LEARNING ABOUT IDENTIFICATION

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ABSTRACT. When Rational Expectations models feature multiple equilibria, stability under learning is often used as an equilibrium selection criterion (e.g., Sargent (1993) and Evans and Honkapohja (2001a)). When a 'large' agent, whose actions influence the data-generating process, confronts model uncertainty, however, we need a selection criterion for multiple learning models, to use a learning model to select a particular equilibrium. If a model is to be used for policy, specification of a model requires identification restrictions. On the equilibrium path, these identification restrictions are untestable. Although alternative identification schemes deliver alternative reduced forms, when the data are endogenous, estimates of each of these reduced forms can adapt to fit the data equally well. In this sense, learning models confront their own multiplicity problem more insidious than the multiplicity of Rational Expectations equilibria.

This paper develops a selection method by introducing off-equilibrium path experimentation into traditional recursive learning models. Although each of several competing identification schemes may fit the data equally well, they are not all equally robust to off-equilibrium path perturbations. By exploiting these differences, we show how even large agents can learn about identification. Essentially, this paper proposes a stability concept for selecting adaptive learning models, which leads to a selection criterion for multiple stable self-confirming equilibria. As an application, we examine the problem of distinguishing between Classical and Keynesian directions of fit in the Phillips Curve (Sargent (1999), King and Watson (1994)), and claim that that the Classical direction of fit is more robust to experimentation, and is eventually selected by our method almost surely.

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What the government observes depends partly on what it believes. – Sargent (1999, p. 75)

1. Introduction

The Rational Expectations revolution of the 1970s produced many changes in economics. From a practical standpoint, perhaps the most important concerned econometric identification methods. Rational Expectations econometrics was designed to address the problem of feedback between beliefs and outcomes. That is what the Lucas Critique was all about. The key to achieving identification in these settings is to exploit the cross-equation restrictions imposed by the Rational Expectations hypothesis. Applying these identification methods allows policy makers to obtain reliable estimates about the effects of policy innovations.

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1Lucas and Sargent (1981) contains a summary of this early work.
Although these methods have been widely applied in the academic literature, they have never really been embraced by practitioners. One could debate why this is, but a likely candidate is lingering doubts about the Rational Expectations hypothesis itself. Policy makers are in the business of confronting and managing model uncertainty, so the Rational Expectations assumption that policy makers somehow know the relevant specifications of tastes and technology has always been treated with skepticism.

Policymakers have been somewhat more receptive to recent work on adaptive learning (Evans and Honkapohja (2001b)). This literature retreats from the Rational Expectations Hypothesis in some important ways. First, it assumes policy makers must learn about the parameters of their models. Since the policy maker’s actions influence the data-generating process, parameter uncertainty produces potential gains from experimentation. The macroeconomic learning literature usually abstracts from this, either because policy makers are assumed to be unaware of this feedback, or because they purposely choose to neglect it. Perhaps the most significant departure from Rational Expectations, however, concerns the identification problem. Of course, policy makers in traditional Rational Expectations models do not confront this problem. They know the model. Only econometricians need cross-equation restrictions, as they must estimate what the policy maker already knows. The policy maker’s job is purely technical, i.e., compute time paths under alternative policy rules. Once you withdraw knowledge of the model, however, the policy maker must confront this problem in order to obtain reliable estimates of the effects of policy innovations. Existing work on adaptive learning has focused exclusively on the problem of estimation, as opposed to inference. Policy makers in learning models do not worry about identification. They simply estimate reduced forms. In most applications this is harmless, since most applications assume the policy maker’s model is correctly specified. However, in more recent applications, which feature model misspecification, identifying assumptions are never questioned (e.g., Sargent (1999)). Since economists spend most of their time arguing about identification, and since the learning literature has always been motivated by the attempt to achieve symmetry between economists and the agents they model, it would seem to be essential to allow agents in these models to learn about identification. Our paper is an attempt to do this. In a sense, we are trying to answer the call of Peter Ireland to develop an ‘Irrational Expectations’ econometrics (Ireland (2003)).

We study a policy maker who has a set of models, each of which is recursively estimated. Although our policy maker does not know the true model, he is quite shrewd. He knows that in drawing inferences about the usefulness of each model, he faces a difficult counterfactual - How would history have been different if some other model had been used to formulate policy? Fitting a model to data that was generated while another model was in use could produce highly misleading inferences about how a candidate model would perform if given the chance. In response, our policy maker does two things. First, he deliberately experiments with alternative models. As discussed in more detail later, since the data are endogenous, it is essential that the policy maker occasionally try new models. Otherwise, he can fall into the trap of of getting stuck in bad self-confirming equilibria (Sargent (1999) and Fudenberg and Levine (2009)). Second, in order to obtain accurate

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estimates of how a model would perform if given the chance, model parameters are only updated while a model is in use.

The key to our analysis is that off-equilibrium path experimentation provides valuable information about the validity of each model’s identifying restrictions. By definition, identifying restrictions are untestable within a given sample. However, when a model is used out-of-sample, which in our case means off-the-equilibrium path since the data are endogenous, then invalid identification restrictions will be revealed by unexpected policy outcomes. By keeping track of these errors, and adjusting each model’s selection probability appropriately, we show that one can eventually learn to impose the right identification restrictions. This idea bears some resemblance to work that studies the empirical relevance of the Lucas Critique. For example, Miller and Roberds (1991) show that reduced form VAR forecasts deteriorate following policy regime changes, which they interpret as evidence supporting the Lucas Critique. However, this work is purely retrospective. It does not recursively compare the relative forecasting performance of alternative reduced forms based on alternative identification restrictions. It is the real-time comparisons that enable our agent to learn about identification.

We should emphasize that while our policy maker is relatively sophisticated in how he approaches the problem of endogenous data, he is not particularly sophisticated as an econometrician. He makes no effort to formally test the specification of his model along other dimensions. In fact, our policy maker adheres to a fallacious ‘Principle of Fit’ (Kocherlakota (2007)) when selecting among models. That is, model selection probabilities are determined by each model’s (relative) forecast errors. As Kocherlakota (2007) points out, there is certainly no guarantee that better-fitting models produce better policies. Kocherlakota shows how a well fitting model, which is based on erroneous identification restrictions, can produce worse predictions about the effects of policy than a poorer fitting model, which is based on identification restrictions that are closer to the truth. In Kocherlakota’s examples, the decision maker never discovers his error because he never entertains the possibility that he could be wrong. If he did, he would explore alternatives. Since these alternatives call for different tax policies, experimenting with alternative models quickly reveals which provides the more reliable policy predictions.

As an illustration, we revisit the analysis in Sargent (1999). Following the lead of King and Watson (1994), Sargent compares two Phillips Curve identification schemes, one based on a ‘Classical’ direction of fit, which places unemployment on the left-hand side and inflation on the right-hand side, and one based on a ‘Keynesian’ direction of it, which does the reverse. The actual data generating process (DGP) is a Lucas-style natural rate model, in which (unbeknowst to the policy maker) only unanticipated inflation influences unemployment. The two specifications embody alternative identification restrictions, based on alternative least-squares orthogonality conditions. Although both are misspecified, since they fail to account for the private sector’s expectations, the Classical fit is closer to the truth. In the Keynesian model, the explanatory variable is correlated with the error term, and this produces a Phillips Curve slope estimate that is flatter and more biased than the Classical model’s. Consequently, the Keynesian model produces worse outcomes.

In related work (Cho and Kasa (2008)) we study a more formal, statistically based, approach to model validation, while abstracting from potential identification problems.
It is interesting that while Sargent, the modeler, worries about identification, the agent in his model doesn’t. Even though the Keynesian fit produces worse outcomes, the policy maker in Sargent’s model would have no way of knowing this unless he explored alternatives. In fact, if model selection is based on fit, as we assume, then he would have no reason to switch, even if he compared notes with a foreign colleague who used the Classical fit, since both fit the data equally well.

Of course, if model selection were based on realized losses, then the superiority of the Classical fit would quickly be discovered by comparing outcomes. However, we feel there are good reasons to assume model selection is based on fit, simply because that seems to be the way the policy making process often works. For example, based on interviews and his own experience, Sims (2002) provides a detailed discussion of model construction and use at several national Central Banks. He identifies three common elements: (1) No Central Bank relies exclusively on a single model, deemed to be the ‘true’ model. Rather, a so-called ‘suite’ of models is used. These models differ in their dimensionality and the degree to which they are structural (i.e., the degree to which they worry about identification), (2) Reliance on models evolves over time, in response to past relative performance, and (3) Central Banks exhibit a clear division of labor between those who build and evaluate models, and those who make policy decisions. Although there are regularly scheduled meetings between the two, both sides acknowledge that policy decisions are based on more than just the recommendations of the research staff. Participants often refer to these non-model, non-quantitative, factors as ‘judgment’. To a certain extent, this judgmental input likely reflects constraints on policy or aspects of the loss function that are only imperfectly known the the research staff. If this is the case, then it is simply not possible to select models based on the policy makers actual loss function. Instead, the staff search for models that are both useful and fit the data reasonably well.

After summarizing model use and construction, Sims (2002) then takes a normative turn, and admonishes Central Bankers (and their staff) for not being more Bayesian. In contrast, our paper pursues a purely positive/descriptive approach, by formalizing the above process, and studying what kind of outcomes it produces. Our main result is to show that even when model selection is based on fit, as long as policy makers occasionally experiment, they can learn to avoid invalid identification restrictions. This experimentation can be interpreted as a form of ‘judgment’. In the case of Sargent’s (1999) model, experimentation means that you occasionally adopt the policy advice from the competing model. For example, if the current model is Classical, experimentation with the Keynesian model produces an increase in inflation. If the current model is Keynesian, then experimentation with the Classical model produces a decrease in inflation. Both models overestimate the unemployment consequences of (discrete) inflation changes. However, because the Keynesian model is more biased, its mistakes are larger, and so eventually the policy maker learns to avoid it.

The remainder of the paper is organized as follows. Section 2 describes the environment. It outlines the policy maker’s problem, and the problem of the research department. The

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4Reifschneider, Stockton, and Wilcox (1997) discuss the evolution of models within the U.S. Federal Reserve
5Of course, locally, in a neighborhood of their respective self-confirming equilibria, each model makes random errors when forecasting unemployment
model class is defined, and experimentation is modeled as a mixed strategy across models. We then introduce the concept of a ‘robust’ self-confirming equilibrium, defined to be a self-confirming equilibrium that is not only stable with respect to the dynamics of a single model’s parameter estimates, but is also robust to off-equilibrium path experimentation with other models. Section 3 uses traditional singular perturbation methods to characterize the invariant distribution across models. Under certain conditions, this invariant distribution eliminates models that impose invalid identification restrictions (i.e., their limiting probability of being selected is zero). Section 4 provides an example, based on the Phillips Curve model in Sargent (1999). We show that only the Classical self-confirming equilibrium is robust. Section 5 contains a few concluding remarks.

2. POLICY, RESEARCH AND EXPERIMENTATION

2.1. Preliminaries. Let $X$ be the state space, assumed to be a $K$-dimensional Euclidean space, where $K$ is the number of variables available to the model builder. Let $U \subset X$ be the set of policy actions, which is assumed to be compact and convex. Models, both actual and perceived, induce joint probability distributions over $X^\infty$. Let $\nu_t(x : u)$ be the time-$t$ probability distribution over $X$ conditioned on $u_t \in U$, which is called the Actual Law of Motion (ALM).

To highlight the key ideas, we limit our attention to the class of linear models and linear-quadratic objective functions. The analysis could in principle be extended to more general settings, featuring nonlinear models or non-quadratic objectives, but this would only serve to obscure and complicate an already difficult identification problem.

By a linear model, we mean a linear restriction over $X^\infty$:

$$\beta X = \varepsilon_t$$

where $\beta = (\beta_1, \ldots, \beta_K) \in \mathbb{R}^K$. That is, models are linear in the parameters, but can be arbitrarily nonlinear in the variables, since $X$ can itself include nonlinear functions.

Let $k_0, k_1, \ldots, k_J$ index different subsets of the elements in $\{1, \ldots, K\}$. A model specification involves the following ingredients:

1. Each model is defined by a list of included variables. Setting $\beta_{k_2} = \cdots = \beta_{k_J} = 0$ excludes subsets of explanatory variables from the right hand side of the regression equation. It is possible that $J = 0$, meaning that the model includes all available variables. We call such a model a maximal model.

2. Each model $k_i, i = 0, 1, \ldots, J$, imposes an identification restriction, taking the form of an orthogonality condition,

$$E(\varepsilon_t \cdot X_{t,-k_i}) = 0,$$

where $X_{t,-k_i}$ denotes the time-$t$ values of the regressors in model $k_i$, and a normalization,

$$\beta_{k_i} = 1,$$

where the corresponding variable $x_{k_i}$ is called the dependent variable. We usually write $x_{k_i}$ as $y$ and place it on the left hand side of the regression equation.

Note that depending on which variable is selected as the dependent variable, there are multiple maximal models. Also note that depending upon the configuration of $\{k_0, k_1, \ldots, k_J\}$,
different regression equations can be generated. In particular, if \( J \) variables are available, then \( L = J \times (2^J - 1) \) different models are possible. For example, if there are four variables, each of which could potentially serve as the dependent variable, then there are 28 possible model specifications. However, in some applications, the decision maker may be constrained as to which variables can serve as controls, and which variables must be forecast. If this is the case, then the number of possible models could be far less.

A generic equation can be written

\[
X_t \beta = \varepsilon_t
\]

with a specification of the dependent variable (\( \beta \) is 1). For each model the regression error, \( \varepsilon_t \), is presumed i.i.d. with

\[
E_t(\varepsilon_t | X_t) = 0.
\]

Note that \( X_t \) may include lags of the dependent variable. Since all we need is consistency, the error does not need to be strictly exogenous.

By a functional form of a regression equation, we mean a specification of (2.2). By assigning different values of \( \beta \) for those elements which are not constrained to be 0, we can generate different linear restrictions, each of which results in a forecasting rule. Let \( \mathcal{M}_\ell \) be the collection of all such functional forms. We call \( \mathcal{M}_\ell \) a model. Define \( \mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_L \)

which is then called the model class.

To simplify the analysis we assume the coefficients are uniformly bounded.

**Assumption 2.1.** \( \forall \ell, \beta \in B_\ell \) where \( B_\ell \) is convex and compact. Then, \( \exists b > 0 \) such that

\[
\max_\ell \sup_\mathcal{M}_\ell \| \beta \| \leq b.
\]

This bound presumes at least some a priori knowledge of the underlying data generating process. Even though the agent does not know the true process, he can exclude outrageous magnitudes, based on common knowledge or other readily available data.

Convexity of \( \mathcal{M}_\ell \) is needed to ensure good boundary behavior of the regression coefficient estimates. A projection facility may be required to constrain the estimates in a pre-specified convex compact set (e.g., Evans and Honkapohja (2001b)).

We think of a monetary authority as consisting of a team, comprised of a policy maker and a research department. The research department tries to identify the best forecasting rule from the given model class, which it then forwards to the policy maker. The research department is assumed not to know the details of the policy maker’s objective function, which means that model selection cannot be based on payoffs. Upon receiving the model, the policy maker uses it to formulate a policy by solving an optimization problem, which then generates a new realization of the state.

Let us now examine each player’s problem more formally.

2.2. **Policy Maker.** Suppose that the policy maker receives a forecasting rule \( \beta_t \in \mathcal{M}_\ell \) in period \( t \). The policy maker solves

\[
\min_{u \in C_t(\beta_t, X_t)} W_t(\beta_t, X_t, u)
\]
where $\mathcal{W}_t(\beta_t, X_t, u)$ is the policy maker’s objective function, and $\mathcal{C}_t(\beta_t, X_t)$ is the set of feasible controls. Note that we allow the policy maker to solve a different control problem, depending on which model the forecasting rule belongs to.

**Assumption 2.2.** The policy maker’s minimization problem has a unique solution, denoted as $b_t^*(\beta_t, X)$, which is Lipschitz continuous with respect to $(\beta_t, X)$.

This condition ensures the dynamics of the estimated $\beta_t$ can be approximated by a deterministic process, as in Marcet and Sargent (1989). We are interested in the dynamics of the selected model $\beta_t$ as well as the asymptotic properties of the state.

### 2.3. Research Department.

The research department’s objective function is to minimize model forecasting errors. If the state evolved exogenously, the research department’s problem would just be to compute least squares estimators and forward them to the policy maker:

$$\min_{\beta} \sigma_\beta^2 = \min_{\beta} \mathbb{E}(X\beta)^2$$

subject to $\beta_{k_0} = 1$. In fact, if stochastic process $X_t$ is stationary, then the best forecaster is the model which minimizes the mean square error.

In conventional models, where model specification goes unquestioned, the behavior of the decision maker can just be described as

$$\min_{\beta \in \mathcal{M}_t} \sigma_\beta^2 = \min_{\beta \in \mathcal{M}_t} \mathbb{E}(X\beta)^2$$

subject to $\beta_{k_0} = 1$, where $\mathcal{M}_t$ is the model imposed upon the decision maker by the modeler.

In contrast, we replace $\mathcal{M}_t$ by $\mathcal{M}$, while allowing the policy maker to solve a different optimization problem conditional on the recommended model:

$$\min_{\beta \in \mathcal{M}} \mathbb{E}(X\beta)^2$$

subject to $k_{t_0} = 1$ where $\ell_0$ is the dependent variable of model $\ell$.

Note that $\mathcal{M}$ can be a very complex object. As a result, the research department’s optimization problem is difficult to analyze. However, we can decompose its problem into two steps, each of which is operational and easy to implement recursively. Since the research department is trying to minimize forecast errors, assuming the data are exogenous (which is false except when the state is near an equilibrium), we can confine attention to the collection of least squares estimators from each individual model. Let $\beta_\ell$ be the least squares estimator in $\mathcal{M}_\ell$. We can write (2.3) as

$$\min_{\ell \in \{1, \ldots, \ell_0\}} \mathbb{E}(X\beta_\ell)^2$$

where $\beta_\ell \in \mathcal{M}_\ell$ satisfies

$$\mathbb{E}(X\beta_\ell)^2 \leq \mathbb{E}(X\beta)^2 \quad \forall \beta \in \mathcal{M}_\ell.$$  

Note that (2.4) is a maximization problem over a finite set of elements, while (2.5) can be computed recursively.

Consider a $\beta_\ell^*$, satisfying

$$\beta_\ell^* \in \arg \min_{\ell} \mathbb{E}(X\beta_\ell)^2$$
where $X_t$ represents the state variables generated by the policy maker when using $\beta_t$.

This is a natural candidate for an equilibrium. However, the following example reveals the need to expand the choice set.

**Example 2.3.** Assume the true data generating process on $\mathbb{R}^2$ is

\begin{align*}
  x_{1,t} &= x_{2,t-1} + \epsilon_{1,t} \\
  x_{2,t} &= x_{1,t-1} + \epsilon_{2,t}
\end{align*}

where $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are mutually independent i.i.d. Gaussian with mean 0 and variances $\sigma_1^2$ and $\sigma_2^2$.

Suppose the research department has two models. Model 1 is

\begin{equation}
  x_{1,t} = \beta_{1,1,t} \hat{x}_{1,t-1} + \beta_{1,2,t} \hat{x}_{2,t-1}
\end{equation}

and model 2 is

\begin{equation}
  x_{2,t} = \beta_{2,1,t} \hat{x}_{1,t-1} + \beta_{2,2,t} \hat{x}_{2,t-1}
\end{equation}

where $\hat{x}_{i,t-1}$ is the most recent observation of $x_{i,t}$ before period $t$:

\begin{equation}
  \hat{x}_{i,t-1} = x_{i,k}
\end{equation}

with

\begin{equation}
  k = \max\{k' : k \leq t - 1\}.
\end{equation}

Suppose each $\beta_{i,t}$ is updated according to a least squares estimation process. The forecaster assumes that $x_{i,t}$ is i.i.d. over time with a Gaussian distribution with mean 0 and variance $\sigma_i^2$.

The forecaster compares the two models using their mean squared forecasting errors, and recommends the one with the smaller mean squared error. Based on the recommended model, the policy maker generates the actual realization of $x_{i,t}$. Note that if model $i$ is selected in period $t$, only $x_{i,t}$ is observed, but not $x_{j,t}$ ($j \neq i$). Thus, the sequence of observed values $\{x_{i,t}\}$ where $t \geq 1$ and $i_t \in \{1, 2\}$ is self-referential.

In order to predict $x_{i,t}$, the most useful data is $x_{j,t-1}$ for $i \neq j$. However, as model $i$ is used for many periods, its mean squared error increases without bound, as its prediction must rely on the increasingly stale data on $x_{i,t-1}$. As a result, model $i$ is replaced by model $j$ within a finite number of periods. In the long run, both models are used with positive frequency almost surely. Moreover, the forecasting errors of both models increases indefinitely. Large samples by themselves do not refine inferences about model selection.

In this particular case, one can easily construct another parametrized model that in the long run generates strictly smaller mean squared errors than either individual model. Define a new random variable

\begin{equation}
  z_t = \begin{cases} 
  x_{1,t} & \text{if } t \text{ is even} \\
  x_{2,t} & \text{if } t \text{ is odd}
  \end{cases}
\end{equation}

Consider

\begin{equation}
  z_t = \gamma_t z_{t-1}
\end{equation}

which alternates between the two stochastic processes $x_{1,t}$ and $x_{2,t}$, generating and exploiting the best possible data for each forecast. Similarly, the forecaster assumes that
the coefficient $\gamma_t$ alternates between the two values, depending upon whether $t$ is even or odd. It is easy to show that the mean squared error of (2.8) is strictly smaller than that of (2.6) or (2.7). In fact, the mean square error of (2.8) is the minimum mean square error among all possible models, including the correctly specified one, which by assumption is not available for the forecaster.

A natural question would be why the decision maker is knowingly to use the data generated by one to update the coefficients of another model. It might appear that we force the decision maker to ignore an evident information that a particular model is used at a specific time, at the same time assuming very much sophisticated computational capability from a decision maker. On the contrary, their behavior is self-confirming and consistent with the model selection behavior. Given that the two models are used with an equal probability, the decision maker is aware that the data is generated by the two models with an equal probability. The forecasting model is a mixture of the two models assigned with an equal probability. In this case, the data generated by any model in the support of probability distribution is a legitimate source of information to update the coefficient of any models used with a positive probability.

This example shows that if no single model is used almost always, one can always construct another model that generates a better forecast than any of the individual models. Moreover, the new model has no extra explanatory variables; it is essentially a convex combination of the two models. Therefore, in the long run, the forecaster ends up with alternating multiple models. Our task is to generalize this example, and by doing so, characterize those models which are selected by the research department with a positive limiting frequency.

Let $\pi \in \Delta^L$ be a probability vector in $L$ dimensional Euclidean space, where $\pi(\ell)$ denotes the probability with which the forecaster recommends $\beta_\ell$ to the policy maker. Abusing notation, let $\beta_\pi$ be a randomized forecasting rule, in which $\beta_\ell$ is recommended with probability $\pi(\ell)$. Similarly, let $X_\theta$ be the state induced when the policy maker uses $\beta_\ell'$ with probability $\theta(\ell')$. Thus,

$$E(X_\theta \beta_\pi)^2 = \sum_\ell \sum_{\ell'} E(X_{\ell'} \beta_\ell)^2 \theta(\ell'|\ell) \pi(\ell).$$

We are ready to state the solution concept informally.

**Definition 2.4.** $\pi^* \in \Delta^L$ is a robust self-confirming equilibrium if

$$E(X_{\pi^*} \beta_{\pi^*})^2 \leq E(X_{\pi^*} \beta_\pi)^2 \quad \forall \pi \in \Delta^L$$

where $\beta_\ell$ is the least squares estimator in $\mathcal{M}_\ell$.

In order to highlight the research department’s decision making process, consider a matrix $\Sigma = (E(X_{\ell'} \beta_\ell)^2)_{\ell,\ell'}$. Imagine that in each period, given $\beta_\ell$, the forecaster recommends $\beta_{\ell'}$ to the policy maker, where

$$\ell' \in \arg\min_{\ell''} \{E(X_{\ell''} \beta_\ell)^2\}_{\ell'' \in \{1, \ldots, L\}}$$

which in turn becomes the research department’s forecasting rule in the next round. Based on this behavior, we can construct a transition matrix $B_\Sigma = (b_{\ell\ell'})$ where $\forall \ell, b_{\ell\ell'} > 0$ only if (2.9) holds.
Let $\pi_t$ be the research department’s randomized forecasting rule in period $t$, which selects $\beta_{t}$ with probability $\pi_t(\ell)$. Then, in the next period, the forecasting rule is determined by

$$\pi_tB_{\Sigma} = \pi_{t+1}.$$ 

We state another informal definition of a robust self-confirming equilibrium:

**Definition 2.5.** $\pi^*$ is a robust self-confirming equilibrium if

$$\pi^* = \pi^*B_{\Sigma}.$$ 

Although $\pi^*$ will be the focal point of our analysis, it should be noted that this is still an incomplete definition, because we have yet to spell out the properties of $\beta_{t}$ and $\Sigma$ associated with $\pi^*$.

### 2.4. Learning and Stability.

In order to understand the stability of robust self-confirming equilibria, let us construct a recursive procedure to approximate $\pi^*$, along with $\Sigma$ and $\beta_{t}$, since they affect the transition rule.

Suppose that in period $t - 1$, the policy maker uses model $\ell_{t-1}$ to generate the data. At the beginning of period $t$, the research department has a profile of least squares estimators

$$(\beta_{1,t-1}, \ldots, \beta_{L,t-1}).$$

Given $t$, let $T(\ell, \ell')$ be the number of periods during which the forecaster uses $\beta_{t}$ but recommends $\beta_{t}'$ to the policy maker to generate the data.

$$\sigma_{\ell\ell',t}^2 = \begin{cases} 
\frac{1}{T(\ell, \ell')} \sum_{t'=t}^{t} (\beta_{t}' - \beta_{t})^2 & \text{if } T(\ell, \ell') > 0 \\
\infty & \text{otherwise.} 
\end{cases}$$

Let $\Sigma_t = (\sigma_{\ell\ell',t})_{\ell, \ell'}$. In period $t$, the research department recommends model $\ell_t$ according to

$$\ell_t = \arg\min_{\ell'} \{\sigma_{\ell_{t-1}\ell'}^2\}.$$ 

Data in period $t$ is generated by the policy maker using $\beta_{t}$. The research department updates model estimates as follows:

$$\beta_{t} = \beta_{t-1} + \eta_{\beta} R_{t-1}^{-1} (y_{\ell_t} - \beta_{t-1}X_{\ell_t}) \quad \forall \ell$$

$$R_{\ell_t} = R_{\ell_{t-1}} + \eta_{\beta} (X_{\ell_t}^T X_{\ell_t} - R_{\ell_{t-1}})$$

and

$$\sigma_{\ell\ell',t} = \begin{cases} 
\sigma_{\ell\ell',t-1} & \text{if } \ell' \neq \ell_t \\
\sigma_{\ell\ell',t-1} + \eta_{\sigma} ((y_{\ell'} - \beta_{t-1}X_{\ell'})^2 - \sigma_{\ell\ell',t}) & \text{if } \ell' = \ell_t 
\end{cases}$$

Note that updating of $\Sigma_t$ is asynchronous, because only the column of $\Sigma_t$ corresponding to the current model is updated. Since the speed at which the forecaster switches from one model to another is much faster than the speed at which $\Sigma_t$ and $\beta_t$ evolve, the average frequency of using model $\ell \in \{1, \ldots, L\}$ is determined by a $\pi^*(\ell)$ solving

$$\pi^* = \pi^*B_{\Sigma},$$

where $B_{\Sigma}$ is the transition matrix satisfying (2.9).
By letting $\eta_\sigma, \eta_\beta \to 0$, we can approximate the evolution of $(\Sigma_t, \beta_t)$ by a deterministic process:

\begin{align}
\dot{\Sigma} &= \Sigma_{\pi} - \Sigma \\
\dot{\beta}_\ell &= \sum_{\ell' = 1}^L \pi(\ell') \Psi_{\ell'\ell}(\beta_{\ell'}) \quad \forall \ell
\end{align}

where $\Psi_{\ell'\ell}(\beta_{\ell'})$ represents the evolution of $\beta_{\ell'}$ when the data are generated by $\beta_{\ell'}$.

We can now define robust self-confirming equilibria in a more convenient form than (2.5).

**Definition 2.6.** $(\pi^*, \beta^*, \Sigma^*)$ is a robust self-confirming equilibrium if $(\pi^*, \beta^*, \Sigma^*)$ is a stationary solution of (2.14), (2.15) and (2.13).

We can also define the stability of robust self-confirming equilibria in terms of the stability of an ordinary differential equation.

**Definition 2.7.** $(\pi^*, \beta^*, \Sigma^*)$ is a stable self-confirming equilibrium in an extended sense if $(\pi^*, \beta^*, \Sigma^*)$ is a stable stationary solution of (2.14), (2.15) and (2.13) in the sense of Lyapunov.

3. Properties

For a given model class, one can have multiple maximal models, depending on the choice of dependent variable. Because maximal models have no restrictions on their coefficients, they should produce the best fits. Thus, if a maximal model is stable in a conventional sense, it would be a leading candidate to be stable in our extended sense.

**Proposition 3.1.** Suppose that $\beta^*_L$ is a conventional stable self-confirming equilibrium of a maximal model, and that no component of $\beta^*_L$ is equal to zero. If $\beta^*_L$ generates the smallest mean squared forecast error among all conventionally stable equilibria of maximal models, then $\pi^*(L) = 1$, $\Sigma^* = \Sigma_{\pi^*}$ and $\beta^* = (\beta^*_1, \ldots, \beta^*_L)$ where $\beta^*_\ell$ is the least squares estimator of $M_\ell$ is a robustly stable self-confirming equilibrium.

**Proof.** Since $\beta^*_L$ is stable, the associated ODE

\begin{equation}
\dot{\beta}_L = \Psi_{LL}(\beta_L)
\end{equation}

has a Hessian, whose eigenvalues are negative at $\beta_L = \beta^*_L$.

Choose $\pi$ close to $\pi^*$ so that the ordering $\sigma_{ik}^2 > \sigma_{ik'}^2$ is preserved $k \neq k'$, $\forall i$. This is possible, because the mean squared forecast error is a continuous function of $\pi$ and $\beta_\ell$. Since no component of $\beta^*_L$ is zero,

\[ \sigma_{iL}^2 < \sigma_{ik}^2 \quad \forall k \neq L. \]

Thus, even if $\pi$ has full support, $\beta^*_L$ continues to generate the smallest mean squared forecast error, as long as $\pi$ is sufficiently close to $\pi^*$.

Choose a small neighborhood of $(\beta^*, \pi^*)$ such that any $(\beta, \pi)$ satisfies the above conditions. It remains to prove that from any initial condition $(\beta, \pi)$, $\pi \to \pi^*$ and for any $\ell$ in the support of $\pi$, $\beta_\ell \to \beta^*_\ell$. 

Note that since $\beta_L$ generates the smallest mean squared forecasting error, $\forall t > 0$, $\pi(t) = \pi^*$, as it assigns full probability to $\beta_L$. Thus,

$$\dot{\pi} = \pi^* - \pi$$

where $\pi^*$ is concentrated at the $L$-th component.

The Hessian of the right hand side of (2.15) and (2.13) is a $2L \times 2L$ upper triangular matrix:

$$\Omega = \begin{bmatrix}
\frac{\partial \Psi_{L1}(\beta_1)}{\partial \beta_1} & 0 & \cdots & 0 & \Psi_{11}(\beta_1) & \Psi_{12}(\beta_1) & \cdots & \Psi_{1L}(\beta_1) \\
0 & \frac{\partial \Psi_{L2}(\beta_2)}{\partial \beta_2} & \cdots & 0 & \Psi_{21}(\beta_2) & \Psi_{22}(\beta_2) & \cdots & \Psi_{2L}(\beta_2) \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{\partial \Psi_{LL}(\beta_L)}{\partial \beta_L} & \Psi_{L1}(\beta_L) & \Psi_{L2}(\beta_L) & \cdots & \Psi_{LL}(\beta_L) \\
0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1
\end{bmatrix}.$$  

Except for $L$ models, the rest of the estimator $\beta_\ell$ ($\ell \neq L$) does not embody feedback, because only $L$-th model is used to generate the data. By the strict convexity of the forecaster’s objective function, $\forall \ell \neq L$, $\frac{\partial \Psi_{LL}(\beta_\ell)}{\partial \beta_\ell}$ is negative definite. By the stability in the conventional sense, $\frac{\partial \Psi_{LL}(\beta_L)}{\partial \beta_L}$ is a negative definite matrix. Thus, $\Omega$ is negative definite. $\square$

Suppose that $\beta^*_{L_1}$ and $\beta^*_{L_2}$ are two maximal models, and that the only difference between $\beta^*_{L_1}$ and $\beta^*_{L_2}$ is in the choice of the dependent variable. Also suppose that both $\beta^*_{L_1}$ and $\beta^*_{L_2}$ are stable self-confirming equilibria in the conventional sense. Proposition 3.1 then implies that both models are robust self-confirming equilibria as defined above. The duration of each model is determined by the large deviation properties of each self-confirming equilibrium.

4. Example

As an illustration of the above analysis, let us return to the model of Sargent (1999). The government is minimizing the social cost of unemployment, $u_t$, and inflation, $y_t$

$$\mathbb{E} \sum_{t=0}^{\infty} \delta^t (u_t^2 + y_t^2)$$

by choosing a target inflation rate $x_t$ which produces the actual inflation rate $y_t$ according to

$$y_t = x_t + v_{2t}$$

where $v_{2t}$ is a mean zero velocity shock. The data-generating process is

$$u_t = u^* - \theta(y_t - x_t) + v_{1t}$$

Both $v_{it}$ are Gaussian white noise with variance $\sigma^2_i$ for $i = 1, 2$. 

12 IN-KOO CHO AND KENNETH KASA
The government does not know the true model is (4.18). Instead, it employs a research department that attempts to model the relationship between inflation and unemployment by fitting regression equations. The research department is unsure of the underlying causal structure. That is, it is unsure about identification. Some members of the department argue that output and employment are largely predetermined by the state of aggregate demand, and that inflation is caused by excess demand in the labor market. They recommend regressing inflation on unemployment. Others claim that it is inflation that is predetermined, largely by the (exogenous) actions of the Central Bank, and that unemployment responds to inflation due to either (temporary) nominal rigidities or adaptive expectations. They recommend regressing unemployment on inflation. Neither camp worries about Rational Expectations or cross-equation restrictions, so both sides are doomed to commit a fundamental misspecification.

Although Sargent (1999) provides a fascinating description of the consequences of these two identification schemes, he assumes that for some reason the research department has committed itself to one or the other. It never entertains the possibility that it could be wrong. This raises an obvious question. Suppose the research department is slightly more sophisticated, and it worries about imposing the wrong identification restrictions. How should it proceed? We show that if it adaptively experiments with both models, as outlined in the previous sections, then it will eventually learn to impose the Classical identification restrictions, as they are closer to the truth.

In contrast to Sargent (1999) then, suppose the research department keeps both models on the table:

**Classical fit.**

\[ M_1 = \{ (\beta_{11}, \beta_{12}) : u_t = \beta_{11} + \beta_{12}y_t \} \]

**Keynesian fit.**

\[ M_2 = \{ (\beta_{21}, \beta_{22}) : y_t = \beta_{21} + \beta_{22}u_t \} \]

With multiple models and endogenous data, the research department confronts a difficult counterfactual - How would a given model perform if given the chance? Fitting a model to data that was generated by the other model could produce very misleading inferences. Although one could argue that actual Central Banks simply do not worry about this problem, it does seem to be the case that, at least informally, inferences about competing models are tempered by the recognition that previous data were generated by alternative ‘regimes’. For example, it is not uncommon to see them dummy out periods featuring alternative targeting procedures, or alternative exchange rate regimes. We take this to the limit, by supposing that the research department is fully aware of endogeneity, and responds to it by only updating a model’s coefficients when it is in use.

Therefore, as long as a model is in use, the research department updates its coefficients using (constant gain) recursive least squares. Estimates are then forwarded to the policy maker each period, where they are used to solve the ‘Phelps problem’ in (4.17). For the

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6See King and Watson (1994) for a detailed account of the debate between these two schools of thought, and the evidence each used to bolster its case. Of course, in practice, it was the Keynesians who ruled the day, as most Central Banks used the former identification scheme.
Classical model this produces the following target inflation rate

\[ x_{C,t} = \frac{-\beta_{11,t}\beta_{12,t}}{1 + \beta_{12,t}} \]

For the Keynesian model, the policy maker must first invert the relationship, since inflation is the choice variable. This produces the following inflation target

\[ x_{K,t} = \frac{\beta_{21,t}}{1 + \beta_{22,t}} \]

Note that in both models the inflation target evolves in response to evolving beliefs about the Phillips Curve. In general, due to feedback, the dynamics of the coefficient estimates are quite complicated, as they feature an interplay between the model’s mean dynamics and its escape dynamics (Cho, Williams, and Sargent (2002)). In our case, however, only the mean dynamics are relevant, since model switching takes place faster than escapes. That is, coefficient estimates never get a chance to escape (in the sense of large deviations) their self-confirming equilibrium values.

Each model’s self-confirming equilibrium is defined to be point where the endogenously generated data satisfy the model’s identification restrictions. At this point the residuals have mean zero and are uncorrelated (in sample) with the regressors. In other words, each model generates unbiased forecasts within sample. For the Classical model, the self-confirming equilibrium is given by

\[ \tilde{\beta}_{12} = -\theta \quad \tilde{\beta}_{11} = (1 + \tilde{\beta}_{12}^2)u^* \]

while for the Keynesian model it is

\[ \tilde{\beta}_{22} = -\frac{\theta\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad \tilde{\beta}_{21} = (\tilde{\beta}_{22}^{-1} - \tilde{\beta}_{22})u^* \]

Notice that the Keynesian model produces a ‘flatter’ Phillips Curve slope estimate. Due to their failure to distinguish between expected and unexpected inflation, both models produce biased estimates of the employment response to systematic inflation. However, the Keynesian model’s estimate is more biased. In the Keynesian model, the regressor is correlated with the error term, since they both contain realizations of \( v_{2t} \). As a result, the Keynesian model produces worse outcomes. For example, using Sargent’s parameters (\( \theta = 1, \sigma_1 = \sigma_2 = 0.3 \)), the Classical model produces an average inflation rate of 5%, whereas the Keynesian model produces an average inflation rate of 10%. In both models unemployment hovers around the natural rate.

The higher inflation rate in the Keynesian model arises from an overly optimistic estimate of the employment consequences of inflation. Its persistence reflects an exaggerated fear of what would happen if inflation were reduced. Without experimentation, there is no way the government can escape this trap. If the Keynesian model has been in use, then fitting the classical model to the historical data will produce a worse apparent fit. Even if the government were slightly more sophisticated, and compared notes with a foreign colleague who employed the Classical model, it might still be reluctant to switch, since both models fit their own endogenously generated data equally well. For example, with Sargent’s parametrization, each produces an \( R^2 = 0.5 \). In this sense, the government really is observing what it believes.
What if the government occasionally experiments? Notice that if the government has been using the Keynesian model, experimenting with the Classical identification scheme calls for a discrete reduction in inflation. On the other hand, if the current model is Classical, then experimenting with the Keynesian identification scheme calls for a discrete increase in inflation. Although each model makes a biased forecast when the other model is used to select the inflation target (since its forecasts are only unbiased around its own SCE), the mistake the Keynesian model makes when the Classical model is used for policy is bigger than the mistake the Classical model makes when the Keynesian model is used for policy. For example, consider the case of Sargent’s parametrization. Suppose the current model is Keynesian, so inflation has been running at 10%, and the estimated Phillips Curve slope is \(-0.5\). Now you decide to experiment with the Classical identification, which calls for a drop in inflation to 5%. Keynesians think this is going to be a disaster. According to the Keynesian model, unemployment is forecast to rise by 10 percentage points, to 15%. On the other hand, with a 5% inflation rate, the Classical model predicts an unemployment rate of 5%, which is an unbiased forecast of what actually happens. In contrast, now suppose the current model is Classical, so inflation has been running at 5%, and the estimated Phillips Curve slope is \(-1.0\). An experiment with the Keynesian model calls for an inflation increase to 10%. Classical economists predict this will cause unemployment to fall 5 percentage points, to 0%. Keynesians, on the other hand, make an unbiased forecast that unemployment will be 5%. Notice that the mistakes the Keynesian model makes are bigger than the mistakes the Classical model makes.

Figure 1 presents a simulation of our model based on Sargent’s parameter values. For simplicity, coefficients of each model are fixed at their self-confirming equilibrium values. The simulation is initialized to reflect doubts about the Classical fit, which is parametrized by an unfavorable initial root-mean-squared error. As in Section 2, the government selects randomly between the Classical and Keynesian directions of fit, where the probability vector, \(\pi_t\), follows the Markov process, \(\pi_{t+1} = \pi_t B(t)\). The transition matrix, \(B(t)\), depends on each model’s recursively estimated (relative) root mean squared error. It is parametrized as follows:

\[
B_{\Sigma}(1,1)(t) = \exp(\alpha_t [\omega_{2,t} + \phi_{11,t}]) / [\exp(\alpha_t [\omega_{2,t} + \phi_{11,t}]) + \exp(\alpha_t \omega_{1,t})]
\]

\[
B_{\Sigma}(2,1)(t) = \exp(\alpha_t [\omega_{2,t} + \phi_{21,t}]) / [\exp(\alpha_t [\omega_{2,t} + \phi_{21,t}]) + \exp(\alpha_t \omega_{1,t})]
\]

where \(\omega_{i,t}\) is the time-\(t\) estimate of model-\(i\)’s root mean squared error. The \(\phi_{i,0}\) parameters are set so that initially the probability of model switching is low, implying \(\phi_{11,0} < 0\) and \(\phi_{21,0} > 0\). The parameter \(\alpha_t\) determines how sensitive model selection is to relative performance. The bigger \(\alpha\) is, the less experimentation there is. In the absence of structural change, it is optimal to let \(\alpha_t \to \infty\), so that eventually experimentation ceases. We assume \(\alpha_t = \gamma \cdot \sqrt{t}\), with \(\alpha_0 = .01\) and \(\gamma = .05\).

Model 1 is defined to be the Classical model, and it is assumed the policy maker is initially using the Keynesian model. For the first 30 periods or so the Keynesian model makes.\(^7\)

\(^7\)Sargent’s ‘Fed watcher’ assumption, according to which the private sector knows the government’s inflation target, is perhaps more suspect here, in the presence of experimentation. The Central Bank must announce it is conducting an experiment, and these announcements must be believed.
continues to be used. Eventually, however, the policy maker experiments with the Classical model, and the results raise doubts about the relative usefulness of the two models. The weight on Model 1 begins to increase. As it does so, the policy maker begins to experiment more frequently, and inflation becomes volatile. This experimental phase, taking place during observations 30-50, is highly informative. The weight on the Classical model rapidly converges to unity. After about 100 periods, the government has learned to impose identification restrictions that are closest to the truth within the confines of its pre-specified model class. Although the details of the convergence rate depend on the details of the process of experimentation, the identity of the robust self-confirming equilibrium does not.

5. Concluding Remarks

Traditional learning models focus on the problem of estimating parameters of correctly specified models. The presence of feedback makes this an interesting and difficult problem. However, identification is not an issue in these settings, since structural parameters can be inferred from estimates of the reduced form. Unfortunately, to the extent these models are motivated by the supposedly implausible informational assumptions of the Rational Expectations Hypothesis, one could argue that once you assume knowledge of the structure, you might as well assume agents know the parameters as well. Compared to model uncertainty, parameter uncertainty is trivial.

Beginning with Sargent (1999), the macroeconomic learning literature has begun to explore the consequences of model misspecification. Here, the identification problem is central. In practice, identification restrictions are the most controversial aspects of model
specification. According to conventional wisdom, these restrictions are untestable. At a given point time, with a given sample, this is certainly true. However, when models are used recursively, in real-time, agents can learn about identification by occasionally experimenting with alternative identification restrictions. This will be the case as long as alternative identification restrictions produce alternative policy recommendations. Of course, if they don’t, then the identification problem is irrelevant from a practical standpoint.

Our paper has provided one possible formalization of this experimentation process, based loosely on insights from the evolutionary game theory literature. The basic idea is to allow decision makers to employ a mixed strategy across models, and to then allow the mixing probabilities to evolve in response to relative performance. The invariant distribution of this process delivers a refinement of the concept of self-confirming equilibria, one that is robust to off-equilibrium path experimentation with competing models.

In previous work (Cho and Kasa (2008)) we have attempted to incorporate traditional model validation procedures into recursive learning models. In that setting, the decision maker regarded his current model as ‘null hypothesis’, and only switched models in response to a test rejection. As a result, model switching is rare (in the large deviations sense). A natural extension of this paper would be to combine the two analyses, by allowing the decision maker to pursue a more traditional, statistical, approach to the identification problem. This would likely enhance the efficiency of the learning process. For example, allowing the research department to compute recursive analogs of Hausman (1978) or Anderson and Rubin (1949) statistics would be a useful extension. In fact, it is interesting to note that our strategy of comparing, in real-time, the policy outcomes produced by two alternative identification schemes bears some resemblance to Hausman’s strategy of comparing estimates, on a given sample, using two alternative estimators.
References


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