. i) If \( s_t = 0 \) then \( l_t = 1 \); ii) There is a \( c^- > 0 \) such that \( l_t = 1 \) whenever \( s_t = 1 \) and \( c_t \leq c^- \); iii) There is a finite \( c^+ < +\infty \) such that \( l_t = 1 \) whenever \( s_t = 1 \) and \( c_t \geq c^+ \).

**Proof:** See the Appendix.

Not surprisingly Lemma 1.i says that an agent in the unemployment state chooses not to work. Also, Lemma 1.iii states that for sufficiently large levels of consumption an employed agent optimally chooses not to work. This is the usual wealth effect typically stressed in the literature on labor supply when leisure is a normal good. Interestingly, Lemma 1.ii states that below some consumption level, it is also optimal no to work. The reason why an agent chooses not to work at low levels of consumption is because the “power” of
her labor supply is very small due to the effect of the nutrition curve. In that case the agent is better off by consuming less, but by enjoying more leisure. We conclude that the nutrition curve dramatically reduces the incentives to supply labor for those agents enjoying a low level of consumption. Since we see very poor people lying around in the streets (in countries without a welfare state), our model seems to match this informally collected bit of reality, while the usual model would predict that very poor agents would always work very hard. A way to summarize Lemma 1 would be to say that “only the middle-class faces uncertainty.”

The introduction of the nutrition curve has also large effects on the consumption sets. Consider momentarily a static version of the previous problem in which we define net wealth as $\tilde{\theta} = \theta + b_t - c_t - q_t b_t$. With this definition the budget constraint of an employed (or productive) agent can be written as $c \leq \tilde{\theta} + P(c)(1 - l)$, and we define the consumption set of the static problem as $X = \{(c, l) \in \mathbb{R}_+ \times [0, 1] : c \leq \tilde{\theta} + P(c)(1 - l)\}$. A first interesting observation is that if $\tilde{\theta} \leq 0$ then $X$ is empty. An implication of this is that it is never optimal for a productive agent to consume only whatever results from supplying labor into the market, i.e., consumption in the productive state is larger than $P(c_t)(1 - l_t)$. Furthermore, we also have

Lemma 2: Assume A1-A5. For $\tilde{\theta} > 0$ the set $X$ is not convex.

Proof: See the Appendix.

The implication of Lemma 2 is that the FOC’s computed above are necessary but not sufficient for optimality, hence there may be more than one pair of consumption and leisure choices satisfying them. Also, notice that in Lemma 1 nothing prevents $c^- \geq c^+$, in which case the problem is uninteresting because, nor uncertainty neither endogenous labor supply, have any bite. Assumptions 6 and 7 below rule out these possibilities and assure that interior solutions are unique.

A6: i) $c^- < c^+$ and for all $c \in [c^-, c^+]$, the identity $u'(c)P(c) - u'(l)(1 - P'(c)(1 - l)) \equiv 0$, implicitly defines a function $l_f : [c^-, c^+] \to A \subseteq [0, 1]$; ii) furthermore, $u'(c)/(1 - P'(c)(1 - l_f(c)))$ is monotonically decreasing in $c$.

A7: Given any $\gamma \in \mathbb{R}_+$, define $c^0$ and $c^1$ (as a function of $\gamma$) by the following equations

$$u'(c^1) = \frac{u'(c^0)}{1 - P'(c^1)(1 - U(c^1))} = \gamma$$

\footnote{It should be clear that Lemma 1 would also obtain with an Inada condition for consumption at the origin combined with the nutrition curve discussed in the previous footnote.}
We require that
\[ c^1 - P(c^1)(1 - l^f(c^1)) \neq c^0, \] (7)
provided that \( u'(c^-) > \gamma > u'(c^+) \).

The identity in A6 i) is simply the result of combining the first order conditions for optimality with respect to consumption and leisure for an agent’s problem when leisure is interior. It ensures that for levels of \( c \in [c^-, c^+] \) there is only one possible level for leisure satisfying the optimality conditions. That is, A6 guarantees that Equations (2) and (3) are necessary and sufficient for an optimum. Notice in particular that the expression in A6 ii) corresponds to the Lagrange multiplier associated to the budget constraint of an agent’s problem. Hence, A6 is automatically satisfied in models abstracting from the nutrition curve.

The importance of A7 is discussed to some extent after Lemma 3 below, and in more detail in the next section. As of now we simply mention that it implies that

**Lemma 3**: Assume A1-A7. For all \( q \geq \beta \), \( \exists b > B \) such that if \( l \in (0,1) \) is optimal in state \( s = 1 \), then \( b_t = b = b_{t-1} \) is optimal in both states \( s = 0 \) and \( s = 1 \).

**Proof**: See the Appendix.

Lemma 3 asserts that for prices in the set of admissible equilibrium prices (Huggett 1993, Aiyagari 1994), there is no asset level such that the agent is willing to keep constant forever provided that in the employment state it is optimal to work (in the jargon of dynamic programming, this means that decision rules for bonds at \( s = 0 \) and \( s = 1 \) do not cross on the 45 degrees line). Given this, the only two possibilities are that i) the agent keeps visiting all possible levels of bond holdings, as in the usual models of incomplete markets, or ii), that bond holdings that the agent is willing to sustain forever are associated to not supplying any labor. Of course, in this case Lemma 1 states that only when the agent is sufficiently poor, or sufficiently rich, he/she chooses optimally not to work. The main result of the paper is to show that this is precisely what happens in any stationary equilibrium.

### 3 Results

We now examine several properties of the stochastic processes that solve the problem in (2) under the assumption that \( B < B = \theta/(q - 1) \). The amount
B corresponds to the maximum debt that a consumer is able to repay with probability 1 in the following period when \( b_{t-1} = B \). This means that the central authority would never extend a credit balance beyond \( B \), and therefore, \( b_t > B \) and \( q_t \gamma_t = \beta E[\gamma_{t+1}] \) in all periods. For future reference we introduce a notion of competitive equilibrium:

Definition: A Stationary Competitive Equilibrium is a price \( q \) and stochastic processes for \( sp_i = \{c^i, l^i, b^i\} \), such that: (i) \( sp_i \) satisfies conditions Eq. (2)-(6) given \( q \) and Eq. (4) holds with equality for each agent \( i \), and such that (ii) \( \int b_idi = 0 \).

The first part of Proposition 1 is a version of the well known result that even with incomplete insurance to idiosyncratic uncertainty asset holdings diverge to infinity whenever its return is larger than the rate of time preference (e.g., Chamberlain and Wilson, 2002). In our model assets continuously grow until consumption reaches the satiation level. Once there, asset accumulation stops growing. Unlike the usual result in incomplete markets models, the second part of the proposition states that in the long run, consumption would be zero if the asset’s return is below the rate of time preference. The intuition for this result is that agents deplete assets to smooth out utility, and at some point, they become sufficiently poor so that the idiosyncratic uncertainty has no bite (because of the nutrition curve, as stated in Lemma 1). Given that \( \beta/q < 1 \), then it is optimal to keep depleting assets over time and the agent observes a consumption sequence that converges to zero (and assets converge to the natural borrowing limit).

**Proposition 1:** Assume A1-A7 and \( \beta/q > 1 \), then \( \lim_{t \to \infty} c_t \to c \geq \bar{c} \), a.s.

ii) Assume A1-A7 and \( \beta/q < 1 \), then \( \lim_{t \to \infty} c_t = 0 \), a.s.

**Proof:** See the Appendix.

An implication of Proposition 1 is that neither \( \beta > q \) nor \( \beta < q \) can be the prices that hold in a stationary equilibrium, since they would imply that for all agents either aggregate assets grow without bound (in the former case), or aggregate assets converge to a negative amount (in the latter). The next proposition describes what happens in the long run when \( \beta = q \).

**Proposition 2:** Assume A1-A7 and \( \beta/q = 1 \). Then,  

i) \( \Pr(b(lim_{t \to \infty} c_t \leq c^-) + \Pr(b(lim_{t \to \infty} c_t \geq c^+) = 1. \)

ii) If in addition \( c_0 \in (c^-, c^+) \), then \( \Pr(b(lim_{t \to \infty} c_t = c^- \text{ or } c^+) = 1. \)

**Proof:** See the Appendix.

The proposition says that if the initial amount of assets is such that \( c_0 \geq c^+ \),
then the agent faces no uncertainty, and the optimal decision entails to keep assets constant, enjoy the full unit of time as leisure, and consume the amount $c_0$ period after period. A similar situation applies when $c_0 \leq c^-$, but with a smaller consumption and assets level. For asset levels such that $c_0 \in (c^-, c^+)$, then for some time the agent keeps increasing consumption and assets when she receives the good shock (and decreasing consumption and assets when she receives the bad shock), but sooner or later the agents becomes sufficiently rich (or poor), so that idiosyncratic uncertainty disappears and the agent optimally decides to keep a constant consumption stream equal to $c^+$ (or $c^-$) forever.

We summarize the implications of Propositions 1 and 2 in the following corollary, and postpone a thorough discussion until Section 4.

**Corollary:** 1) Only if $\beta/q = 1$ there can be a Stationary Competitive Equilibrium.
2) In any Stationary Competitive Equilibrium the distribution of wealth is polarized and without mobility.
3) There exists many stationary distributions compatible with Stationary Competitive Equilibrium.

*Proof:* See the Appendix.

### 4 Discussion

1. **No mobility.** We obtain that in the long run society is fully polarized in rich and poor households, and that there is no mobility at all between these two groups. This result is in sharp contrast with previous findings in the literature because in our model markets are nearly complete: All workers are paid their marginal productivity of labor and there is competitive price taking in asset markets. **WHAT ARE THE RELEVANT REFERENCES HERE??**

2. **Existence of equilibria.** The existence of a Stationary Competitive Equilibrium is by no means guaranteed. The reason is that the assets level required to sustain $c^-$, call it $b^-$, is not necessarily negative, hence in that case all agents would hold positive assets and thus $q = \beta$ would not clear the assets market. The amount $b^-$ depends on the precise shape of $u$, $v$, $P$, and on $\beta$ and $\theta$, so it is straightforward to construct examples where $b^-$ is negative, zero, or is strictly positive. If the combination of underlying functions and parameter values guarantees that $b^-$ is negative with $q = \beta$, 

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then it is also straightforward to construct stationary equilibria by conveniently reallocating mass of agents in the two mass-point distributions. If $b^-$ is known to be positive when $q = \beta$, then it would be possible to study whether equilibrium dynamics are monotone, cyclical, or display a chaotic behavior. The study of the dynamic equilibrium is certainly interesting but beyond the scope of the current paper.

3. **About assumption A6.** Incomplete markets models with idiosyncratic shocks in the Macro literature (e.g. Aiyagari, Huggett, and Krusell and Smith) share the fact that the derivative of the value function (or the Lagrange multiplier associated to the budget constraint) is monotonically decreasing in assets. This is related to A6 because in those models consumption can be shown to be strictly increasing in assets and marginal utility of consumption is decreasing. These facts follow directly from the strict concavity of the instantaneous return function and from the convexity of the consumption set, and they guarantee the existence of (optimal) policy functions (as opposed to correspondences). The presence of the nutrition curve delivers non convex sets (Lemma 2), hence A6 helps to simplify our analysis because it eliminates the possibility of multiple solutions (i.e., that for a given level of assets, several leisure/asset levels are mutually compatible). A monotonically decreasing Lagrange multiplier is obtained in the bad state ($s = 0$) under general strictly increasing and strictly concave utility functions. In the good state ($s = 1$), the technical requirement for a decreasing Lagrange multiplier when the solution for leisure is interior can be written as condition

$$C1: \frac{dl}{dc} \frac{v''(l)}{v'(l)} \leq \frac{P'(c)}{P(c)},$$

with

$$\frac{dl}{dc} = \frac{u''(c)P(c) + u'(c)P'(c) + v'(l_f(c))(1 - l_f(c))P''(c)}{v''(l_f(c))(1 - P'(c)(1 - l_f(c))) + v''(l_f(c))P'(c)}.$$  

Since the nutrition curve changes the incentives to work at low levels of consumption (and thus $dl/dc$ may be negative), it is difficult to verify C1 (hence A6) at the same level of generality as with $s = 0$. In case this condition does not hold, the analysis is more involved because we need also to check the associated sufficient conditions, but the previous results still go through because the necessary conditions we relayed on must still be satisfied.

4. **About assumption A7.** A second property of the standard incomplete markets models abstracting from the nutrition curve that has to do with the derivative of the value function is that in the good state it is smaller than in the bad state (for a given level of assets). This second fact is an
equilibrium property, and by means of the Benveniste-Scheikman result, it can be restated as saying that the marginal valuation of consumption is always larger in the bad state than in the good state. Viewed in this way, the Euler Equation dictates that a certain proportion must hold between current consumption’s valuation and its expected present value in the following period. Hence, optimal decisions entail depleting assets in the bad state (because the current valuation of assets is larger than its expectation in the next period when wealth is kept constant) and increasing them in the good state (because the current valuation of assets is smaller than its expectation in the next period if wealth was constant). Our concern here is that the nutrition curve may be able to reverse the label “good” state from $s = 1$ to $s = 0$ and “bad” state from $s = 0$ to $s = 1$, precisely because consumption is extremely costly when assets are small (see Eq. (2)), and that if this happens, then the incentives for asset accumulation will also be reversed.

Consider the following example characterized with $u(c) = \log(\bar{c} + c)$, with $\bar{c} = .08$, $v(l) = \phi \log l$, with $\phi = .09$, $P(c) = pc/(1+c)$, with $p = .15$, together with $\pi_{1|1} = \pi_{1|0} = .5$, $\theta = .22$, and $\beta = .99$. Under these assumptions we fix $q = \beta$, and investigate whether stationary equilibria exist or not. Figure 1 depicts optimal decision rule for leisure in $s = 1$ as a function of assets. The figure shows that leisure displays a U-shape. In particular, for asset levels below $b^- = -0.448$ labor supply is equal to zero ($l = 1$), and for asset levels above $b^+ = 15.192$, leisure is again equal to one. Since $b^- < 0$ and $b^+ > 0$, it is possible to construct stationary distributions such that $q = \beta$ is a Stationary Competitive Equilibrium.

We now look at the general shape of $\gamma$ in each state. In particular, we checked that these multipliers are both monotonically decreasing (hence the example satisfies all A1-A6), although the differences between them are rather small. Figure 2 plots the difference between $\gamma^1$ (as described in A6.ii) and $\gamma^0$ (from Eq. (4)) as a function of $b$: $\Gamma(b) = \gamma^1(b) - \gamma^1(b)$. A first thing to notice is that unlike other incomplete markets models, for $b > b^-$ this difference is initially positive. $\Gamma(b)$ then equals zero for some $\hat{b}$ (which in the example is about 6.244), and finally it becomes negative afterwards, as standard theory would predict. It follows from the Euler Equation (4) that $b_{t+1} = b_t$ in both $s = 1$ and $s = 0$ when $b_t = \hat{b}$, hence $c^0 = c^1 - P(c^1)(1 - l_f(c^1))$. Thus, in

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8 We solve for decision rules iterating on an initial guess for policy functions using Eq. (4) and a search method on a grid of states. See the Appendix for further details.
this example \( \hat{b} \) violates A7.

*** INSERT FIGURE 2 ABOUT HERE ***

The implications of A7 not holding are most clearly seen in the next figure. Figure 3 displays the differences \( b_{t+1} - b_t \) for \( s_t = 1 \) and \( b_{t+1} - b_t \) for \( s_t = 0 \). These differences vanish below \( b^- \), above \( b^+ \), but also at \( \hat{b} \). Unlike other incomplete market models, in this example agents deplete assets in the “bad” \( (s = 0) \) state for \( b \in (\hat{b}, b^+) \) but accumulate them for \( b \in (b^-, \hat{b}) \) (precisely the opposite of the decision rule in the “good” state \( (s = 1) \)). That is, there is a reversion over the exogenous states \( s \) in the pattern of accumulation/decumulation of assets. More importantly, here \( b^- \), \( b^+ \), and \( \hat{b} \) are all absorbing states, and thus, in this example there is polarization but equilibrium distributions are three mass-point distributions.

*** INSERT FIGURE 3 ABOUT HERE ***

The example above suggests that it may be possible to construct examples with many mass-point distributions. We conclude the discussion about this issue with an analytical example in which only two mass point distributions are possible. Consider the case of linear leisure \( v(l) = \phi l \) (with the same other functions as before). In this case we have that if there was a \( \hat{b} \) such that \( b_{t+1} = b_t \) in both \( s = 1 \) and \( s = 0 \) when \( b_t = \hat{b} \), and \( c^0 = c^1 - P(c^1)(1 - l_f(c^1)) \) with \( l_f(c^1) \in (0, 1) \) then we would have

\[
\gamma^1(c^1) = \frac{\phi(1 + c^1)}{pc^1},
\]

and

\[
\gamma^0(c^0) = \frac{1}{\epsilon + c^0} = \frac{\phi(\bar{c} + c^1)}{\phi(\bar{c} + c^1)[\bar{c} - (c^1)^2] + (c^1)^2p} = \gamma^0(c^1).
\]

To simplify notation, denote \( c^1 \) imply by \( c \) and let \( \Gamma(c) = \gamma^1(c) - \gamma^0(c) \). It should be clear that \( \Gamma(c^-) = \Gamma(c^+) = 0 \), except in the limiting case of \( c^- = 0 \) in which \( \Gamma \) is not well defined (but still, if assets are such that the optimal consumption is zero then the best the agent can do is to choose \( l = 1 \) in both states). Notice that if \( \bar{c} = 0 \) then \( c^- = 0 \) (this is found by solving Eq. (2) and Eq. (3) under the assumption of \( l = 1 \), which delivers \( c(\phi c + (\phi - p)) = 0 \). Thus, \( c^- = 0 \) and \( c^+ = (p - \phi)/\phi > 0 \) because we purposely choose \( p > \phi \). Some algebra reveals that

\[
\Gamma'(c) = \frac{\phi}{pc^2(cp - \phi c^2)^2}(-\phi^2c^4),
\]

hence, \( \Gamma'(c) < 0 \) for all \( c > 0 \). Since \( \Gamma(c) \) is continuous for \( c > 0 \), the
implication is that there is no $c$ such that $\gamma^1(c) = \gamma^0(c)$, hence, in this example equilibrium distributions are two mass-point distributions.9

5. Alternative specifications. The key mechanism driving our results is that both at low and high levels of wealth, uninsured idiosyncratic uncertainty has no effect. The model with the nutrition curve seems appropriate if one thinks of very poor societies and of economies at early stages of their development process. We present here an alternative story and suggest other mechanisms that may have similar effects but on developed societies. We think of a laziness model in which agents care about leisure and in which utility from leisure is subject to stochastic shocks.10 In particular, we assume that

\textbf{AL}: Utility in a given period is given by $u(c_t) + v(l_t, s_t)$. The utility from consumption satisfies $u_c > 0$ and $u_{cc} < 0$ and the usual Inada conditions. The utility from leisure satisfies $v_l > 0$ and $v_{ll} < 0$, and there is an $0 < \bar{l} < 1$ such that $v_l(l, s_1) = v_l(l, s_2)$ if $l \in [0, \bar{l}]$, and $v_l(l, s_1) < v_l(l, s_2)$ if $l \in (\bar{l}, 1]$. We think of a laziness shock as a bad shock such that both the utility from leisure and its marginal valuation is larger than in the good shock, but only in as much as leisure is above some threshold level $\bar{l}$. When leisure is below that level, i.e., when the agent decides to supply a substantial amount of labor, the bad shock looses its bite and thus the utility from leisure is state independent. The model can be completed with a production sector, in which competitive firms hire capital and labor at market prices. 

Like in the nutrition curve model, poor agents would optimally choose to have very low leisure, so that the realization of the shock will have no effect and agents face no uncertainty. Also, at high levels of wealth leisure is again one (because of the wealth effect). It follows that only steady states with polarized societies (poor workers that work a lot and rich families that do not work at all) are possible (see the Appendix for a further results from this model).

The laziness example suggests that polarization is possible even if there is no uncertainty about labor productivity or other sources of real income. In developed societies education and accumulation of human capital may play a role similar to that of the nutrition curve

6. Policy implications. Polarization can be avoided by any policy interven-\hfill

9Not only that: in this example agents deplete assets whenever they are employed, and increase their assets whenever they are unemployed.

10Shocks to preferences have widely used and investigated in economics. See for instance Lucas 1978, Hall 1984, or more recently().

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tion guaranteeing $c_t \geq c_{\text{min}}$ a.s.

5 Appendix

A. Proofs

Proof of Lemma 1: The arguments are static in nature, so in what follows we drop the time subscripts to simplify notation. i) Equation (3) reads $v'(l) \leq \eta$ when $s = 0$. Since $v'(l) > 0$ for $l < \infty$, then $\eta > 0$ and thus, $l = 1$ from the complementary slackness in Equation (6). ii) Take now $s = 1$, and assume that $l < 1 \forall c > 0$. Combining Equation (2) and Equation (3) we get $v'(c)P(c) = v'(l)(1 - P'(c)(1 - l)) > v'(1)(1 - P'(c)(1 - l))$, where the inequality follows from A3 and the hypothesis. Rearranging produces

$$l < 1 + \frac{u'(c)P(c) - v'(1)}{v'(1)P'(c)} = L(c).$$

We have that $\lim_{c \to 0} L(c) = 1 - 1/P'(0) < 0$, and that $\lim_{c \to +\infty} L(c) = 1 - 1/(v'(1)P'(\infty))$ which diverges to $-\infty$, by A5. By continuity, then, there is $c^-$ such that $l = 1 \forall c \leq c^-$. Also by continuity, there is $c^+$ such that $l = 1 \forall c \geq c^+$.

Proof of Lemma 2: Take $x_1$ and $x_2$ in $X$ with $c_1 < c_2$ and satisfying the budget constraint with equality, choose $\alpha \in [0, 1]$ to compute $\bar{c} = \alpha c_1 + (1 - \alpha)c_2$ and $\bar{l} = \alpha l_1 + (1 - \alpha)l_2$, and define $F(\alpha) = \alpha(1-l_1)/(P(c_1) - P(\bar{c})) + (1 - \alpha)(1-l_2)/(P(c_2) - P(\bar{c}))$. We will show that $\bar{x} = (\bar{c}, \bar{l}) \not\in X$ for some $\alpha \in (0, 1)$ by showing that $F(\alpha) > 0$ for some $\alpha$. The budget constraint implies that $1 - l_t = (c_1 - \bar{\theta})/P(c_1)$, which we substitute in $F(\alpha)$ to obtain $F(\alpha) = (c_2 - \bar{\theta})(1 - \alpha)/(P(c_2) - P(\bar{c}))/P(c_2) - (c_1 - \bar{\theta})\alpha(P(\bar{c} - P(c_1))/P(c_1)$.

Notice that since $c_1 < c_2$, the result would follow if we are able to show that $(1 - \alpha)(P(c_2) - P(\bar{c}))/P(c_2) > \alpha(P(\bar{c}) - P(c_1))/P(c_1)$ for some $\alpha$. If this were not true, then $(1 - \alpha)(P(c_2) - P(\bar{c}))/P(c_2) \leq \alpha(P(\bar{c}) - P(c_1))/P(c_1)$ for all $\alpha$. This would deliver $P(c_1)P(c_2) \leq P(\bar{c})(\alpha P(c_2) + (1 - \alpha)P(c_1)) < P(\bar{c})P(\alpha c_2 + (1 - \alpha)c_1)$, where the strict inequality follows from $P$ being strictly increasing and strictly concave. By continuity, this would result in a contradiction for all $\alpha$’s sufficiently close to 0 and to 1. Hence, for those $\alpha$’s, the desired result follows because the first term in $F(\alpha)$ is strictly larger than the second.

Proof of Lemma 3: We proceed by contradiction. To this end assume that there is $b > B$ such that $b_t = b = b_{t-1}$ is optimal for both $s = 0$ and $s = 1$, implying respectively, that $c^+_t = \theta + b(1-q)$ and $c^+_t = \theta + b(1-q) + P(c^+_t)(1-
\( l^u(c_t^\gamma) \), or \( c_t^\gamma = c_t^u + P(c_t^\gamma)(1 - l^u(c_t^\gamma)) \). But if \( \beta/q = 1 \) this contradicts A7, since \( l^u(c_t^\gamma) \in (0, 1) \) (by hypothesis), \( c_t^\gamma \in (c^-, c^+) \) (by Lemma 1), and Eq. (4) evaluated at \( s = 1 \) and \( s = 0 \) implies that \( \gamma_t^\gamma = \gamma_t^u \). If \( \beta/q < 1 \) then Eq. (4) evaluated at \( s = 1 \) implies that \( \gamma_t^\gamma < \gamma_t^u \), but the same equation evaluated at \( s = 0 \) implies \( \gamma_t^u < \gamma_t^\gamma \), another contradiction. The same argument can be used to rule out the possibility when \( \beta/q > 1 \).

We will use the following definitions: \( \Omega^- = \{ \omega \in \Omega : c_t(\omega) = c^- \text{ for some } t \} \), and \( \Omega^+ = \Omega \setminus \Omega^- \).

**Proof of Proposition 1**

From Eq. (4) we have that \( \gamma_t \geq \beta/q E_t[\gamma_{t+1}] \). Multiplying both sides by \( (\beta/q)^t \) and defining \( \lambda_t = (\beta/q)^t \gamma_t \), it follows that \( \lambda_t = E_t[\lambda_{t+1}] \) forms a non-negative martingale and thus it converges to a finite random variable with probability one.

\( \beta/q > 1 \): In this case \( \lim_{t \to \infty} (\beta/q)^t \) diverges to infinity, hence \( \lim_{t \to \infty} \gamma_t = 0 \). It follows from Eq. (2) that in the limit \( u'(c) = 0 \). By A2, this can only happen if \( c \geq \bar{c} \).

\( \beta/q < 1 \): Eq. (4) implies that \( \gamma_0 = (\beta/q)^t E_0[\gamma_{t+1}] \). Since \( (\beta/q)^t \) converges to zero, then it must be the case that \( E_0[\gamma_{t+1}] \) diverges to infinity. It follows from Eq. (2) that \( c_t = 0 \) for some finite \( t \). Once this happens the FOC for capital \( k_t \) is a non-negative martingale, and by A2 and A6, it is bounded, so that it converges a.s. to a random variable \( \gamma \). We now prove that consumption converges a.s.. Notice that either \( \omega \in \Omega^- \), or that \( \omega \in \Omega^+ \). If \( \omega \in \Omega^- \), then \( c_{t+j} = c^- \) for \( j = 0, 1, 2, ..., \) by P1. Assume now that \( \omega \in \Omega^+ \). Then \( b_{-1} \in (b^-, b^+) \) and R2 implies that \( c_t(\omega) > 0 \) for all \( t \). Therefore,

\[
\frac{u'(c_t(\omega))}{1 - s_t(\omega)P_t(c_t(\omega))(1 - l_t(\omega))} = \gamma_t(\omega), \forall t,
\]

and \( c_t \) converges a.s. to the set \( \{ c^1(\gamma(\omega)), c^0(\gamma(\omega)) \} \), where each element of this set is defined analogously as in A7. To prove (??) note that if \( u'(c^-) > \gamma(\omega) > u'(c^+) \), by assumption A7 deficit uncertainty is positive so that, by the usual argument, the budget constraint would explode and the consumption would be unfeasible or suboptimal, so that this is impossible. The only possibilities are that \( u'(c^-) = \gamma(\omega) \), or that \( \gamma(\omega) = u'(c^+) \), and in these cases consumption must converge to \( c^- \) or to \( c^+ \). Notice that in both cases, labor supply is zero.

ii) Again, \( k_t \) is a bounded martingale, so it converges a.s. to \( \gamma \). Since convergence a.s. implies convergence in probability, then \( \gamma_t \to \gamma \) in probability.
The Euler equation implies
\[ \gamma_0 = (\beta/q)^t E_0 (\gamma_t) \]
for all \( t \) and, since \((\beta/q)^t \to \infty\) then \( \gamma_t \) converges in mean to 0. Since, by the Chebyshev inequality, convergence in absolute mean implies convergence in probability, we have \( \gamma_t \to 0 \) in probability. Therefore, \( \gamma_t \to 0 \) a.s., so that \( u'(c) \to 0 \) which can only occur if consumption goes to saturation.

iii) Now \((\beta/q)^t \to 0\) so \( E_0 (\gamma_t) \to \infty\). This implies that \( Prb (\nu_t = 0 \text{ at some } t) > 0\), otherwise, \( E_0 (\gamma_t) = E_0 \left( \frac{u'(c_t)}{1-\delta_t P(c_t) (1-l_t)} \right) < u'(0) < \infty\). The fact that \( P(\nu_t = 0 \text{ at some } t) > 0\) implies that \( b(\cdot, 0) \) is flat at \( b = -r\Omega \), otherwise it is impossible that the positivity constraint on consumption is binding. This implies that we can go to zero consumption in finitely many steps so we go there with probability one (this argument must be completed ...).

B. The Laziness model

An agent’s problem, given the price \( q \geq \beta \) reads:

\[
\max_{\{c_t, l_t, b_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(l_t, s_t)\} \tag{8}
\]
\[
\text{s. to } c_t + qb_t = w(1-l_t) + b_{t-1}, \tag{9}
\]
\[
c_t \geq 0, l_t \in [0,1], b_t \geq -B. \tag{10}
\]

The solution to the previous problem can be stated in terms of a value function \( v(b, s) \), and policy functions for consumption \( c(b, s) \), leisure \( l(b, s) \), and assets \( b(b, s) \). Under the maintained assumptions

R1 (Huggett 1993, and MOW): \( c(b, s) \) is strictly increasing in \( b \), \( l(b, s) \) is increasing in \( b \), and \( b(b, s) \) is increasing in \( b \). For a given \(-B\), there is an asset level \( b^- \) such that \( b(b, s) = -B \) for all \( b \in [-B, b^-] \).

Notice that the natural borrowing limit in this model is given by \( w/(q-1) \).

If an agent was born with \( b = w/(q-1) \), that agent could just keep constant her debt level by choosing \( l = 0 = c \) in every period. For borrowing limits above the natural level but close to it, it follows from R1 that there are asset levels for which consumption and leisure can be made arbitrarily small. It is also straightforward to use the FOC for optimality corresponding to the previous problem to show that:

R2: There is \( c^- > 0 \) such that \( l \leq \bar{l} \) for all \( 0 < c \leq c^- \).
The implication of R2 is that the borrowing limit can be chosen so that there will be asset levels such that consumption and leisure decisions will be state independent. At those asset levels, assets in the following period will also be state independent. We therefore have the following

**Lemma A1:** If \( c(-B, s_i) \leq c^* \) for \( q = \beta \), then \( q = \beta \) is an equilibrium price. The equilibrium is characterized by full polarization, with some agents enjoying \( l = 1 \) and some agents enjoying \( l = l \) in all periods.

**Proof:** Follows from Proposition 1 and 3 above.

It should be clear that in general, equilibrium prices depend on \(-B\). It could be, therefore, that for very stringent borrowing limits \( l > l \). In this case we have the following result:

**Lemma A2:** If \( l(-B, s_i) > l \), then the equilibrium is characterized \( q > \beta \) as in the Aiyagari-Huggett world: the equilibrium distribution is unique and there is no polarization.

**Proof:** Follows from Proposition 1 and 3 above.

### C. Numerical simulations

For simplicity we explain the solution algorithm for the laziness model (the nutrition curve is a bit more involved because \( P \) adds substantial non-linearities and a slightly different strategy is needed). In the language of dynamic programming, the solution of the the laziness model entails decision rules for asset holdings and leisure such that the following FOC for optimality is satisfied:

\[
 u'(b + (1 - l(b, s)) - qb(b, s)) \geq \frac{\beta}{q} E \left[ u'(b(b, s) + (1 - l(b(b, s), s')) - qb(b(b, s), s')) | b, s \right],
\]

which holds with equality whenever \( b(b, s) > -B \). We discretize the state space in the \( b \) dimension to form a grid \( B = \{b_1, b_2, ..., b_n\} \). For each \( b \in B \) and \( s \), we postulate an initial guess for \( b(b, s) \) and for \( l(b, s) \). For a given \( b \) and \( s \) we then seek for the \( b_j, b_{j+1} \in B \) such that

\[
 qu'(b + (1 - l(b, s)) - qb_j) < \beta \sum_{s'} \pi_{s'|s} u'(b_j + (1 - l(b_j, s')) - qb(b_j, s')) ,
\]

\[
 qu'(b + (1 - l(b, s)) - qb_{j+1}) > \beta \sum_{s'} \pi_{s'|s} u'(b_{j+1} + (1 - l(b_{j+1}, s')) - qb(b_{j+1}, s')) ,
\]

so that we can be sure that the \( b' \) that satisfies Equation (11) lies between \( b_j \) and \( b_{j+1} \). This value is found using one of the above equations and linearly interpolating between \( l(b_j, s') \) and \( l(b_{j+1}, s') \) and between \( b(b_j, s') \) and
This linear interpolation delivers one equation in one unknown that can be easily solved up to arbitrary accuracy with standard methods (e.g., bisection or Newton-Raphson). The procedure is repeated for every point in the grid, and the new policy values are updated at the end of each complete iteration. Notice that this procedure is not the same as iterating directly on the derivative of the value function (used for instance in Huggett 1993, and in MOW). Both procedures allow for choosing asset levels not in the grid, but it seems that the procedure described above is substantially more accurate than the one iterating on the derivative of the value function. We thank Josep Pijoan-Mas for suggesting us this approach.
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Figure 1: Policy function for leisure $l(b, 1)$. 
Figure 2: Difference between the Lagrange multiplier in the good and bad state as a function of assets: $\gamma^1 - \gamma^0$. 
Figure 3: Accumulation and decumulation of assets in the good and bad state. The thinner line measures $b(b, 0) - b$, and the thicker line measures $b(b, 1) - b$. 