Solving the Procyclical News Shock Problem*

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Abstract

This paper constructs model economies that are capable of producing positive comovement in consumption, labor hours, and investment, in response to positive news shocks about future technology. In contrast, the standard neo-classical model cannot. We prove that a single departure from the neo-classical model - a strictly convex production frontier between consumption and investment - along with a high intertemporal consumption elasticity, is sufficient to support a technology news-driven boom. While procyclical, the boom in key variables is quantitatively small. The further addition of vintage capital magnifies the boom significantly. By vintage capital, we mean that the pre-existing capital stock's efficiency grows by relatively less than does future capital investment in response to the shock.

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1 Introduction

In a boom, investment, consumption, and hours worked all increase relative to their trends. In a recession, each falls. The sequence of alternating booms and recessions is called a business cycle. Several early studies, e.g. Frisch (1933) and Pigou (1927), proposed an impulse and propagation theory to explain these cycles. According to this theory, impulses, or external shocks, impact an economy and these impulses propagate through to macroeconomic variables by economic actors producing and interacting through markets. A more modern discussion of this impulse and propagation approach begins with Friedman and Schwartz (1963) and Lucas (1975).

One potential source of impulses, or external shocks, is the arrival of news about the future, such as news about changes in technology or tax rates in the future. Such "news shocks" might cause forward-looking economic actors to adjust their current behavior, resulting in a boom or bust today, in response to an event that is months or years away. While this news shock hypothesis has intuitive appeal, researchers in the 1990/2000s have shown that the hypothesis fails in the context of a modern macroeconomics touchstone: the basic neo-classical growth model. Specifically, for news shocks that this research has considered, consumption, investment and hours do not comove. For example, Danthine, Donaldson and Johnsen (1999) demonstrate this failure using a positive news shock about future technology. The news shock generates a positive wealth effect today, which leads households to desire greater consumption and leisure. The desired increase in leisure is equivalent to desire for fewer hours worked. With the production possibilities currently unchanged (because the technology increase occurs in the future and the capital stock is fixed), a decline in hours along with an increase in consumption requires a reduction in investment. This investment decline as well as consumption increase and hours reduction will be optimal. This is because, while the desired capital stock will eventually increase, agents will choose to put off capital accumulation until the actual technology increase arrives.

In this paper, we prove that a single modification of the neo-classical growth model can generate procyclical consumption, investment and hours in response to positive news regarding future technology. This modification is a sufficiently convex production frontier between consumption and investment, i.e. complementarity in production. In the standard model, the marginal rate of transformation between consumption and investment is fixed at one. Here, this marginal rate of transformation depends upon the consumption-investment ratio.

Beaudry and Portier (2007) first use production complementarities to generate news-driven cycles. Our work builds on theirs. They do this by studying the impulse responses from a simulated calibrated DSGE model. The first part of our paper further formalizes their findings by proving both sufficient and necessary conditions for news-driven cycles, where the procyclical comovements extend beyond the time of news arrival up until the time of the news realization.

We then add vintage capital to the production complementarity model. This magnifies the quantitative size of the boom in all three variables. Adding vintage capital amplifies the investment response because the planner increases investment by more, since the existing capital stock is less efficient relative to capital investments following the news. Greater investment reduces the labor cost of producing consumption because of production complementarity and, therefore, consumption increases as well.

Our model successfully achieves the three goals of: (i) making consumption, investment and hours procyclical in response to positive news, and (ii) doing so in quantitatively significant way, and (ii) maintaining the qualitative response to contemporaneous technology shocks that occur in the standard neoclassical model. We view (iii) as important, because many researchers view contemporaneous technology shocks are important and that the standard neoclassical model works well empirically for the contemporaneous shock.

Two recent published papers study the procyclical news shock problem; however, each meets only a subset of these goals. First, Beaudry and Portier (2007) add production complementarities of the form considering our paper. They simulate their model's response to an news shock about a future tax cut. While their simulation meets goals (i) and (iii), it does not meet (ii) because the quantitative response of the three variables are very small. Our paper futher achieves (ii) by adding vintage capital to the production complementarties model. Second, Jaimovich and Rebelo (2008) develop and simulate a variant of the neoclassical model that generates a business cycle in response to news of a future technology increase. Their departure from the neo-classical model includes: investment adjustment costs, variable capital utilization and a new form of time non-separable preferences. While the paper achieves (i) and (ii), it does not meet (iii). For example, the response of hours worked to a contemporenous technology shock is extremely lagging. The maximal response of hours occurs XXX years following the shock and the half-life of the response is XXX years./footnoteSimulations available from Dupor and Mekhari on request.

The intuition for our finding has three components. One is related to production complementarity and two are related the intertemporal elasticity of substitution for consumption (IES). First, the wealth effect of positive news about future technology increases desired current consumption. With a convex production frontier, greater consumption increases the marginal product of labor towards investment. This effect tends to increase investment and labor upon the arrival of the news. In an extreme but illustrative case, if consumption and investment are produced in a Leontief manner, then they comove perfectly in an optimal policy. The complementarity effect, by itself however, is insufficient to generate procyclical news shocks.

In addition, we require a high IES. This contributes in two ways. For preferences that satisfy balanced growth, the IES is intrinsically linked to the cross-partial between consumption and labor effort in the utility function. Of particular relevance, the marginal disutility of work is decreasing in consumption if the IES is greater than one. Although there is a negative wealth effect on labor supply due to a good news shock, the resulting consumption boom partially offsets this labor supply effect. The IES also determines the strength of the consumption-smoothing motive. If this motive is too strong, the planner will always prefer to increase consumption and decrease investment (that is, holding hours worked fixed) in response to positive news about technology.

To allow for a complete dynamic theoretical analysis of this model we mainly rely on a continuous time model. In continuous time, the method of Laplace Transform avails itself to us. The Transform is useful for studying linear differential equations with constant coefficients and exogenous (non-homogeneous) terms with discontinuities. Once we log-linearize the growth model, our differential equations take exactly this form. Later in this paper we also include labor adjustment costs and durability in consumption goods in a discrete time version of this model. Our main motivation for looking at these additions is to generate quantitatively larger impulse responses, so as qualify news shocks in our model as a possibly empirically important source of business cycle fluctuations.

Other recent work on news-driven cycles can be roughly divided into two categories: theoretical and calibration/simulation. For theory-based, work Beaudry and Portier (2007), in addition to the simulation results discussed previously, further establish a necessary condition for news-driven cycles: production complementarities of the kind studied in our paper. Their theoretical work does not explore the full equilibrium dynamics or sufficiency conditions for news-driven cycles.

Research using numerical/simulation methods, besides those discussed earlier, includes Beaudry and Portier (2004), which generates news-driven cycles by modeling final consumption as a function of non-durables and the capital stock. Christiano, Motto and Rostagno (2005) use investment adjustment costs and habit persistence to generate news-driven business cycles. Nah (2009) uses production complementarities such as ours and financial frictions. Gunn and Johri (2009) and Qureshi (2009) calibrate a neoclassical growth model with learning-by-doing. In response to news about future technological improvement, forward-looking agents increase hours worked and investment immediately in order to build up their stock of knowledge. Gunn and Johri (2009) show that learning-by-doing combined with variable capital utilization can generate procyclical stock prices. Qureshi (2009) shows that learning-by-doing along with an intratemporal adjustment cost can generate sectoral comovement in response to news about neutral and sector-specific technologies. Schmitt-Grohe and Uribe (2008) estimate a business cycle model augmented with real frictions for news (anticipated) and current (unanticipated) shocks of different varieties, including neutral and investment-specific technology. Their estimates imply that news shocks explain a greater fraction of output volatility than current shocks.

Our paper differs from the above numerical/simulation-based results in that it provides necessary and sufficient conditions for procyclical news shocks within a class of neoclassical models;[1] moreover, our paper contains the minimal departure, we believe, from the neoclassical model relative to the above, existing results. Each of the above, existing results uses departures that contain one or both of the following: (i) multiple modifications of the neoclassical model; (ii) a modification that introduces a new dynamic variable to the system. Gunn and Johri (2009) add dynamic learning-by-doing, an example of (ii). Jaimovich and Rebelo (2008) adds two dynamic state variables, lagged investment through adjustment costs and time non-separable preferences, plus variable capital utilization; thus, it contains both (i) and (ii). To get comomvent, there is a single departure in our paper: production complementarity between investment and consumption. It is also a static friction. Saying the departure is minimal is different than saying it is correct or has the most empirical support among alternatives. Rather, as a theoretical enterprise as well as due to Occams

[1] Thus far we have only done simulations of the vintage capital augmented model. We are currently attempting to extend our theoretical results to the vintage capital case.

razor, we believe our exercise is a useful one. As we explain in the paper, we find some empirical support in that the required conditions for procyclical news shocks are satisfied for parameter values that have been used in existing work. Finally, by expanding the model to include vintage capital, we significantly increase the expansion caused by a positive news shock.

In the next section, we describe the model and characterize an equilibrium. In section 3, we prove the paper's lemmas and theorems. In section 4, we show how adding vintage capital increases substantially the magnitude of the responses to a news shock. Section 5 concludes.

2 Business Cycles with a Convex Investment-Consumption Frontier

A social planner has the following preferences

$$U = \int_0^\infty e^{-\rho t} U \left[C(t), N(t) \right] dt$$

over time paths for consumption C and hours worked N. To preserve balanced growth, we assume $U(C, N) \equiv (C \exp(-N))^{1-\sigma} / (1-\sigma)$, $\rho = 1/\beta - 1 > 0$ and $\sigma \geq 0$.

The planner is subject to the following constraints:

$$F\left[C(t), I(t)\right] = K(t)^{\alpha} \left[A(t) N(t)\right]^{1-\alpha} \tag{2.1}$$

$$\dot{K}(t) = I(t) - \delta K(t) \tag{2.2}$$

Here K, I and A represent capital, investment and technology. The path of technology and the initial capital stock is exogenous. The depreciation rate, δ, and the elasticity of output with respect to capital, α, both lie between zero and one.

The sole departure of this environment from the neo-classical model pertains to F, which represents the production possibility frontier for consumption and investment given the amount of inputs and level of technology. We assume

$$F(C, I) \equiv \left[\theta C^{\upsilon} + (1-\theta) I^{\upsilon}\right]^{1/\upsilon}$$

where $\theta \in (0,1)$ and $\upsilon \geq 1$. When $\upsilon = 1$, the equation collapses to the standard neo-classical model, which has infinite substitutability between the two goods. As υ increases, the complementarity between the production of the two goods increases. If $\upsilon = \infty$, the production frontier takes a Leontief form. Figure 2 shows how the complementarity between consumption and investment change for a fixed output level, as υ changes.

An intuitive way to interpret the υ parameter above is to think of it as measuring the factor substitutability between the consumption and investment sectors of the economy. In the basic neo-classical model ($\upsilon = 1$), factors are equally productive in both the consumption and investment sectors. As a result the relative price of consumption to investment remains constant irrespective how many resources are being devoted to consumption vs. investment sector. In this general model, factors are not equally productive in both sectors. As υ increases they become less and less productive in the other sector. For example, a worker that produces n units in the consumption sector, when moved to the investment sector will produces less than n units. The magnitude of the decrease will depend on υ. The lack of equal productivity causes the relative price of consumption to investment to increase as more and more consumption goods are produced. This in turn causes a force in the economy that pushes the investment to consumption ratio back to its steady-state level.

Next, let us define the sole exogenous process - the level of technology. We will consider two types of technology shocks that occur at time zero: contemporaneous improvements, i.e. a *current shock*, and news of future improvements, i.e. a *news shock*. For both types of shocks, suppose the capital stock is at an initial steady state consistent with a particular fixed and unchanged level of technology $\bar{A}$. For the current shock, the planner has perfect foresight regarding the path of technology:

$$A(t) = 1.01 \times \bar{A} \text{ for all } t \geq 0$$

In the case of the future shock, the planner again has perfect foresight, with

$$A(t) = \begin{cases} \bar{A} & \text{for } t \in [0,T) \\ 1.01 \times \bar{A} & t \geq T \end{cases}$$

The social planner chooses C, I, K and N to maximize U subject to (2.1) and (2.2) taking as given the initial condition $K(0)$ and time path of technology. We can express the problem as a current value Hamiltonian:

$$H = U(C,N) + \Lambda(I - \delta K) + \Phi\left[F(C,I) - K^{\alpha}(AN)^{1-\alpha}\right]$$

The first-order necessary conditions at an interior solution satisfy the following:

$$-\frac{U_N}{U_C} = (1-\alpha)\frac{F}{N}(F_1)^{-1} \tag{2.3}$$

$$\frac{U_C}{\Lambda} = \frac{F_1}{F_2} \tag{2.4}$$

$$\frac{\dot{\Lambda}}{\Lambda} - \rho = \delta - \alpha\frac{F}{K}(F_2)^{-1} \tag{2.5}$$

along with our initial condition on capital and a transversality condition on Λ.

Equation (2.3) is the intratemporal Euler equation between consumption and labor hours. The left-hand side equals the marginal rate of substitution and the right-hand side (excluding the $1/F_1$ term) is the marginal output product of labor. The $1/F_1$ is required because the ability to transform output into consumption depends upon the slope of the consumption-investment production frontier.

Equation (2.4) is the intratemporal Euler equation between consumption and investment. The left-hand side equals the marginal rate of substitution between consumption and capital. In the typical neo-classical case, the right-hand side equals one. When the consumption-investment frontier is strictly convex, such as in this model, the marginal rate of transformation between the two goods will depend upon F_1/F_2. This ratio represents the relative cost of capital, which is augmented by increased current investment accomplished by forgoing consumption.

Equation (2.5) is the optimal capital accumulation equation. The left-hand side is the intertemporal Euler equation for capital (at an instant). The right-hand side is the marginal rate of transformation between capital over time. The term δ appears because increasing the current capital stock implies less undepreciated capital in the future. The second term is the marginal output product of capital. Once again the $1/F_2$ is required here because the ability to transform output into capital/investment depends upon the slope of the consumption-investment production frontier.

3 Theory & Dynamics Close to the Steady State

Given: (i) our initial condition that $K(0)$ is the steady-state capital stock consistent with the constant technology $\bar{A}$, and (ii) an exogenous stochastic process for $A(t)$ that remains sufficiently close to $\bar{A}$, we consider only the log-linearized equations (2.1) through (2.5).

Letting lower case letters denote the log deviations of variables from their initial steady-state values, these equations can be written as:

$$(1 - s_I)\, c + s_I i = \alpha k + (1 - \alpha)(a + n) \tag{3.1}$$

$$\upsilon s_I (i - c) = n \tag{3.2}$$

$$\lambda = (1 - \upsilon)(c - i) - zn - \sigma c \tag{3.3}$$

$$\dot{k} = \delta (i - k) \tag{3.4}$$

$$\dot{\lambda} = -(\rho + \delta)\left[\upsilon (1 - s_I)\, c + \left[\upsilon (1 - s_I) + 1 \right] s_I i - k \right] \tag{3.5}$$

Here, $s_I = (\alpha\delta) / (\rho + \delta)$ and $z = (1 - \sigma)(1 - \alpha) / (1 - s_I)$.

Definition: A *procyclical technology news shock* is exogenous news received at time zero about a permenant innovation in technology at time T that results in (i) comovement between consumption, investment, and labor hours for all time before T, and (ii) procyclical movement between consumption, investment, labor hours, and expectations of future technological innovations before T.[2] That is, in response to positive news about future technology consumption, investment, labor hours all continually rise for all periods before T.

Given the above definition, establishing the existence of procyclical technology news shocks requires showing both the comovement between consumption, investment, and labor hours, and the procyclicality of this comovement with expectations of future technological inno-

[2]Clearly, (ii) implies (i), however, it is important to note the difference given that (i) does not imply (ii). As a result showing just (i) is not sufficient for showing procyclical technology driven news shocks under our definition.

vations. We subdivide our proof into first establishing the procyclical comovement at time zero, and then establishing the procyclical comovement at time $t \in (0, T)$.

Lemma 1: Suppose at time zero, the planner receives news that the level of technology will permanently increase to a new, higher level at $T > 0$. Consumption, investment and hours will comove at time zero (on impact of the news) if and only if $\upsilon (1 - \alpha) > 1$

Proof. All proofs are contained in the main appendix. □

Lemma 2: Suppose at time zero, the planner receives news that the level of technology will permanently increase to a new, higher level at $T > 0$. Consumption, investment and hours will comove procyclically, along with expectations of technology, at time zero if and only if $\upsilon (1 - \alpha) > 1$ and $\lambda(0) > 0$.

The intuition for Lemmas 1 and 2 can be understood using Figure 3. Figure 3 plots the solution to the static consumption-investment decision holding fixed the marginal utility of capital. It plots this for two cases: (a) no production complementarity; (b) positive production complementarity.

Once we substitute out the optimal labor from the production equation (3.1), we get:

$$\alpha k + (1 - \alpha) a = (1 - \phi_I) c + \phi_I i$$

Here $\phi_I = (1 - \upsilon (1 - \alpha)) s_I$.

This is plotted as L_1 in Figure 3(a) and Figure 3(b). In the absence of any production complementarity and any change in technology, this is a downward-sloping line.[3] This is seen in Figure 3(a). Intuitively, when consumption rises, hours cannot optimally rise because leisure is a normal good. Therefore, investment must fall. With production complementarity, this becomes an upward-sloping line. This is seen in Figure 3(b). With complementarity, the increase in consumption raises the marginal product of labor in producing the investment good. This higher marginal product of labor leads the social planner to increase hours even though leisure is a normal good.

[3] We are also assuming that capital is predetermined at the steady-state level at time zero.

Next, consider L_2 the consumption-investment Euler equation with optimal hours substituted out:

$$\gamma_I i - (\sigma + \gamma_I)\, c = \lambda$$

Here $\gamma_I = (\upsilon - 1) - (\upsilon\,(1-\alpha)\,(1-\sigma)\, s_I) \,/\, (1 - s_I)$.

In general, the slope of L_2 can be either positive or negative. The slope depends most crucially on the degree of complementarity ν and the utility parameter σ. In order to generate procyclical news shocks, both ν must be large and σ must be small. To understand why L_2 can be upward-sloping, consider the consumption-leisure Euler equation. For our preference specification, this implies that consumption equals the real wage (ignoring complementarity in production). Because the real wage is simply labors share in production, we know that hours worked is a linear function of the output-consumption ratio. Then, if the planner decided to increase investment relative to consumption, hours worked increases. This is seen in equation (3.2). Note that adding production complementarity (i.e. setting $\nu > 1$) increases the hours effect because it increases the marginal product of hours in producing the consumption good.

Establishing that hours increases is not sufficient. The remaining step in showing that L_2 is upward-sloping is establishing that an increase in an investment-consumption ratio increases consumption. To do this, consider equation (3.3), the consumption-investment Euler equation. It states:

$$\text{marginal utility of consumption } = \sigma c - zn = \lambda + (\upsilon - 1)(i - c)$$

An increase in $i - c$ reduces the price of consumption, which implies a lower marginal utility of consumption at an optimum. Holding hours fixed, this implies that consumption rises. In that case, consumption and the investment-consumption ratio move together; therefore, L_2 is upward-sloping. However, hours increase with the investment-consumption ratio. As long as z is negative or n does not increase too much, then this hours effect will not reverse the consumption response. A sufficiently small σ guarantees that z is negative. This restriction is part of Lemma 2 and is reflected in Figure 3(b).

Next, suppose we consider an increase in the marginal utility of investment (λ_0). For now, we take the increase in λ_0 as given. Later, in Lemma 4, we provide a condition for which a positive technological improvement at T result in an increase in λ_0. First, an increase in λ_0 does not shift L_1 either without complementarity (in Figure 3(a)) or with complementarity (in Figure 3(b)). Because our preferences imply that labor is stationary along a balanced growth path, the consumption-leisure Euler equation, and therefore L_1 does not depend on the co-state variable. Second, an increase in λ_0 induces a shift leftward of L_2 either with or without complementarity. As the marginal utility of investment increases, the social planner shifts away from consumption for a given level of investment. Even though L_2 moves in the same direction in either case, the implication for the optimal investment-consumption pair is different between the two cases. Because L_1 is downward sloping without complementarity, investment rises but consumption falls; however, L_1 is upward sloping with complementarity and both investment and consumption rise. Intuitively, the increase in investment raises the marginal product of labor towards consumption when there is production complementarity. The fall in the relative price of consumption leads the planner to increased hours worked.

Lemma 3: Suppose at time zero, the planner receives news that the level of technology will permanently increase to a new, higher level at $T > 0$. Also, assume that $\upsilon > \upsilon_c = (1-\alpha)^{-1}$. Consumption, investment and hours will comove procyclically for all time $t < T$ if $\forall t < T$, $\dot{\lambda} \geq 0$ and $\dot{k} \geq 0$.

Figure 4 shows how in our baseline model L_1 and L_2 move when $\dot{\lambda} \geq 0$ and $\dot{k} \geq 0$. $\dot{k} \geq 0$ causes L_1 to progressively shifts left, while $\dot{\lambda} \geq 0$ causes L_2 to progressively shift right. This causes consumption and investment to continue increasing for all time $t < T$.

To understand fully the time path of the key aggregate variables we must now study the dynamic system in light of Lemmas 1-3. The log-linearized dynamic system for our model economy is:

$$\begin{bmatrix} \dot{\lambda}(t) \\ \dot{k}(t) \end{bmatrix} = \begin{bmatrix} \Gamma_{\lambda,\lambda} & \Gamma_{\lambda,k} \\ \Gamma_{k,\lambda} & \Gamma_{k,k} \end{bmatrix} \begin{bmatrix} \lambda(t) \\ k(t) \end{bmatrix} + \begin{bmatrix} b_\lambda \\ b_k \end{bmatrix} a(t) \tag{3.6}$$

Here $\Gamma_{\lambda,\lambda}$, $\Gamma_{k,\lambda}$, and b_k are all positive, while the values of $\Gamma_{k,k}$, $\Gamma_{\lambda,k}$, and b_λ depend on the value of σ.[4]

[4]For the conditions on σ see Lemma B.12 in the Supplementary Appendix.

In the presence of a news shocks there is a discontinuous forcing term in the dynamic system. In equation (3.6), $a(t)$ is a step function which takes on the value zero for all time $t < T$ and a value of $ln(1.01)$ for all time $t \geq T$. Standard solution techniques no long work. Laplace transforms lend themselves nicely here. Using these transforms, we can map our problem into the frequency domain where the problem is continuous and solvable using standard techniques. Once we have solved the dynamic system in this new domain we can then map the solution back into the time-domain (using techniques outlined in Boyce and Diprima (1969)). The resulting time-paths of $k(t)$ and $\lambda(t)$ for our system:

$$k(t) = \begin{cases} \frac{\Gamma_{k,\lambda}\lambda(0)+\left(\mu_1-\Gamma_{\lambda,\lambda}\right)k(0)}{\mu_1-\mu_2}e^{\mu_1 t} + \frac{\Gamma_{k,\lambda}\lambda(0)+\left(\mu_2-\Gamma_{\lambda,\lambda}\right)k(0)}{\mu_2-\mu_1}e^{\mu_2 t} & \text{for } t \in [0,T) \\ \frac{\Gamma_{k,\lambda}\lambda(0)+\left(\mu_1-\Gamma_{\lambda,\lambda}\right)k(0)}{\mu_1-\mu_2}e^{\mu_1 t} + \frac{\Gamma_{k,\lambda}b_\lambda-\Gamma_{\lambda,\lambda}b_k}{\mu_1\mu_2} + \frac{\Gamma_{k,\lambda}b_\lambda+\left(\mu_1-\Gamma_{\lambda,\lambda}\right)b_k}{\mu_1(\mu_1-\mu_2)}e^{\mu_1(t-T)} & t \geq T \end{cases} \tag{3.7}$$

$$\lambda(t) = \begin{cases} \frac{\Gamma_{\lambda,k}k(0)+\left(\mu_1-\Gamma_{k,k}\right)\lambda(0)}{\mu_1-\mu_2}e^{\mu_1 t} + \frac{\Gamma_{\lambda,k}k(0)+\left(\mu_2-\Gamma_{k,k}\right)\lambda(0)}{\mu_2-\mu_1}e^{\mu_2 t} & \text{for } t \in [0,T) \\ \frac{\Gamma_{\lambda,k}k(0)+\left(\mu_1-\Gamma_{k,k}\right)\lambda(0)}{\mu_1-\mu_2}e^{\mu_1 t} + \frac{\Gamma_{\lambda,k}b_k-\Gamma_{k,k}b_\lambda}{\mu_1\mu_2} + \frac{\Gamma_{\lambda,k}b_k+\left(\mu_1-\Gamma_{k,k}\right)b_\lambda}{\mu_1(\mu_1-\mu_2)}e^{\mu_1(t-T)} & t \geq T \end{cases} \tag{3.8}$$

where μ_1 and μ_2 are the eigenvalues of the Γ matrix. In the appendix, we prove that μ_1 and μ_2 are real with one being positive and the other negative. Without loss of generality, let $\mu_1 < 0$ and $\mu_2 > 0$.

The solutions for the time paths of $k(t)$ and $\lambda(t)$ show that the dynamics of the system before time T are being determined not only by the stable eigenvalue, but also the unstable eigenvalue. This allows the possibility for the system to generate positive comovement before time T, even though the new steady state marginal utility of investment is lower. Note that after time T, the system is on a new stable path and the dynamics are then determined by the new stable manifold.

As we seek a stable solution to the system it can be shown that:

$$\frac{\Gamma_{k,\lambda}\lambda(0) + (\mu_2 - \Gamma_{\lambda,\lambda})\,k(0)}{(\mu_2 - \mu_1)}e^{\mu_2 t} = \frac{(\Gamma_{k,\lambda}b_\lambda - \Gamma_{\lambda,\lambda}b_k)}{(\mu_1\mu_2)}$$

$$\frac{\Gamma_{\lambda,k}k(0)+(\mu_2-\Gamma_{k,k})\,\lambda(0)}{(\mu_2-\mu_1)}e^{\mu_2 t}=\frac{(\Gamma_{\lambda,k}b_k-\Gamma_{k,k}b_\lambda)}{(\mu_1\mu_2)}.$$

That is, the discontinuity of $a(t)$ does not appear as a discontinuity in the time paths of $k(t)$ and $\lambda(t)$. Instead, the discontinuity of the exogenous process shows up in the time paths as a non-differentiability (kink) at time T.

Using the time paths of $k(t)$ and $\lambda(t)$ above, along with the restrictions for a stable solution, we can further our analysis of the model dynamics:

Lemma 4: Suppose at time zero, the planner receives news that the level of technology will permanently increase to a new, higher level at $T > 0$. Also, assume that $\upsilon > \upsilon_c = (1-\alpha)^{-1}$. $\dot{\lambda} \geq 0$ and $\dot{k} \geq 0$ $\forall t < T$ if and only if $\lambda(0) > 0$.

Lemma 5: Suppose at time zero, the planner receives news that the level of technology will permanently increase to a new, higher level at $T > 0$. Also, assume that $\upsilon > \upsilon_c = (1-\alpha)^{-1}$. $\lambda(0) > 0$ if and only if $\mu_2 < (\rho + (1-\alpha)\,\delta)\,\upsilon/\,(\gamma_I + \sigma)$.

The condition that $\mu_2 < (\rho + (1-\alpha)\,\delta)\,\upsilon/\,(\gamma_I + \sigma)$ implies a relatively high intertemporal elasticity of substitution. In our baseline calibration from the next section, this condition translates to requiring the IES to be greater than 1.79.

Figure 5 plots the phase diagram for three cases: (a) no production complementarity, (b) complementarity with a low IES, (c) complementarity with a high IES. Only case (c) results in procyclical news shocks. Initially before the news shock, in Figure 5(a), the capital-multiplier pair (k_{ss}, λ_{ss}) lie on the initial manifold M and are at the steady state. Upon the time zero news arrival, the multiplier falls below λ_{ss} because there is an immediate consumption boom. Because capital is a stock variable, $k_0 = k_{ss}$. After time zero and before T (which is the instant of the technology arrival), the capital stock falls indicating that investment is below the steady-state. The consumption boom comes at the expense of investment. This is one indicator of the comovement problem in the standard model: investment declines upon the arrival of good news.

Note that between time zero and time T, the capital-multiplier pair flow in the opposite direction of the stable manifold. This is due to influence of the explosive root (μ_2) before the technology change occurs. At time T, the model is on the new stable manifold M' and the system then converges monotonically to the new steady-state. The M and M' manifolds are parallel to each other because the technology shock does not change the coefficients multiplying the endogenous variables.

Figure 5(b) contains the phase diagram with production complementarity. The IES is too low for procyclical news shocks to occur. As in the case without complementarity, investment initially falls in response to the news. Investment eventually increases once the technology change actually occurs. At this point, the system is on the manifold and capital converges monotonically to the new higher steady state. The desire for smooth consumption, due to the low IES, is evident in the path of the multiplier. It jumps downward on impact and then moves monotonically to the new steady state.

Figure 5(c) contains the phase diagram for the case of greatest interest. Both conditions of our theorem are satisfied. These are strong production complementarities and a high IES. First, the multiplier jumps up rather than down in response to the news. An increase in the multiplier is not due to a fall in consumption. In fact, consumption increases as is required in a news-driven boom. The increase in the multiplier is due to a change in the relative price of investment to consumption.

Recall the equation defining L_2 in our static analysis:

$$\lambda = \gamma_I i - (\sigma + \gamma_I)\, c$$

The consumption boom puts downward pressure on the multiplier. If investment increases more than consumption, then the relative price of investment rises. This second effect tends to increase the multiplier. Note that capital increases monotonically from time zero and onward, which is due to the investment boom.

Although investment jumps up at time zero, the new steady state must involve $k'_{ss} > k_{ss}$ and $\lambda'_{ss} < \lambda_{ss}$. This occurs in case (c) because the new manifold eventually crosses into the fourth quadrant of the phase space.

Theorem 1: A procyclical technology news shock occurs in our model economy if and only if $\upsilon\,(1-\alpha) > 1$ and $\mu_2 < (\rho + (1-\alpha)\,\delta)\,\upsilon/\,(\gamma_I + \sigma)$

Corollary: A technology news shock is procyclical if and only if $\upsilon > \upsilon_c = (1-\alpha)^{-1}$ and $\sigma < \sigma_c$.

Here σ_c is such that:

$$\mu_2 = \frac{(\rho + (1-\alpha)\,\delta)\,\upsilon}{\gamma_I + \sigma} \tag{3.9}$$

Using numerical simulation, we can show that $\sigma_c \in [0,1]$. The intuition for why we need a relatively high intertemporal elasticity of substitution, that is $\sigma < \sigma_c \in [0,1]$, has two parts. First, for preferences that satisfy balanced growth, the value of σ determines the sign of the cross-partial between consumption and labor effort in the utility function. Of particular relevance, the marginal disutility of work is increasing in consumption if the σ is less than one. Although there is a negative wealth effect on labor supply due to a good news shock, the resulting consumption boom partially offsets this labor supply effect. Second, the value of σ also determines the strength of the consumption-smoothing motive. A low value of σ weakens the consumption smoothing motive, this increases the marginal utility of investment and give the planner incentive to devote some resources to the production of investment goods. Mathematically, this results in $\lambda(0) > 0$. Again, Figure 5 shows the dynamics of k and λ, for when $\sigma < \sigma_c \in [0,1]$ causing $\lambda(0) > 0$ and for when $\sigma > \sigma_c \in [0,1]$ causing $\lambda(0) < 0$. Figure 5 further shows how the dynamics of k and λ differ between our model and the standard neo-classical model where $\upsilon = 1$ and $\sigma = 1$.

As seen in equation (3.9), changes in the model parameters influence the lower bound on the intertemporal elasticity of consumption (IES) required to generate procyclical news shocks. Figure 6 demonstrates how this IES changes with two parameters by numerical simulation. Figure 6(a) shows that greater consumption-investment complementarity reduces the lower bond on the IES required for procyclical news shocks. For example, if the complementary parameter equals 1.8, as in our benchmark specification, then the IES must be at least 1.79. If the complementary is stronger (e.g. $\nu = 3$), then the lower bound on the IES falls to 1.60. Intuitively, one benefit of an investment increase is that it raises the marginal product of labor in the production of consumption. The initial investment comes at the cost of less smooth consumption. If the benefit of investment increases because of stronger complementarity, then the cost of less smooth consumption can be larger while maintaining the optimality of an investment boom. A reduction in the IES will increase this cost, which implies a reduction in the lower bound.

Figure 6(b) demonstrates that faster depreciation of the capital stock requires a larger lower bound on the IES in order to generate procyclical news shocks. Intuitively, an investment increase in response to the news shock is optimal because the planner accumulates capital to expedite the economy's transition to a new higher target capital stock (because the technology level will eventually be permanently higher). This initial investment comes at the cost of less smooth consumption. A high depreciation rate implies that this transition will require the social planner to devote a large amount of output to investment, making consumption less smoother. It thus becomes optimal to attempt to smooth consumption now, and then transition to the new higher capital stock once the technology level has increased. A high technology level will cause output to increase. The cost of consumption smoothing can be mitigated by an increase in IES, making it optimal once more for the social planner to devote resources to expediting the economy's transition to a higher capital stock.

Our theorem is robust to the value of T as neither υ_c nor σ_c depends on T. In response to a higher T, $\lambda(0)$ adjusts in magnitude but not in sign. This can be readily seen by looking at the equation for λ_0:

$$\lambda_0 = -\left[\frac{\Gamma_{k,\lambda} b_\lambda + (\mu_2 - \Gamma_{\lambda,\lambda})\, b_k}{\Gamma_{k,\lambda}\mu_2}\right] e^{-\mu_2 T}$$

Next, we prove that our model also preserves the ability of contemporaneous technology shocks ($T = 0$) to generate procyclical comovement with respect to technology in time zero.

Theorem 2: If $\upsilon\,(1-\alpha) > 1$ and $\mu_2 < (\rho + (1-\alpha)\,\delta)\,\upsilon/\,(\gamma_I + \sigma)$ then consumption, investment, and labor hours all increase at time zero in response to a contemporaneous technology shock.

Finally, in our model the price of investment, or Tobin's Q, is equal to $((1-\theta)\,/\theta)\,(i/c)^{\upsilon-1}$. Log-linearizing this equation, we have:

$$q = (\upsilon - 1)\,(i - c) \tag{3.10}$$

From (3.1), whenever consumption increases, the corresponding increase in investment is of a larger magnitude. As a result, the relative price of investment rises and falls with consumption.

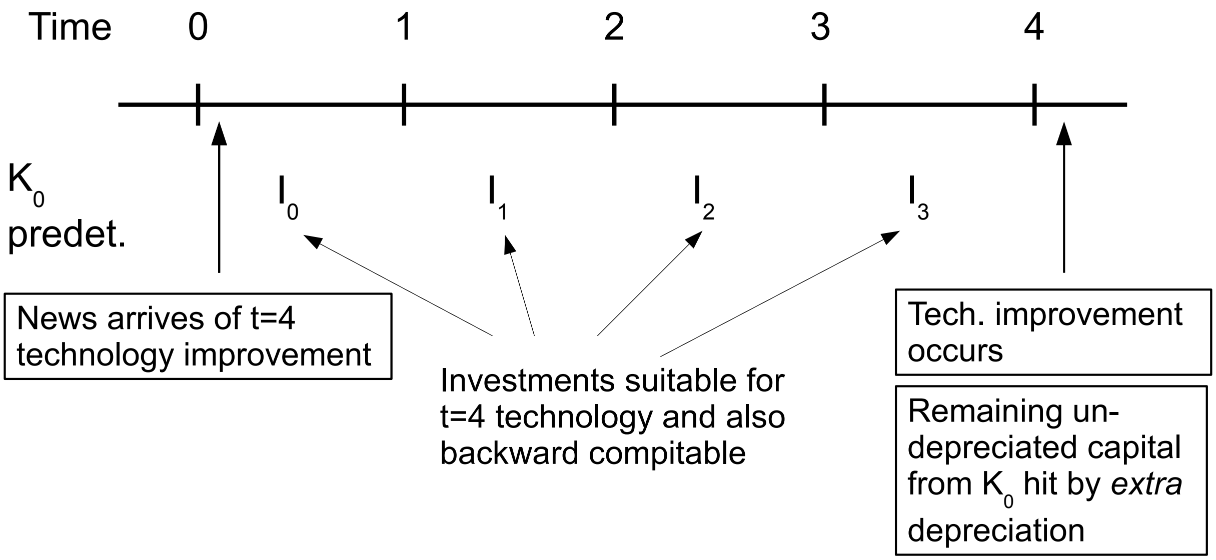
4 Quantitative Results and the Addition of Vintage Capital

We parametrize our model such that it meets the two conditions: $\upsilon > \upsilon_c$ and $\sigma < \sigma_c$, for procyclical technology news shocks. First, υ is chosen to be 1.8. Valles (1997) also calibrates this parameter to this value in order to match estimated responses of investment to various shocks. Sims (1989) uses a similar F function and chooses $\upsilon = 3$. The value of σ in our baseline calibration is 0.5, which is lower than the oft used 1.0 (log utility). However, $\sigma = 0.5$ is within the range of some empirical estimates in the literature (Mulligan (2002), Attanasio and Webber (1989)). Also, as shown by Nah (2009), σ can be parameterized to the more conventional 1.0 with the addition of financial frictions in a business cycle model similar to ours. The remaining parameter values are given in Table 5. Under this baseline parameterization, the structure of the news shock is as follows: At time zero, the economy is in a steady state. Then, agents receive news that technology will be one percent permanently higher at $t = 3$. The impulse responses associated with this news shock to our baseline model are given in Figure 7. As can be seen in this figure in response to the news shock, consumption, hours, investment, capital, and the price of investment all increase. Figure 8 reports the time zero to time 3-ϵ window of the experiment to give a clearer picture of the actual increases that take place before the technology improvements arrive at time 3.

As shown in the previous section, our results are robust for $T \in [0, \infty)$. As T increases, the initial response at time zero decreases; however, the direction of the response remains the same. Also, the value of the firm is procyclical because of a procyclical price of investment.

At time 3, when the actual technology increase takes place, the economy exhibits a large increase in consumption, investment, and hours. One criticism of the news shock explanation of business cycles is that news shocks are unable to generate large enough fluctuations so as to be able to explain real world business cycle fluctuations.

Figure 1: Sequence of events in model augmented with vintage capital



Suppose output is given by

$$F\left(C_t, I_t\right) = \left(K_t\right)^{\alpha} \left(A_t N_t\right)^{1-\alpha} \tag{4.1}$$

where A_t is the time t technology level. Technology evolves according to: $A_t/A_{t-1} = \exp\left(\varepsilon_t\right)$. Here, ε_t is a mean zero, unit variance iid variable that becomes known to the planner at time $t-T$.

The sequence of events within a period consists of the following four stages, which appears in figure **??**:

(i) The planner enters period t with capital K_t.

(ii) The current technology level A_t is determined by ε_t.(which became known to the planner at time $t-T$) and the pervious period technology level. The planner also learns the technology innovation ε_{t+T}, which will determine A_{t+T}. Thus, his current information about the exogenous process is $\{\varepsilon_0, \ldots, \varepsilon_{t+T}\}$ and as a result $\{A_0, \ldots, A_{t+T}\}$.

(iii) The planner chooses (C_t, I_t, N_t) to satisfies (4.1).

(iv) The period ends with the determination of K_{t+1}. This is given by the equation:

$$K_{t+1} = \sum_{j=0}^{T-1} (1-\delta)^j I_{t-j} + (1-\delta)^T \left(1 - \gamma\varepsilon_t\right) K_{t-T+1}$$

where $\gamma \geq 0$. This is the non-standard part of the model because it will implement the vintage capital idea. In stage (ii) of period $t+1$, technological change will occur. The investments $\{I_{t-T}, I_{t-T+1}, \ldots, I_t\}$ will be fully adapted to the new technology because they were created after the news arrived at $t-T$. The capital that was in existence before the news arrived, that is K_{t-T-1}, will be relatively less adapted to the new technology. Intuitively a relatively higher growth rate at ε_t implies that the new technology is less compatible with the old capital and, thus, the old capital depreciates more.

This completes the description of the model's timing.

There are two key features about technology:

(i) Technological change has a greater impact on the efficiency of new capital investments relative to the efficiency of the pre-existing capital stock.

(ii) Investment carried out after the arrival of news at time $t-T$ will match the efficiency of the capital investment created once the technology actually arrives.

The vintage capital model adds a single parameter γ. A larger γ implies a stronger vintage effect. If $\gamma = 0$, the model collapses to the model presented earlier in the paper. If $\gamma = (1-\alpha)/\alpha$, then a technology shock is neutral with respect to the pre-existing capital stock. That is, if capital is fixed at its pre-news level and labor is also fixed, then current output will not change in response to a technology improvement.

5 Conclusion

TO BE COMPLETED

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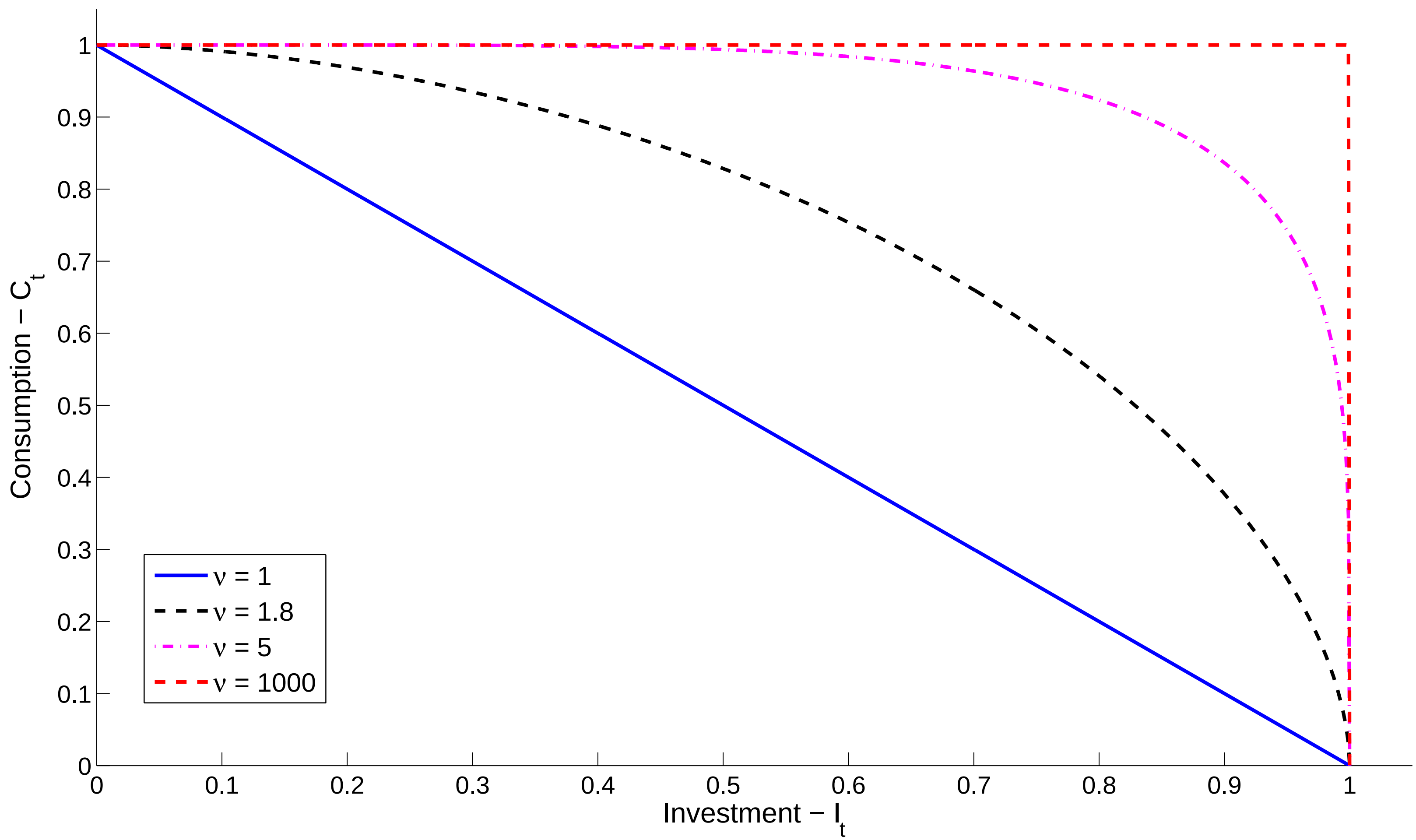
Table 1: Parameterization for the Baseline Model Simulation

Parameter	Value	Description
β	0.985	Subjective discount factor.
α	0.33	Labor share in production is $(1-\alpha)$.
δ	0.025	Depreciation rate of capital.
υ	1.8	Degree of complementarity between c and i ($> \upsilon_c = (1-\alpha)^{-1}$).
σ	0.5	Inverse of intertemporal elasticity of substitution ($< \sigma_c \approx 0.56$).
θ	0.253	Calibrated to match a price of investment of 1 in steady state.
T	3	Time of the expected technology increase.

Figure 2: Production frontier between consumption and investment as ν changes.



Note: $\theta = 1/2$ and the fixed output level is equal to $(1/2)^{1/\nu}$.

Figure 3: L_1 and L_2 in i-c Space, and an increase in the marginal utility of investment (λ)

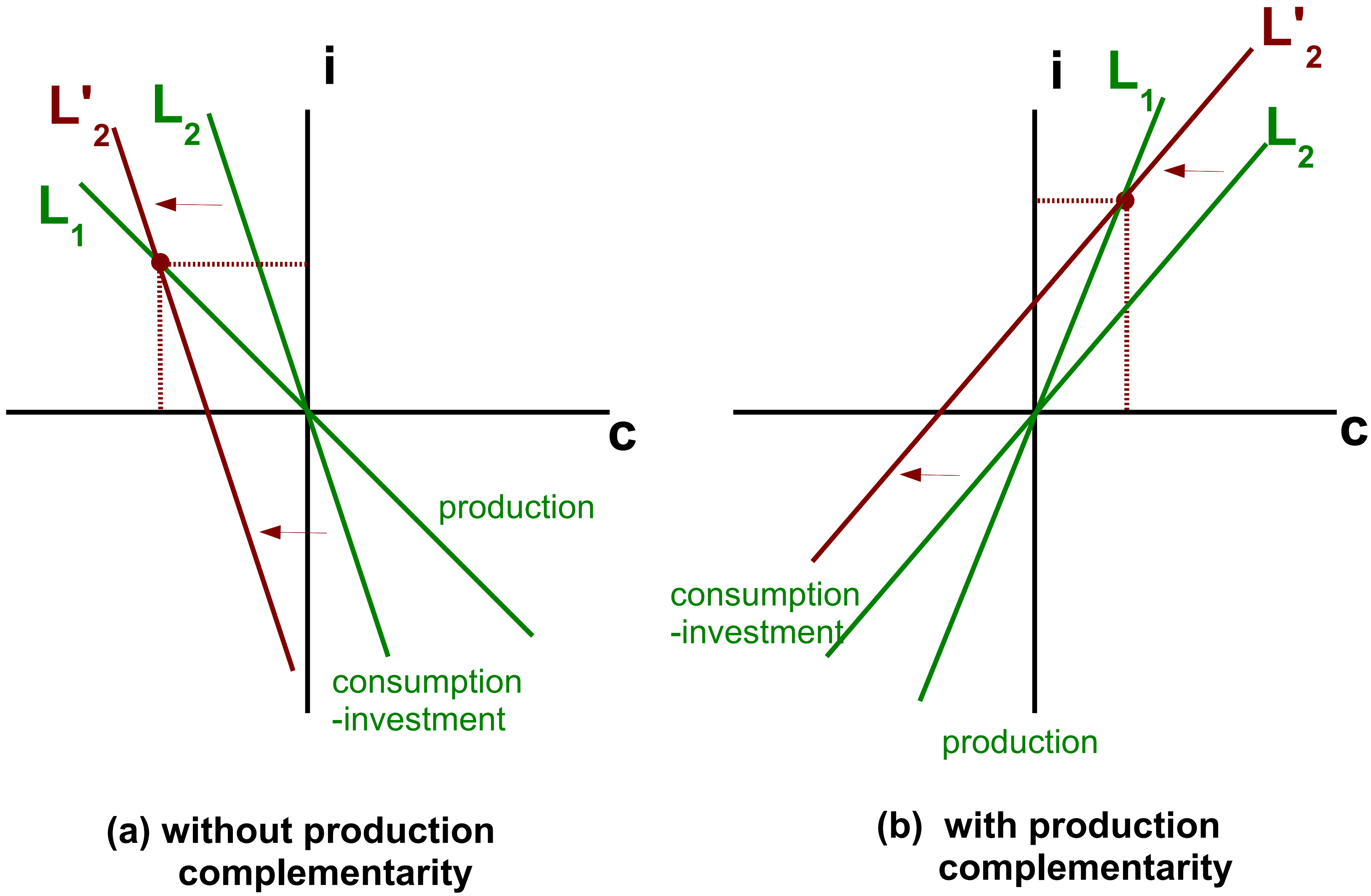


Figure 4: L_1 and L_2 in i-c Space, and an increase in both the marginal utility of investment (λ) and capital (k)

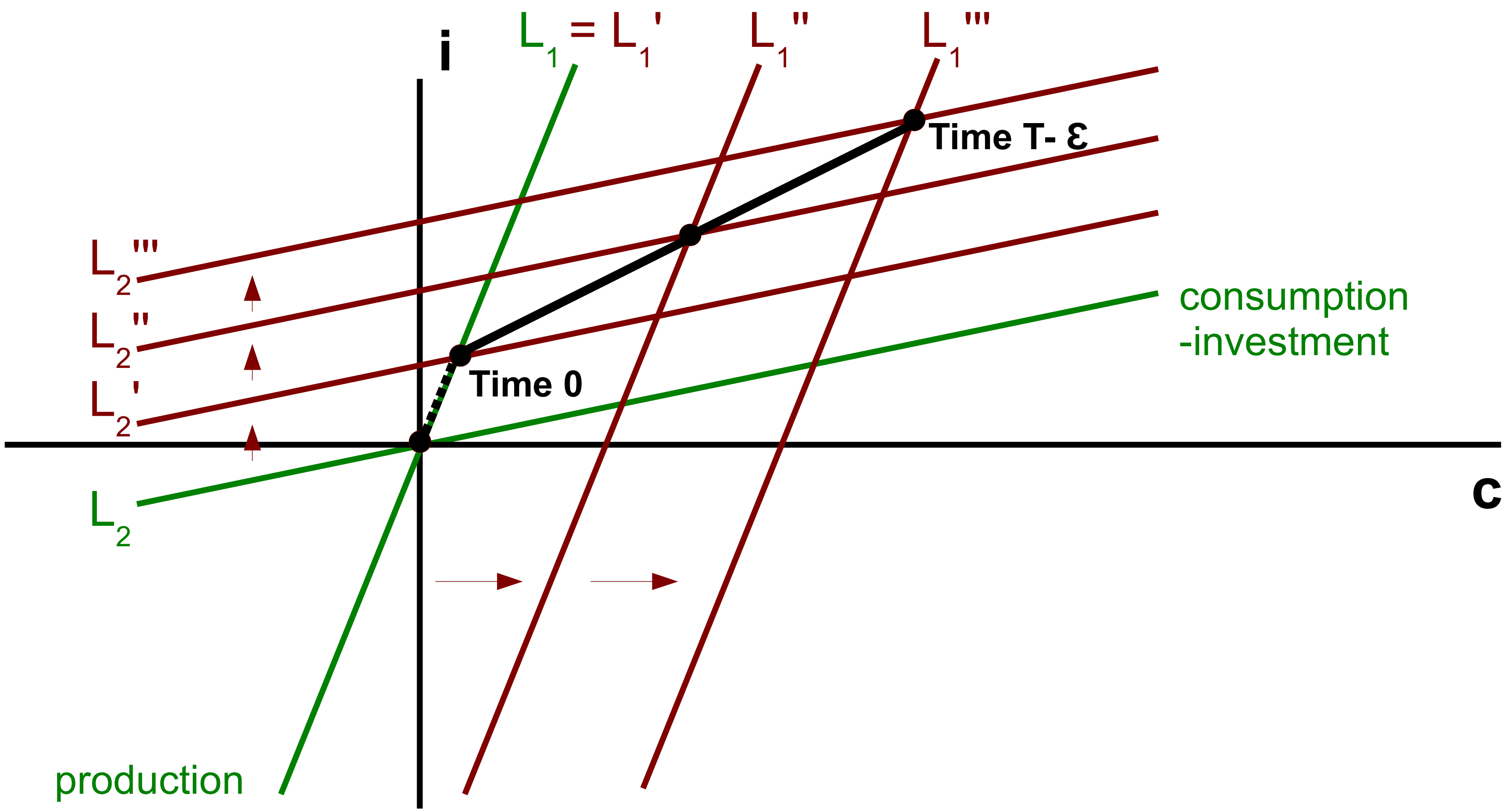
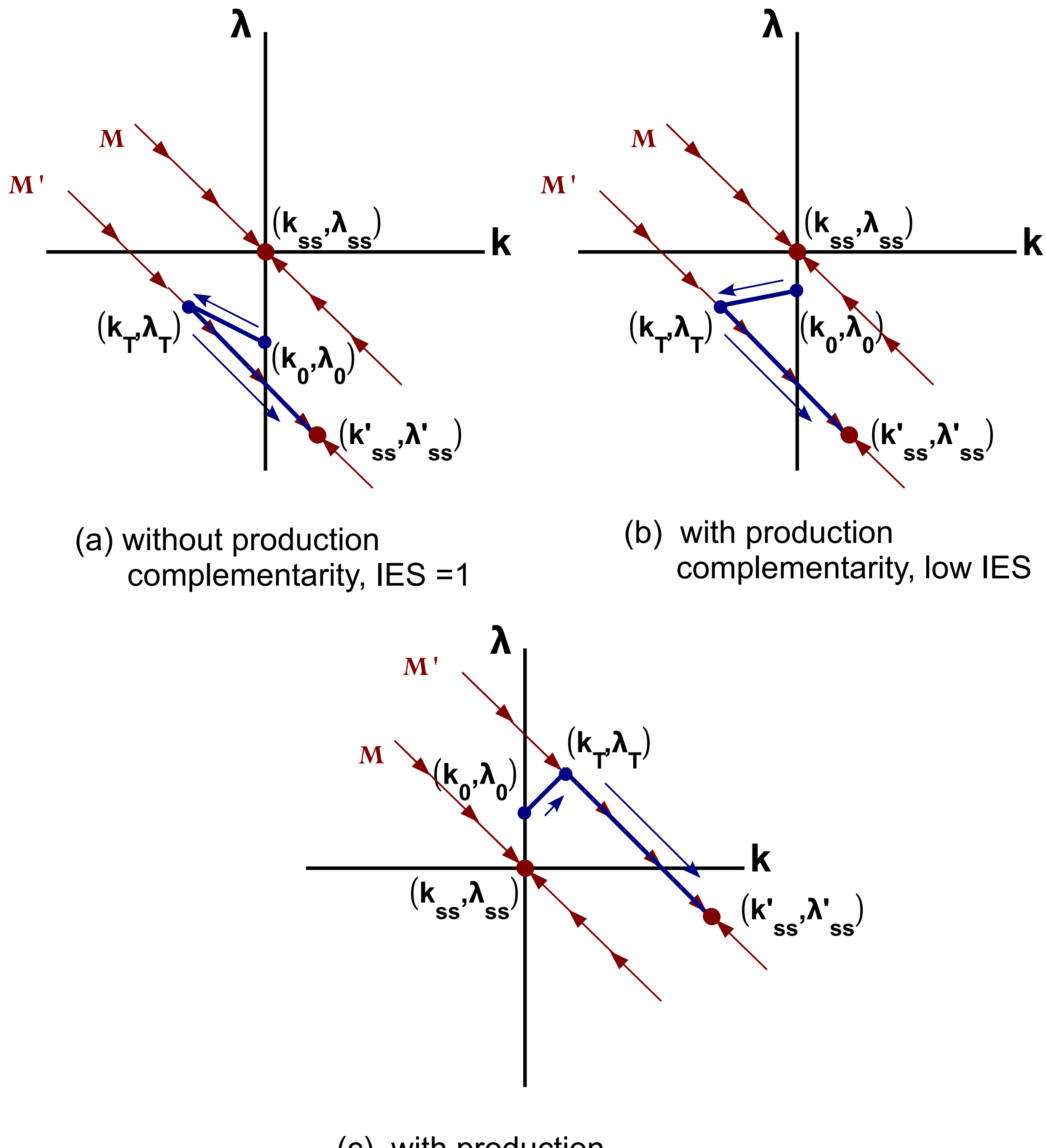


Figure 5: k - λ dynamics in response to news shock, varying complementarity and IES

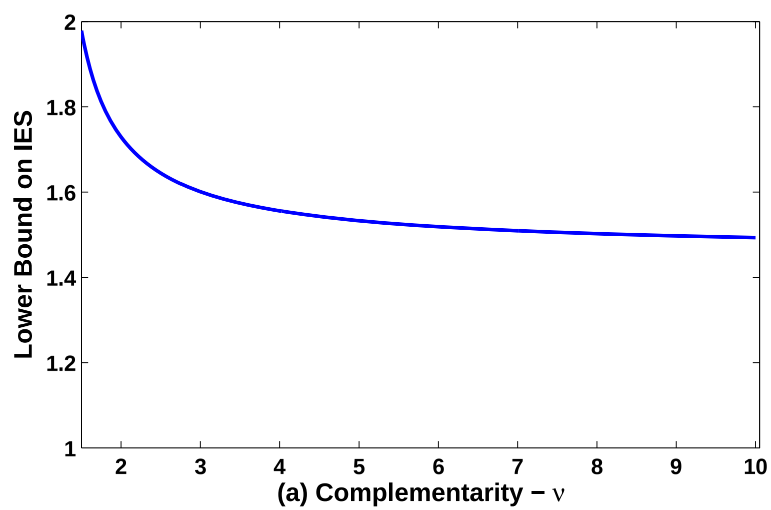


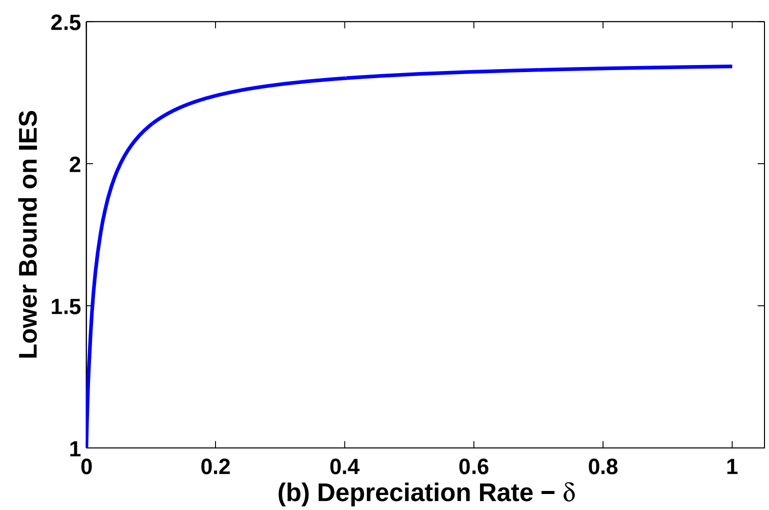
(a) without production complementarity, IES =1

(b) with production complementarity, low IES

(c) with production complementarity, high IES

Figure 6: Lower Bound on IES (σ_c^{-1}) to generate a procyclical news shock, as a function of other parameters





Note: Other parameters are set equal to: $\beta = 0.985$, $\upsilon = 1.8$, $\alpha = 1/3$, $\delta = 0.025$.

Figure 7: Baseline Model - Impulse Responses

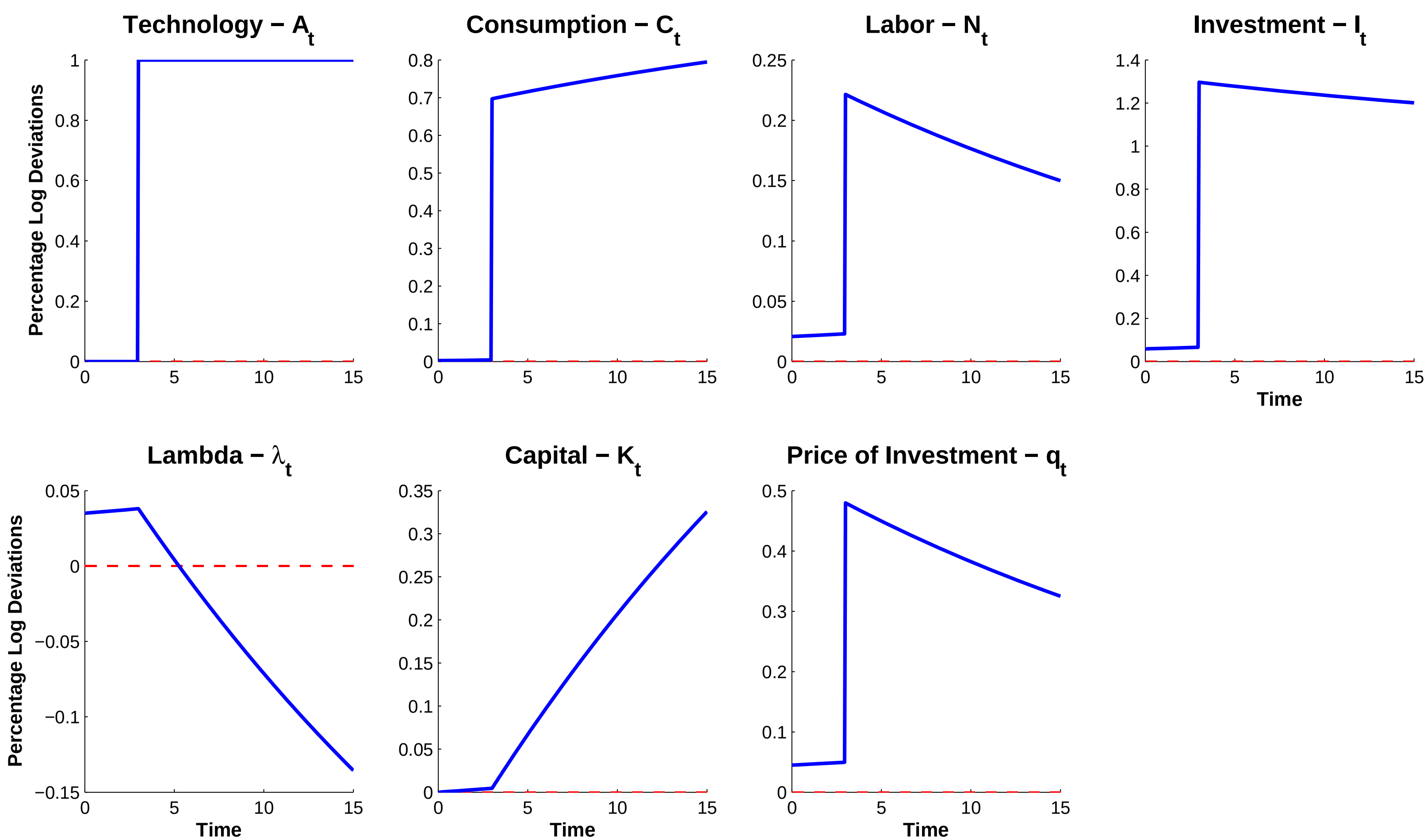


Figure 8: Baseline Model - Impulse Responses before Time T

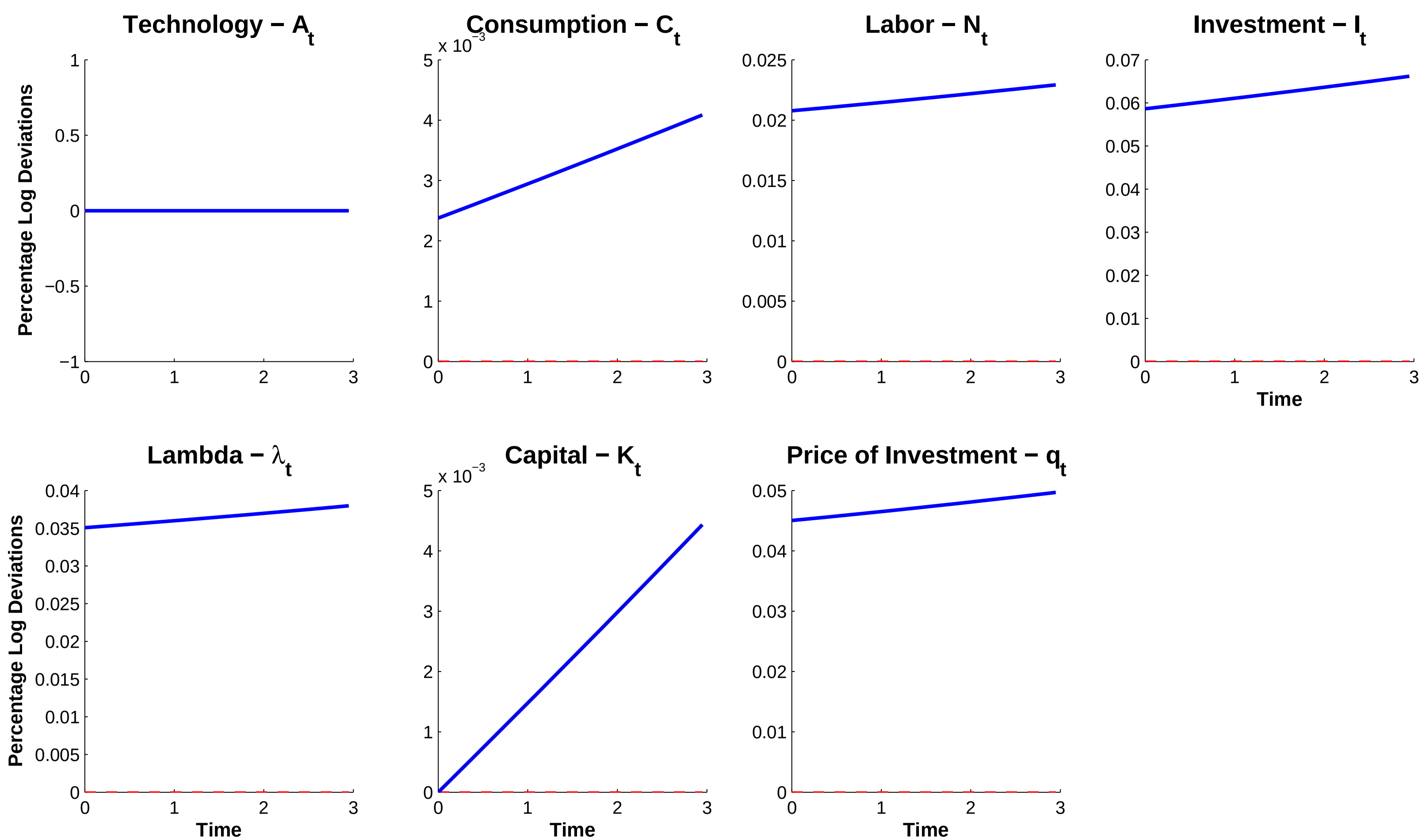
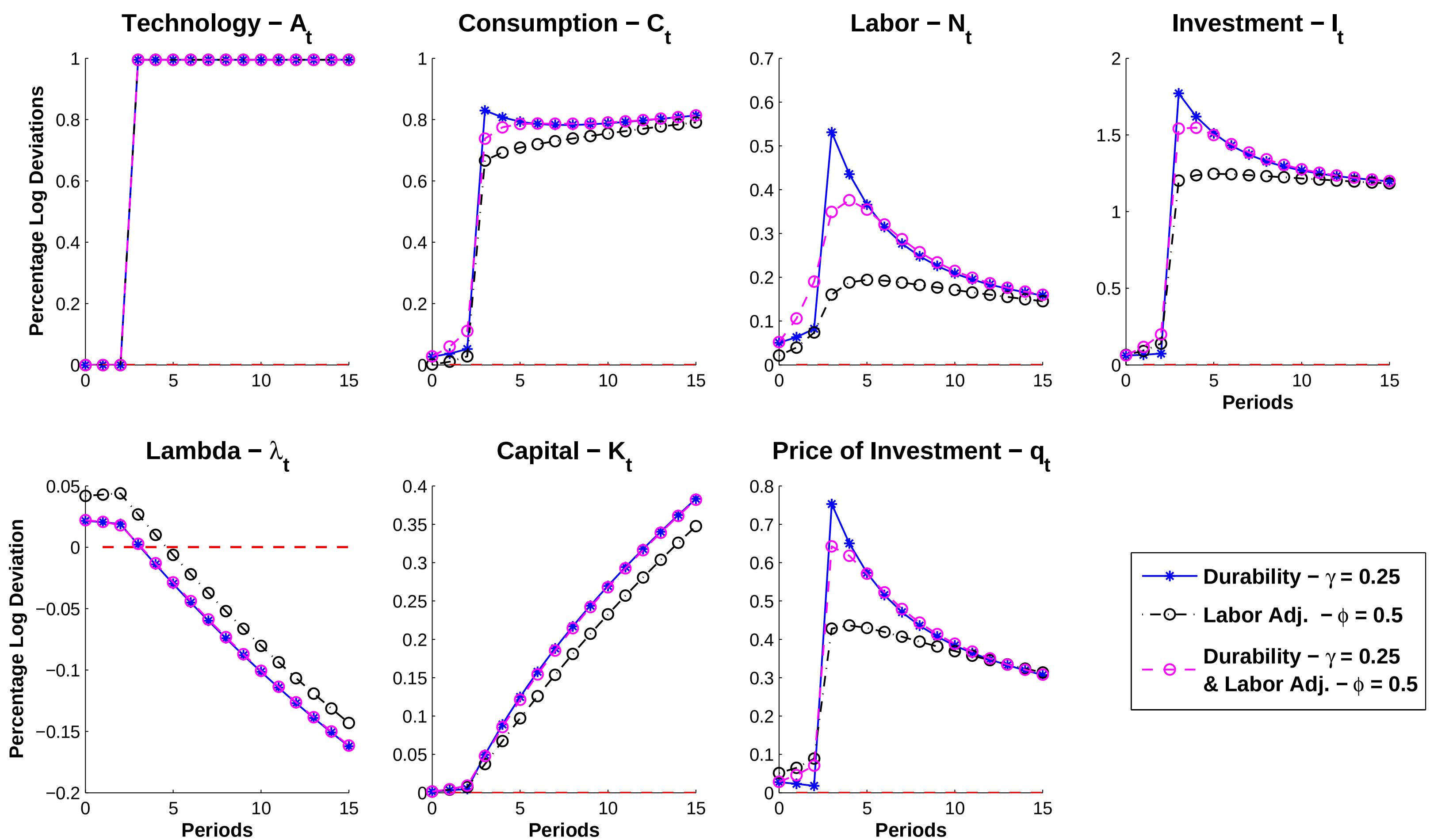


Figure 9: Full Model - Impulse Responses



A Main Appendix (Intended for Publication)

Proof of Lemma 1. Substituting (3.2) into (3.1):

$$\begin{aligned}\alpha k + (1-\alpha)\,a &= (1-(1-\upsilon(1-\alpha))\,s_I)\,c + (1-\upsilon(1-\alpha))\,s_I i \\ &= (1-\phi_I)\,c + \phi_I i \end{aligned} \tag{A.1}$$

Here $\phi_I = (1-\upsilon(1-\alpha))\,s_I$.

$k(0) = 0$ and $a(0) = 0$, therefore (A.1) can be written as:

$$0 = (1-\phi_I)\,c(0) + \phi_I i(0)$$

If $\upsilon(1-\alpha) > 1$, then $\phi_I < 0$ & $(1-\phi_I) > 0$. Therefore, if $\upsilon(1-\alpha) > 1$ and $c(0)$ increases then for (A.1) to hold $i(0)$ must also increase.

Further, if $\upsilon(1-\alpha) > 1$, then $(1-\phi_I) = (-\phi_I + 1) > -\phi_I$. Therefore, if $c(0)$ increases, then for (A.1) to hold $i(0)$ must increase by a larger magnitude than $c(0)$, this implies $(i(0)-c(0))$ increases when $c(0)$ increases, which in turn due to (3.2) implies that $n(0)$ must increase.

Therefore, if $\upsilon(1-\alpha) > 1$, then consumption, investment, and hours will comove at time zero.

If $\upsilon(1-\alpha) < 1$, then $\phi_I > 0$ and $(1-\phi_I) > 0$. Therefore, if $c(0)$ increases then for (A.1) to hold $i(0)$ must decrease. Therefore, if $\upsilon(1-\alpha) < 1$, then consumption, investment and hours will not comove at time zero. □

Proof of Lemma 2. Substituting (3.2) into (3.3):

$$\gamma_I i - (\sigma + \gamma_I)\,c = \lambda \tag{A.2}$$

Here $\gamma_I = (\upsilon - 1) - (\upsilon(1-\alpha)(1-\sigma)\,s_I)\,/\,(1-s_I)$. Also, if $\upsilon(1-\alpha) > 1$ then $\gamma_I > 0$.[5]

From the proof of Lemma 1, we further know that if $\upsilon(1-\alpha) > 1$ then c and i comove and that the magnitude of the increase in $i(0)$ is always greater than the increase in $c(0)$. Given

[5] For the proof see Lemma B.2 in the Supplementary Appendix.

this fact and from (A.2), we know that if $\lambda(0) > 0$, then $c(0)$ and $i(0)$ both increase, that is they are both greater than zero. This follows from the fact that $\gamma_I > 0$, therefore, $\gamma_I + \sigma > 0$.

As a result, if $\upsilon(1-\alpha) > 1$ and $\lambda(0) > 0$, then consumption and investment will comove procyclically at time zero in response to a news shock about technology in time $T > 0$. And again by (3.2) and (A.1), hours also comove procyclically at time zero.

Finally, by Lemma 1 we know that if $\upsilon(1-\alpha) < 1$ and $\lambda(0) > 0$, then consumption and investment will not comove at time zero. □

Proof of Lemma 3. Solving (A.1) and (A.2) simultaneously for the values of c and i:

$$c = \tau_{c,k}k + \tau_{c,\lambda}\lambda + \tau_{c,a}a \tag{A.3}$$

$$i = \tau_{i,k}k + \tau_{i,\lambda}\lambda + \tau_{i,a}a \tag{A.4}$$

Here $\tau_{c,k}, \tau_{c,\lambda}, \tau_{c,a}, \tau_{i,k}, \tau_{i,\lambda}$, and $\tau_{i,a}$ are all positive.[6]

It follows directly that if $\dot{\lambda} \geq 0$ and $\dot{k} \geq 0$ $\forall t < T$ then $\dot{c} \geq 0$ and $\dot{i} \geq 0$ for all $t < T$. Again, remember for $\forall t < T$, $a(t) = 0$. □

Proof of Lemma 4. Recall $k(0) = 0$. As a result, the time derivatives of the $k(t)$ and $\lambda(t)$ paths for all $t < T$:

$$\dot{k}(t) = \frac{\Gamma_{k,\lambda}\left(\mu_2 e^{\mu_2 t} - \mu_1 e^{\mu_1 t}\right)}{\mu_2 - \mu_1}\lambda(0)$$

$$\dot{\lambda}(t) = \left[\frac{(\mu_2 - \Gamma_{k,k})}{\mu_2 - \mu_1}\mu_2 e^{\mu_2 t} - \frac{(\mu_1 - \Gamma_{k,k})}{\mu_2 - \mu_1}\mu_1 e^{\mu_1 t}\right]\lambda(0)$$

First, for $0 \leq t < T$: $\left(\Gamma_{k,\lambda}\left(\mu_2 e^{\mu_2 t} - \mu_1 e^{\mu_1 t}\right)\right)/\left(\mu_2 - \mu_1\right)$ is positive as $\Gamma_{k,\lambda} > 0$, and we know that $\mu_2 > 0$ and $\mu_1 < 0$. Therefore the $sign(\dot{k}(t)) = sign(\lambda_0)$.

Second, for the $\dot{\lambda}$ equation: $\left(\mu_2 - \Gamma_{k,k}\right)\mu_2 e^{\mu_2 t}/\left(\mu_2 - \mu_1\right)$ is positive because $\mu_2 - \Gamma_{k,k} =$

[6] For exact values and proofs see Lemmas B.8 and B.9 in the Supplementary Appendix.

$\Gamma_{\lambda,\lambda} - \mu_1$[7] , and we know $\mu_1 < 0$ and $\Gamma_{\lambda,\lambda} > 0$. $(\mu_1 - \Gamma_{k,k})\,\mu_1 e^{\mu_1 t}/\,(\mu_2 - \mu_1)$ may be either positive or negative. First, if $\mu_1 - \Gamma_{k,k} > 0$, then the second term on the right-hand side is positive. In this case, $\dot{\lambda}\,(t) > 0$. However, if $\mu_1 - \Gamma_{k,k} < 0$, then $(\mu_1 - \Gamma_{k,k})\,\mu_1 e^{\mu_1 t}/\,(\mu_2 - \mu_1)$ is negative. In this case, we must show that $(\mu_2 - \Gamma_{k,k})\,\mu_2 e^{\mu_2 t}/\,(\mu_2 - \mu_1)$ is larger than $(\mu_1 - \Gamma_{k,k})\,\mu_1 e^{\mu_1 t}/\,(\mu_2 - \mu_1)$ in order that $\dot{\lambda}\,(t) > 0$. Because $\mu_2 > 0 > \mu_1$, in this second case, the smallest value for $\dot{\lambda}\,(t)$ occurs at $t = 0$.

$$\begin{aligned}
\dot{\lambda}\,(0) &= \frac{\lambda(0)}{\mu_2 - \mu_1}\left[\mu_2\left(\mu_2 - \Gamma_{k,k}\right) - \mu_1\left(\mu_1 - \Gamma_{k,k}\right)\right] \\
&= \lambda_0\left[\mu_2 + \mu_1 - \Gamma_{k,k}\right] \\
&= \lambda_0\left[\Gamma_{k,k} + \Gamma_{\lambda,\lambda} - \Gamma_{k,k}\right] \\
&= \lambda_0 \Gamma_{\lambda,\lambda}
\end{aligned}$$

As $\Gamma_{\lambda,\lambda} > 0$, this establishes that $sign(\dot{\lambda}\,(t)) = sign(\lambda_0)$. □

Proof of Lemma 5. Recall $\mu_2 > 0$ and

$$k\,(t) = \begin{cases} \frac{\Gamma_{k,\lambda}\lambda(0) + \left(\mu_1 - \Gamma_{\lambda,\lambda}\right)k(0)}{\mu_1 - \mu_2} e^{\mu_1 t} + \frac{\Gamma_{k,\lambda}\lambda(0) + \left(\mu_2 - \Gamma_{\lambda,\lambda}\right)k(0)}{\mu_2 - \mu_1} e^{\mu_2 t} & \text{for } t \in [0, T) \\ \frac{\Gamma_{k,\lambda}\lambda(0) + \left(\mu_1 - \Gamma_{\lambda,\lambda}\right)k(0)}{\mu_1 - \mu_2} e^{\mu_1 t} + \frac{\Gamma_{k,\lambda} b_\lambda - \Gamma_{\lambda,\lambda} b_k}{\mu_1 \mu_2} + \frac{\Gamma_{k,\lambda} b_\lambda + \left(\mu_1 - \Gamma_{\lambda,\lambda}\right) b_k}{\mu_1\left(\mu_1 - \mu_2\right)} e^{\mu_1 (t - T)} & t \geq T \end{cases}$$

Then as $k(0) = 0$ a non-explosive path for $[\lambda\ k]'$ requires that we choose $\lambda(0)$ such that the terms involving the explosive root μ_2 in the exponential are 'zeroed out' for all $t > T$. Otherwise the path for $k(t)$ will be explosive. This imposes the following restriction on $\lambda(0)$:

$$\left(\frac{\Gamma_{k,\lambda}}{\mu_2 - \mu_1}\right)\lambda_0 = -\frac{\Gamma_{k,\lambda} b_\lambda + \left(\mu_2 - \Gamma_{\lambda,\lambda}\right) b_k}{\mu_2\left(\mu_2 - \mu_1\right)} e^{-\mu_2 T}$$

This can be re-written as:

$$\lambda_0 = -\left[\frac{\Gamma_{k,\lambda} b_\lambda + \left(\mu_2 - \Gamma_{\lambda,\lambda}\right) b_k}{\Gamma_{k,\lambda}\mu_2}\right] e^{-\mu_2 T} \tag{A.5}$$

Because $\Gamma_{k,\lambda} > 0$, $\lambda(0) > 0$ if and only if $\Gamma_{k,\lambda} b_\lambda + \left(\mu_2 - \Gamma_{\lambda,\lambda}\right) b_k < 0$. Also, $\Gamma_{k,\lambda} b_\lambda + \left(\mu_2 - \Gamma_{\lambda,\lambda}\right) b_k < 0$ if and only if $\mu_2 < \left(\rho + (1 - \alpha)\,\delta\right) \upsilon/\left(\gamma_I + \sigma\right)$. □

[7] This follows because $tr(\Gamma) = \mu_1 + \mu_2$.

Proof of Theorem 1. $\Leftarrow$. If $\upsilon(1-\alpha) > 1$ and $\mu_2 < (\rho + (1-\alpha)\delta)\upsilon/(\gamma_I + \sigma)$, then a technology news shock is procyclical. Lemmas 2 and 5 prove the procyclical comovement at $t = 0$, while Lemmas 3, 4 and 5 establish the procyclical comovement for $0 < t < T$.

$\Rightarrow$. If $\upsilon(1-\alpha) < 1$ or $\mu_2 < (\rho + (1-\alpha)\delta)\upsilon/(\gamma_I + \sigma)$, then a technology news shock is not procyclical. This follow trivially from Lemma 2, as the procyclical comovement will not occur at time $t = 0$ if either of the above conditions are met. □

Proof of Theorem 2. From (A.3) and (A.4) we know that if $\lambda(0) > 0$ then $c(0) > 0$ and $i(0) > 0$, as $k(0) = 0$ and $a(0) = log(1.01)$. Therefore, to prove the above theorem, it suffices to show that in response to a contemporaneous technology shock we still have $\lambda(0) > 0$ given $\upsilon(1-\alpha) > 1$ and $\mu_2 < (\rho + (1-\alpha)\delta)\upsilon/(\gamma_I + \sigma)$.

In response to a contemporaneous technology shock we can describe the time path that λ will take by the following second order differential equation:

$$\ddot{\lambda} - (\Gamma_{\lambda,\lambda} + \Gamma_{k,k})\dot{\lambda} + (\Gamma_{k,k}\Gamma_{\lambda,\lambda} - \Gamma_{k,\lambda}\Gamma_{\lambda,k})\lambda + (\Gamma_{k,k}b_\lambda - \Gamma_{\lambda,k}b_k) = 0$$

Solving this second order differential equation for $\lambda(t)$ and imposing the condition that the solution does not diverge we get:

$$\lambda(t) = \left(\frac{\Gamma_{\lambda\lambda}\sigma - b_\lambda}{\Gamma_{\lambda\lambda} - \mu_1}\right)\exp(\mu_1 t) - \sigma$$

As the matrix Γ is the same as before the eigenvalues remain the same here as in the news shock case, with $\mu_1 < 0$. Given this non-divergent solution for $\lambda(t)$, we have $\lambda(0) > 0$ if and only if

$$\frac{\Gamma_{\lambda\lambda}\sigma - b_\lambda}{\Gamma_{\lambda\lambda} - \mu_1} - \sigma > 0$$

When $\upsilon(1-\alpha) > 1$ this condition can be shown to simplify to

$$\mu_2 < \frac{(\rho + (1-\alpha)\delta)\upsilon}{\gamma_I + \sigma}$$

□

B Supplementary Appendix (Intended for On-line Publication)

B.1 The Economy Log Linearized and Simplified

As shown in the main paper our model economy can be described by the following five log linearized equations [8]:

$$(1 - s_I)\, c + s_I i = \alpha k + (1 - \alpha)\,(a + n) \tag{B.1}$$

$$\upsilon s_I\,(i - c) = n \tag{B.2}$$

$$\lambda = (1 - \upsilon)\,(c - i) - \sigma c - zn \tag{B.3}$$

$$\dot{k} = \delta\,(i - k) \tag{B.4}$$

$$\dot{\lambda} = -\,(\rho + \delta)\,[\upsilon\,(1 - s_I)\,(c - i) + i - k] \tag{B.5}$$

Here, $s_I = \frac{\alpha\delta}{\rho+\delta}$ and $z = \frac{(1-\sigma)(1-\alpha)}{(1-s_I)}$

We can substitute (B.2) into (B.1) to get:

$$(1 - \phi_I)\, c + \phi_I i = \alpha k + (1 - \alpha)\, a \tag{B.6}$$

Here, $\phi_I = (1 - (1 - \alpha)\,\upsilon)\, s_I$

We can also substitute (B.2) into (B.3) to get:

$$\gamma_I i - (\sigma + \gamma_I)\, c = \lambda \tag{B.7}$$

Here, $\gamma_I = (\upsilon - 1) - \frac{\upsilon(1-\alpha)(1-\sigma)s_I}{(1-s_I)}$.

[8] These are equations (3.1) - (3.5) in the paper

Equations (B.6) and (B.7) now give us a system of equations in i and c (treating λ, k, and a as exogenous). Solving this system, we get:

$$c = \left(\frac{\phi_I}{\phi_I \sigma + \gamma_I}\right)\left(\frac{\gamma_I \alpha}{\phi_I} k + \frac{\gamma_I (1-\alpha)}{\phi_I} a - \lambda\right) \tag{B.8}$$

$$i = \left(\frac{\gamma_I + \sigma}{\phi_I \sigma + \gamma_I}\right)(\alpha k + (1-\alpha) a) + \left(\frac{1-\phi_I}{\phi_I \sigma + \gamma_I}\right)\lambda \tag{B.9}$$

Further, substituting these above equations into (B.2), we also get an expression for n in terms of λ, k, and a:

$$n = \frac{\upsilon s_I}{\phi_I}\left(\left(1 - \frac{\gamma_I}{\phi_I \sigma + \gamma_I}\right)(\alpha k + (1-\alpha) a) + \frac{\phi_I}{\phi_I \sigma + \gamma_I}\lambda\right) \tag{B.10}$$

Finally, we can also simplify our two dynamic equations (B.4) and (B.5) in terms of λ, k, and a, by using (B.8), (B.9), and (B.10):

$$\dot{k} = \frac{\delta}{\phi_I \sigma + \gamma_I}\left[-\left((1-\alpha)\gamma_I + \phi_I \sigma - \alpha\sigma\right) k + (1-\phi_I)\lambda + (1-\alpha)(\gamma_I + \sigma) a\right] \tag{B.11}$$

$$\begin{aligned} \dot{\lambda} \;=\; & \frac{\rho+\delta}{\phi_I \sigma + \gamma_I}\left[\left((1-\alpha)\gamma_I + \phi_I \sigma - \alpha\sigma + \alpha\sigma\upsilon(1-s_I)\right) k\right] + \\ & \frac{\rho+\delta}{\phi_I \sigma + \gamma_I}\left[\left(\phi_I - (1-\upsilon(1-s_I))\right)\lambda + (1-\alpha)\left(\sigma(\upsilon(1-s_I)-1) - \gamma_I\right) a\right] \end{aligned} \tag{B.12}$$

Equations (B.9) - (B.12) now give us a simplified system of equations that define a dynamic stochastic general equilibrium for our model economy. For ease of use in this appendix we take this one step further and rewrite these equations as follows:

$$c = \tau_{c,k} k + \tau_{c,\lambda}\lambda + \tau_{c,a} a$$

$$i = \tau_{i,k} k + \tau_{i,\lambda}\lambda + \tau_{i,a} a$$

$$n = \tau_{n,k} k + \tau_{n,\lambda}\lambda + \tau_{n,a} a$$

$$\dot{k} = \Gamma_{k,k}k + \Gamma_{k,\lambda}\lambda + b_k a$$

$$\dot{\lambda} = \Gamma_{\lambda,k}k + \Gamma_{\lambda,\lambda}\lambda + b_\lambda a$$

Here,

$$\tau_{c,k} = \frac{\partial c}{\partial k} = \frac{\gamma_I \alpha}{\phi_I \sigma + \gamma_I} \qquad \Gamma_{k,k} = \frac{\partial \dot{k}}{\partial k} = \frac{-\delta((1-\alpha)\gamma_I + \phi_I \sigma - \alpha\sigma)}{\phi_I \sigma + \gamma_I}$$

$$\tau_{c,\lambda} = \frac{\partial c}{\partial a} = \frac{-\phi_I}{\phi_I \sigma + \gamma_I} \qquad \Gamma_{k,\lambda} = \frac{\partial \dot{k}}{\partial \lambda} = \frac{\delta(1-\phi_I)}{\phi_I \sigma + \gamma_I}$$

$$\tau_{c,a} = \frac{\partial c}{\partial \lambda} = \frac{\gamma_I(1-\alpha)}{\phi_I \sigma + \gamma_I} \qquad \Gamma_{\lambda,k} = \frac{\partial \dot{\lambda}}{\partial k} = \frac{(\rho+\delta)((1-\alpha)\gamma_I + \phi_I \sigma - \alpha\sigma + \alpha\sigma\upsilon(1-s_I))}{\phi_I \sigma + \gamma_I}$$

$$\tau_{i,k} = \frac{\partial i}{\partial k} = \frac{\alpha(\gamma_I + \sigma)}{\phi_I \sigma + \gamma_I} \qquad \Gamma_{\lambda,\lambda} = \frac{\partial \dot{\lambda}}{\partial \lambda} = \frac{(\rho+\delta)(\phi_I - (1-\upsilon(1-s_I)))}{\phi_I \sigma + \gamma_I}$$

$$\tau_{i,\lambda} = \frac{\partial i}{\partial a} = \frac{1-\phi_I}{\phi_I \sigma + \gamma_I} \qquad b_k = \frac{\partial \dot{k}}{\partial a} = \frac{\delta(\gamma_I + \sigma)}{\phi_I \sigma + \gamma_I}(1-\alpha)$$

$$\tau_{i,a} = \frac{\partial i}{\partial \lambda} = \frac{(1-\alpha)(\gamma_I + \sigma)}{\phi_I \sigma + \gamma_I} \qquad b_\lambda = \frac{\partial \dot{\lambda}}{\partial a} = \frac{(\rho+\delta)(1-\alpha)(\sigma(\upsilon(1-s_I)-1)-\gamma_I)}{\phi_I \sigma + \gamma_I}$$

$$\tau_{n,k} = \frac{\partial n}{\partial k} = \frac{\upsilon s_I \alpha \sigma}{\phi_I \sigma + \gamma_I}$$

$$\tau_{n,\lambda} = \frac{\partial n}{\partial a} = \frac{\upsilon s_I}{\phi_I \sigma + \gamma_I}$$

$$\tau_{n,a} = \frac{\partial n}{\partial \lambda} = \frac{\upsilon s_I (1-\alpha)\sigma}{\phi_I \sigma + \gamma_I}$$

Recall: $s_I = \frac{\alpha\delta}{\rho+\delta}$, $\phi_I = (1-(1-\alpha)\upsilon)s_I$, and $\gamma_I = (\upsilon - 1) - \frac{\upsilon(1-\alpha)(1-\sigma)s_I}{(1-s_I)}$

B.2 The Coefficients

In this section we attempt to sign the coefficients of the system of equations described in the previous section and some other useful expressions.

Given Lemma 1 we will make the following assumption for all the remaining proofs in this appendix:

Assumption: $\upsilon > \upsilon_c$, where $\upsilon_c = (1-\alpha)^{-1}$

Lemma B.1: $\phi_I < 0$

Proof. Follows trivially from Assumption 1

$$\phi_I = (1 - (1-\alpha)\,\upsilon)\, s_I < 0$$

□

Lemma B.2: $\gamma_I > 0$

Proof.

$$\begin{aligned}
\gamma_I &= (\upsilon - 1) - \frac{\upsilon\,(1-\alpha)\,(1-\sigma)\, s_I}{(1-s_I)} \\
&= (\upsilon - 1) - \frac{(1-\alpha)\,\delta}{(\rho + (1-\alpha)\,\delta)}\,(1-\sigma)\,\upsilon\alpha \\
&> (\upsilon - 1) - \upsilon\alpha \\
&= \upsilon\,(1-\alpha) - 1 > 0
\end{aligned}$$

□

Lemma B.3: $\phi_I\sigma + \gamma_I > 0$

Proof.

$$\begin{aligned}
\phi_I\sigma + \gamma_I &= (1 - (1-\alpha)\,\upsilon)\, s_I\sigma + (\upsilon - 1) - \frac{\upsilon\,(1-\alpha)\,(1-\sigma)\, s_I}{(1-s_I)} \\
&= s_I\sigma - (1-\alpha)\,\upsilon s_I\sigma + (\upsilon - 1) - \frac{((1-\alpha)\,\upsilon s_I - (1-\alpha)\,\upsilon s_I\sigma)}{1-s_I} \\
&= s_I\sigma + (\upsilon - 1) - \frac{(1-\alpha)\,\upsilon s_I}{1-s_I} + \frac{(1-\alpha)\,\upsilon s_I^2\sigma}{1-s_I} \\
&= \upsilon - (1-\sigma s_I)\left[1 + (1-\alpha)\,\upsilon\frac{s_I}{1-s_I}\right] \\
&= \upsilon\left\{1 - (1-\sigma s_I)\left[\frac{1}{\upsilon} + (1-\alpha)\,\frac{s_I}{1-s_I}\right]\right\} > 0
\end{aligned}$$

The last inequality follows from the following observations:

If $(1-\sigma s_I)<0$, then we are done. If $(1-\sigma s_I)>0$, then we can define:

$$\chi(\upsilon)=\left\{1-(1-\sigma s_I)\left[\frac{1}{\upsilon}+(1-\alpha)\frac{s_I}{1-s_I}\right]\right\}$$

Now,

$$\begin{aligned}\chi(\infty) &= \left\{1-(1-\sigma s_I)(1-\alpha)\frac{s_I}{1-s_I}\right\} \\ &= \left\{1-(1-\sigma s_I)\alpha\frac{(1-\alpha)\delta}{\rho+(1-\alpha)\delta}\right\}>0\end{aligned}$$

$$\begin{aligned}\chi(\upsilon_c) &= \left\{1-(1-\sigma s_I)\left[(1-\alpha)+(1-\alpha)\frac{s_I}{1-s_I}\right]\right\} \\ &= \left[1-(1-\sigma s_I)\frac{(1-\alpha)}{1-s_I}\right] \\ &= \left[1-(1-\sigma s_I)\frac{(1-\alpha)\rho+(1-\alpha)\delta}{\rho+(1-\alpha)\delta}\right]>0\end{aligned}$$

$$\chi'(\upsilon) = \frac{(1-\sigma s_I)}{\upsilon^2}>0$$

Therefore, for $\upsilon\in(\upsilon_c,\infty)$, $\chi(\upsilon)>0$. Given Assumption 1 this translates to $\chi(\upsilon)>0$ □

Lemma B.4: If $\sigma\in[0,1]$, then $(1-\alpha)\gamma_I+\phi_I\sigma-\alpha\sigma>0$

Proof.

$$\left((1-\alpha)\gamma_I+\phi_I\sigma-\alpha\sigma\right)|_{\sigma=0} = (1-\alpha)\gamma_I|_{\sigma=0}>0$$

$$\begin{aligned}\left(\left(1-\alpha\right)\gamma_I+\phi_I\sigma-\alpha\sigma\right)|_{\sigma=1} &= \left(1-\alpha\right)\left(\upsilon-1\right)+\left(1-\left(1-\alpha\right)\upsilon\right)s_I-\alpha\\ &= \left(\left(1-\alpha\right)\upsilon-1\right)\left(1-s_I\right)>0\end{aligned}$$

Furthermore, $\left(1-\alpha\right)\gamma_I+\phi_I\sigma-\alpha\sigma$ is linear in σ, therefore the expression is positive as long as $\sigma\in[0,1]$. □

Lemma B.5: If $\sigma\in[0,1]$ then $\gamma_I+\sigma\left(1-\upsilon\left(1-s_I\right)\right)>0$

Proof.

$$\left(\gamma_I+\sigma\left(1-\upsilon\left(1-s_I\right)\right)\right)|_{\sigma=0} = \gamma_I|_{\sigma=0}>0$$

$$\begin{aligned}\left(\gamma_I+\sigma\left(1-\upsilon\left(1-s_I\right)\right)\right)|_{\sigma=1} &= \left(\upsilon-1\right)+\left(1-\upsilon\right)+\upsilon s_I\\ &= \upsilon s_I\end{aligned}$$

Furthermore, $\gamma_I+\sigma\left(1-\upsilon\left(1-s_I\right)\right)$ is linear in σ; therefore, the expression is positive as long as $\sigma\in[0,1]$. □

Lemma B.6: $1-\upsilon\left(1-s_I\right)<0$

Proof.

$$\begin{aligned}\left(1-\upsilon\left(1-s_I\right)\right) &= 1-\frac{\upsilon\left(\rho+\left(1-\alpha\right)\delta\right)}{\rho+\delta}\\ &= -\frac{\left(\upsilon-1\right)\rho+\left(\upsilon\left(1-\alpha\right)-1\right)\delta}{\rho+\delta}<0\end{aligned}$$

□

Lemma B.7: $\phi_I - (1 - \upsilon(1 - s_I)) > 0$

Proof.

$$\begin{aligned}\phi_I - (1 - \upsilon(1 - s_I)) &= \frac{(1 - \upsilon(1 - \alpha))\alpha\delta}{\rho + \delta} + \frac{(\upsilon - 1)\rho + (\upsilon(1 - \alpha) - 1)\delta}{\rho + \delta} \\ &= \frac{(\upsilon - 1)\rho + (1 - \alpha)(\upsilon(1 - \alpha) - 1)\delta}{\rho + \delta} > 0\end{aligned}$$

□

These lemmas now allow us to sign $\tau_{x,z}$, $\Gamma_{x,z}$, and b_x's in the system of equations that define the general equilibrium of the model. Here, $x = c, i,$ or n and $z = k,$ or λ.

At this stage it is important to note that due to the restrictive nature of Lemma 4 and 5 they are only used to sign the $\tau_{x,x}$'s, $\Gamma_{x,x}$'s, and b_x's. They are not used to construct the proofs of the theorems.

Lemma B.8: $\tau_{c,k} > 0, \tau_{c,\lambda} > 0$, and $\tau_{c,a} > 0$

Proof. $\tau_{c,k} > 0$ follows from Lemmas B.2 and B.3.
$\tau_{c,\lambda} > 0$ follows from Lemmas B.1 and B.3.
$\tau_{c,a} > 0$ follows from Lemmas B.2 and B.3.

□

Lemma B.9: $\tau_{i,k} > 0, \tau_{i,\lambda} > 0$, and $\tau_{i,a} > 0$

Proof. $\tau_{i,k} > 0$ follows from Lemmas B.2 and B.3.
$\tau_{i,\lambda} > 0$ follows from Lemmas B.1 and B.3.
$\tau_{i,a} > 0$ follows from Lemmas B.2 and B.3.

□

Lemma B.10: $\tau_{n,k} > 0, \tau_{n,\lambda} > 0$, and $\tau_{n,a} > 0$

Proof. $\tau_{n,k} > 0$ follows from Lemma B.3.
$\tau_{n,\lambda} > 0$ follows from Lemma B.3.
$\tau_{n,a} > 0$ follows from Lemma B.3.

□

Lemma B.11: $\Gamma_{k,\lambda} > 0$, $\Gamma_{\lambda,\lambda} > 0$ and $b_k > 0$

Proof. $\Gamma_{k,\lambda} > 0$ follows from Lemmas B.1 and B.3.
$\Gamma_{\lambda,\lambda} > 0$ follows from Lemma B.3 and B.7.
$b_k > 0$ follows from Lemmas B.2 and B.3.

□

Lemma B.12: If $\sigma \in [0, 1]$ then $\Gamma_{k,k} < 0$, $\Gamma_{\lambda,k} > 0$ and $b_\lambda < 0$

Proof. $\Gamma_{k,k} < 0$ follows from Lemmas B.3 and B.4.
$\Gamma_{\lambda,k} > 0$ follows from Lemmas B.3 and B.4.
$b_\lambda < 0$ follows from Lemmas B.3 and B.5.

□

B.3 The Dynamic System

Let us now look at the dynamic system:

$$\begin{bmatrix} \dot{\lambda}(t) \\ \dot{k}(t) \end{bmatrix} = \begin{bmatrix} \Gamma_{\lambda,\lambda} & \Gamma_{\lambda,k} \\ \Gamma_{k,\lambda} & \Gamma_{k,k} \end{bmatrix} \begin{bmatrix} \lambda(t) \\ k(t) \end{bmatrix} + \begin{bmatrix} b_\lambda \\ b_k \end{bmatrix} a(t) \tag{B.13}$$

To solve this system the first step is to determine the eigenvalues of Γ.

B.3.1 Eigenvalues of Γ

The sum of the eigenvalues is given by the trace of Γ:

$$\begin{aligned}
\operatorname{tr}(\Gamma) &= \Gamma_{\lambda,\lambda} + \Gamma_{k,k} \\
&= \frac{\delta\left(\left(\alpha-1\right)\gamma_I + \alpha\sigma - \phi_I\sigma\right) - \delta\left(1-\upsilon\left(1-s_I\right)-\phi_I\right) - \rho\left(1-\upsilon\left(1-s_I\right)-\phi_I\right)}{\phi_I\sigma+\gamma_I} \\
&= \frac{-\delta\left(1-\alpha\right)\gamma_I + \sigma\delta\left(\alpha-\phi_I\right) + \left(\rho+\delta\right)\phi_I - \left(\rho+\delta\right)\left(1-\upsilon\left(1-s_I\right)\right)}{\phi_I\sigma+\gamma_I} \\
&= \frac{-\delta\left(1-\alpha\right)\gamma_I + \sigma\delta\left(\alpha-\phi_I\right) + \left(1-\left(1-\alpha\right)\upsilon\right)\alpha\delta + \left(\upsilon-1\right)\rho + \left(\upsilon\left(1-\alpha\right)-1\right)\delta}{\phi_I\sigma+\gamma_I} \\
&= \frac{-\delta\left(1-\alpha\right)\gamma_I + \sigma\delta\left(\alpha-\phi_I\right) + \left(\upsilon-1\right)\rho + \left(1-\alpha\right)\left(\upsilon\left(1-\alpha\right)-1\right)\delta}{\phi_I\sigma+\gamma_I} \\
&= \frac{-\delta\left(1-\alpha\right)\gamma_I + \sigma\delta\left(\alpha-\phi_I\right) + \gamma_I\left(\rho+\left(1-\alpha\right)\delta\right) - \sigma\left(\upsilon\left(1-\alpha\right)\alpha\delta\right)}{\phi_I\sigma+\gamma_I} \\
&= \frac{\rho\gamma_I + \sigma\delta\left(\alpha-\phi_I-\upsilon\alpha\left(1-\alpha\right)\right)}{\phi_I\sigma+\gamma_I} \\
&= \frac{\rho\gamma_I + \sigma\delta\left(\alpha\left(1-\upsilon\left(1-\alpha\right)\right)-\phi_I\right)}{\phi_I\sigma+\gamma_I} \\
&= \frac{\rho\gamma_I + \sigma\delta\left(\left(\alpha-s_I\right)\left(1-\upsilon\left(1-\alpha\right)\right)\right)}{\phi_I\sigma+\gamma_I} \\
&= \frac{\rho\left(\gamma_I + \sigma s_I\left(1-\upsilon\left(1-\alpha\right)\right)\right)}{\phi_I\sigma+\gamma_I} \\
&= \frac{\rho\left(\phi_I\sigma+\gamma_I\right)}{\phi_I\sigma+\gamma_I} \\
&= \rho > 0
\end{aligned}$$

The product of the eigenvalues is given by the determinant of Γ:

$$\begin{aligned}
\det(\Gamma) &= \frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I\sigma+\gamma_I\right)^2}\left[\left(\alpha\left(\gamma_I+\sigma\right)-\phi_I\sigma-\gamma_I\right)\left(\left(1-\upsilon\left(1-s_I\right)\right)-\phi_I\right)\right] \\
&\quad + \frac{-\delta\left(\rho+\delta\right)}{\left(\phi_I\sigma+\gamma_I\right)^2}\left[\left(1-\phi_I\right)\left(\left(1-\alpha\right)\gamma_I - \left(\alpha-\phi_I\right)\sigma + \alpha\upsilon\left(1-s_I\right)\sigma\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-\delta(\rho+\delta)}{(\phi_I\sigma+\gamma_I)^2}\left[((\alpha-1)\gamma_I-(\phi_I-\alpha)\sigma)((1-\upsilon(1-s_I))-\phi_I)\right] \\
&\quad + \frac{-\delta(\rho+\delta)}{(\phi_I\sigma+\gamma_I)^2}\left[(1-\phi_I)((1-\alpha)\gamma_I-(\alpha-\phi_I)\sigma+\alpha\upsilon(1-s_I)\sigma)\right] \\
&= \frac{-\delta(\rho+\delta)}{(\phi_I\sigma+\gamma_I)^2}\left[((1-\alpha)\gamma_I-(\alpha-\phi_I)\sigma)\upsilon(1-s_I)-(1-\phi_I)\alpha\sigma\upsilon(1-s_I)\right] \\
&= \frac{-\delta(\rho+\delta)}{(\phi_I\sigma+\gamma_I)^2}\left[((1-\alpha)\gamma_I-(\alpha-\phi_I)\sigma)\upsilon(1-s_I)-(1-\phi_I)\alpha\sigma\upsilon(1-s_I)\right] \\
&= \frac{-\delta(\rho+\delta)}{(\phi_I\sigma+\gamma_I)^2}\left[\upsilon(1-s_I)(1-\alpha)(\gamma_I+\sigma\phi_I)\right] \\
&= \frac{-\delta(\rho+\delta)}{(\phi_I\sigma+\gamma_I)}\left[\upsilon(1-s_I)(1-\alpha)\right] < 0
\end{aligned}
$$

Let μ_1 and μ_2 represent the two eigenvalues of Γ. From above, we know that:

1. $\mu_1 + \mu_2 > 0$

2. $\mu_1 \mu_2 < 0$

Therefore, one of the eigenvalues must be negative and the other positive. Without loss of generality, we will assume henceforth that $\mu_1 < 0$ and $\mu_2 > 0$.

B.3.2 Solution: Permanent Future Technology Increase

We now introduce a permanent future increase in technology. Specifically,

$$
a(t) = w(t) = \begin{cases} 0 & \text{for } t \in [0, T) \\ 1 & t \geq T \end{cases} \tag{B.14}
$$

To analyze the resulting system, it will be useful to introduce the Laplace transform operator.

The Laplace transform of a function $p(t)$ is:

$$
\mathcal{L}[p(t)] = \bar{P}(s) = \int_0^\infty e^{-st} p(t)\, dt
$$

We will use $\bar{P}$ rather than P to distinguish the Laplace transform of the log deviation of a variables from the level of said variable.

Moreover, we know from Theorem 6.3 from Boyce and Diprima (1969), that

$$\mathcal{L}\left[p'(t)\right] = s\mathcal{L}\left(p(t)\right) - p(0)$$

Taking the Laplace transform of the differential equations in $\left[\begin{array}{cc} \lambda & k \end{array}\right]'$ and applying this theorem, we get:

$$\left[\begin{array}{c} \bar{\Lambda}(s) \\ \bar{K}(s) \end{array}\right] = (sI - \Gamma)^{-1}\left\{\left[\begin{array}{c} \lambda(0) \\ k(0) \end{array}\right] + \left[\begin{array}{c} b_\lambda \\ b_k \end{array}\right] W(s)\right\} \tag{B.15}$$

Given (B.14), it can be shown that

$$\bar{W}(s) = \mathcal{L}\left[w(t)\right] = \frac{1}{s}e^{-sT}$$

Rewriting equation (B.15), we get:

$$\left[\begin{array}{c} \bar{\Lambda}(s) \\ \bar{K}(s) \end{array}\right] = \frac{1}{(s-\mu_1)(s-\mu_2)}\left[\begin{array}{cc} s - \Gamma_{k,k} & \Gamma_{\lambda,k} \\ \Gamma_{k,\lambda} & s - \Gamma_{\lambda,\lambda} \end{array}\right]\left\{\left[\begin{array}{c} \lambda(0) \\ k(0) \end{array}\right] + \left[\begin{array}{c} b_\lambda \\ b_k \end{array}\right] W(s)\right\} \tag{B.16}$$

Remember from the previous section, μ_1 and μ_2 are the eigenvalues of Γ, and $\mu_1 < 0$ and $\mu_2 > 0$.

The lower row of (B.16) gives:

$$\bar{K}(s) = \frac{\Gamma_{k,\lambda}\lambda(0) + (s - \Gamma_{\lambda,\lambda})\,k(0)}{(s-\mu_1)(s-\mu_2)} + \left[\frac{\Gamma_{k,\lambda}b_\lambda + (s-\Gamma_{\lambda,\lambda})\,b_k}{s(s-\mu_1)(s-\mu_2)}\right]e^{-sT}$$

Next, we take the inverse Laplace transform of $K(s)$ to recover k as a function of time. After some algebra,

$$\begin{aligned} k(t) &= \frac{\Gamma_{k,\lambda}\lambda(0)}{\mu_1 - \mu_2}e^{\mu_1 t} + \frac{\Gamma_{k,\lambda}\lambda(0)}{\mu_2 - \mu_1}e^{\mu_2 t} + \frac{(\mu_1 - \Gamma_{\lambda,\lambda})\,k(0)}{\mu_1 - \mu_2}e^{\mu_1 t} + \frac{(\mu_2 - \Gamma_{\lambda,\lambda})\,k(0)}{\mu_2 - \mu_1}e^{\mu_2 t} + u_T(t) \times \\ &\left[\frac{\Gamma_{k,\lambda}b_\lambda - \Gamma_{\lambda,\lambda}b_k}{\mu_1\mu_2} + \frac{\Gamma_{k,\lambda}b_\lambda + (\mu_1 - \Gamma_{\lambda,\lambda})\,b_k}{\mu_1(\mu_1 - \mu_2)}e^{\mu_1(t-T)} + \frac{\Gamma_{k,\lambda}b_\lambda + (\mu_2 - \Gamma_{\lambda,\lambda})\,b_k}{\mu_2(\mu_2 - \mu_1)}e^{\mu_2(t-T)}\right] \end{aligned}$$

where $u_T(t)$ is a step function that takes on a value of one for all $t \geq T$, and zero otherwise.

Recall that we assume the initial capital stock is at the steady-state level associated with the pre-shock technology level. As such, $k(0) = 0$:

$$
\begin{aligned}
k(t) \;=\;& \frac{\Gamma_{k,\lambda}\lambda(0)}{\mu_1 - \mu_2}e^{\mu_1 t} + \frac{\Gamma_{k,\lambda}\lambda(0)}{\mu_2 - \mu_1}e^{\mu_2 t} + u_T(t) \times \\
& \left[\frac{\Gamma_{k,\lambda}b_\lambda - \Gamma_{\lambda,\lambda}b_k}{\mu_1\mu_2} + \frac{\Gamma_{k,\lambda}b_\lambda + (\mu_1 - \Gamma_{\lambda,\lambda})\, b_k}{\mu_1(\mu_1 - \mu_2)}e^{\mu_1(t-T)} + \frac{\Gamma_{k,\lambda}b_\lambda + (\mu_2 - \Gamma_{\lambda,\lambda})\, b_k}{\mu_2(\mu_2 - \mu_1)}e^{\mu_2(t-T)}\right]
\end{aligned}
\tag{B.17}
$$

This gives us the solution to a differential equation with one undetermined variable $\lambda(0)$. We now seek a path for $\left[\begin{array}{cc} \lambda & k \end{array}\right]'$ that is not explosive. In order to achieve this, we choose $\lambda(0)$ such that the explosive root μ_2 is 'zeroed out' for all $t > T$. Otherwise, the path for $k(t)$ will be explosive. This restriction on $\lambda(0)$ is:

$$
\left(\frac{\Gamma_{k,\lambda}}{\mu_2 - \mu_1}\right)\lambda(0) = -\frac{\Gamma_{k,\lambda}b_\lambda + (\mu_2 - \Gamma_{\lambda,\lambda})\, b_k}{\mu_2(\mu_2 - \mu_1)}e^{-\mu_2 T}
$$

This can be re-written as:

$$
\lambda(0) = -\left[\frac{\Gamma_{k,\lambda}b_\lambda + (\mu_2 - \Gamma_{\lambda,\lambda})\, b_k}{\Gamma_{k,\lambda}\mu_2}\right]e^{-\mu_2 T}
$$

Let us also solve the second half of our laplace transform. This will allow us to study the path of $\lambda(t)$ over time. The first row of (B.16) gives us:

$$
\bar{\Lambda}(s) = \frac{\Gamma_{\lambda,k}k_0 + (s - \Gamma_{k,k})\,\lambda_0}{(s - \mu_1)(s - \mu_2)} + \left[\frac{\Gamma_{\lambda,k}b_k + (s - \Gamma_{k,k})\, b_\lambda}{s(s - \mu_1)(s - \mu_2)}\right]e^{-sT}
$$

Similar to before, we can again take the inverse Laplace transform of $\Lambda(s)$ to recover λ as a function of time. After some algebra,

$$
\begin{aligned}
\lambda(t) \;=\;& \frac{\Gamma_{\lambda,k}k(0)}{\mu_1 - \mu_2}e^{\mu_1 t} + \frac{\Gamma_{\lambda,k}k(0)}{\mu_2 - \mu_1}e^{\mu_2 t} + \frac{(\mu_1 - \Gamma_{k,k})\,\lambda(0)}{\mu_1 - \mu_2}e^{\mu_1 t} + \frac{(\mu_2 - \Gamma_{k,k})\,\lambda(0)}{\mu_2 - \mu_1}e^{\mu_2 t} + u_T(t) \times \\
& \left[\frac{\Gamma_{\lambda,k}b_k - \Gamma_{k,k}b_\lambda}{\mu_1\mu_2} + \frac{\Gamma_{\lambda,k}b_k + (\mu_1 - \Gamma_{k,k})\, b_\lambda}{\mu_1(\mu_1 - \mu_2)}e^{\mu_1(t-T)} + \frac{\Gamma_{\lambda,k}b_k + (\mu_2 - \Gamma_{k,k})\, b_\lambda}{\mu_2(\mu_2 - \mu_1)}e^{\mu_2(t-T)}\right]
\end{aligned}
$$

where $u_T(t)$ is the same step function as defined before.

Recall that for our solution the initial capital stock is at the steady-state level associated with the pre-shock technology level, that is $k(0) = 0$:

$$\lambda(t) = \frac{(\mu_1 - \Gamma_{k,k})\lambda(0)}{\mu_1 - \mu_2} e^{\mu_1 t} + \frac{(\mu_2 - \Gamma_{k,k})\lambda(0)}{\mu_2 - \mu_1} e^{\mu_2 t} + u_T(t) \times \left[\frac{\Gamma_{\lambda,k} b_k - \Gamma_{k,k} b_\lambda}{\mu_1 \mu_2} + \frac{\Gamma_{\lambda,k} b_k + (\mu_1 - \Gamma_{k,k}) b_\lambda}{\mu_1 (\mu_{1-}\mu_2)} e^{\mu_1(t-T)} + \frac{\Gamma_{\lambda,k} b_k + (\mu_2 - \Gamma_{k,k}) b_\lambda}{\mu_2 (\mu_2 - \mu_1)} e^{\mu_2(t-T)}\right] \tag{B.18}$$

B.3.3 Solution: Permanent Instantaneous Technology Increase

We will now solve our model for when there is a contemporaneous technological improvement. That is, when $w(t) = 1$ for all t. To do this we will use the standard methodology. This methodology can be broken down into four steps:

Step 1: Express λ as a non-homogeneous second order DFQ with a constant coefficient.
Step 2: Find the general solution of $\lambda(t)$ that does not diverge.
Step 3: Find the implied equation in $k(t)$ that has one undetermined coefficient.
Step 4: Solve for the undetermined coefficient and then solve for $\lambda(t)$.

Step One:

Rearranging the $\dot{\lambda}$ equation in (B.13),

$$k = \frac{1}{\Gamma_{\lambda,k}} \left(\dot{\lambda} - \Gamma_{\lambda,\lambda}\lambda - b_\lambda\right)$$

Differentiate this with respect to time.

$$\dot{k} = \frac{1}{\Gamma_{\lambda,k}} \left(\ddot{\lambda} - \Gamma_{\lambda,\lambda}\dot{\lambda}\right) \tag{B.19}$$

Plugging (B.19) into the second equation of (B.13),

$$\frac{1}{\Gamma_{\lambda,k}} \left(\ddot{\lambda} - \Gamma_{\lambda,\lambda}\dot{\lambda}\right) = \Gamma_{k,\lambda}\lambda + \frac{\Gamma_{k,k}}{\Gamma_{\lambda,k}} \left(\dot{\lambda} - \Gamma_{\lambda,\lambda}\lambda - b_\lambda\right) + b_k$$

We can rewrite this equation as:

$$\ddot{\lambda} - (\Gamma_{\lambda,\lambda} + \Gamma_{k,k})\dot{\lambda} + (\Gamma_{k,k}\Gamma_{\lambda,\lambda} - \Gamma_{k,\lambda}\Gamma_{\lambda,k})\lambda + (\Gamma_{k,k} b_\lambda - \Gamma_{\lambda,k} b_k) = 0 \tag{B.20}$$

Step Two:

The general solution to equation (B.20) is the sum of a particular solution to the equation and the general solution to (B.20) without the constant term.

A particular solution to (B.20) is

$$\lambda\left(t\right) = -\frac{\Gamma_{k,k}b_{\lambda} - \Gamma_{\lambda,k}b_{k}}{\Gamma_{k,k}\Gamma_{\lambda,\lambda} - \Gamma_{\lambda,k}\Gamma_{k,\lambda}}$$

It can be shown that this particular solution to (B.20) can be simplified further:

$$\lambda\left(t\right) = -\sigma$$

Next, the general solution to (B.20) without the constant term is:

$$\lambda\left(t\right) = c_1 \exp\left(\mu_1 t\right) + c_2 \exp\left(\mu_2 t\right)$$

where μ_1 and μ_2 are the eigenvalues of the Γ matrix defined in the previous subsections.

Thus, the complete general solution to (B.20) is:

$$\lambda\left(t\right) = c_1 \exp\left(\mu_1 t\right) + c_2 \exp\left(\mu_2 t\right) - \sigma$$

Next, we are looking for a solution where $\lambda\left(t\right)$ does not diverge as $t \to \infty$. Previously, we established that one eigenvalue is positive and one eigenvalue is negative, and that $\mu_2 > 0$.

Then, our solution which does not diverge is:

$$\lambda\left(t\right) = c_1 \exp\left(\mu_1 t\right) - \sigma \tag{B.21}$$

where, again, c_1 is a coefficient that remains to be determined.

Step Three:

From the original k equation,

$$k = \frac{1}{\Gamma_{\lambda,k}} \left[\dot{\lambda} - \Gamma_{\lambda,\lambda}\lambda - b_\lambda\right]$$

Using (B.21),

$$k(t) = \frac{1}{\Gamma_{\lambda,k}} \left[c_1 (\mu_1 - \Gamma_{\lambda,\lambda}) \exp(\mu_1 t) + \Gamma_{\lambda,\lambda}\sigma - b_\lambda\right]$$

Step Four:

Next, we must solve for c_1 by using a boundary condition. We have $k(0) = 0$. Then,

$$0 = c_1 (\mu_1 - \Gamma_{\lambda,\lambda}) + \Gamma_{\lambda,\lambda}\sigma - b_\lambda$$

Rearranging,

$$c_1 = \frac{\Gamma_{\lambda,\lambda}\sigma - b_\lambda}{\Gamma_{\lambda,\lambda} - \mu_1}$$

Given our initial condition on c_1 and (B.21), our equation for $\lambda(t)$ is

$$\lambda(t) = \left(\frac{\Gamma_{\lambda,\lambda}\sigma - b_\lambda}{\Gamma_{\lambda,\lambda} - \mu_1}\right) \exp(\mu_1 t) - \sigma$$

Lemma B.13: $\mu_2 < \frac{(\rho+(1-\alpha)\delta)\upsilon}{\gamma_I+\sigma}$ if and only if $\left(\frac{\Gamma_{\lambda,\lambda}\sigma - b_\lambda}{\Gamma_{\lambda,\lambda}-\mu_1}\right) - \sigma > 0$.

Proof.

$$\begin{aligned}
& \frac{\mu_1\sigma - b_\lambda}{\Gamma_{\lambda,\lambda} - \mu_1} > 0 \\
\Leftrightarrow \quad & \frac{\mu_1 \frac{\Gamma_{k,k}b_\lambda - \Gamma_{\lambda,k}b_k}{det(\Gamma)} - b_\lambda}{\Gamma_{\lambda,\lambda} - \mu_1} > 0 \\
\Leftrightarrow \quad & \frac{\frac{\left(\Gamma_{k,k}b_\lambda - \Gamma_{\lambda,k}b_k\right)}{\mu_2} - b_\lambda}{\Gamma_{\lambda,\lambda} - \mu_1} > 0 \\
\Leftrightarrow \quad & \frac{\frac{\left(\Gamma_{k,k}b_\lambda - \Gamma_{\lambda,k}b_k\right)}{\mu_2} - b_\lambda}{\mu_2 - \Gamma_{k,k}} > 0
\end{aligned}$$

$$
\begin{aligned}
\Leftrightarrow\quad & \frac{-\Gamma_{\lambda,k}\Gamma_{k,\lambda}}{\Gamma_{k,\lambda}\mu_2\left(\mu_2-\Gamma_{k,k}\right)}b_k-\frac{b_\lambda}{\mu_2}>0\\
\Leftrightarrow\quad & \frac{-\mu_1\mu_2+\Gamma_{\lambda,\lambda}\Gamma_{k,k}}{\Gamma_{k,\lambda}\mu_2\left(\mu_2-\Gamma_{k,k}\right)}b_k+\frac{b_\lambda}{\mu_2}<0\\
\Leftrightarrow\quad & \frac{-\mu_2\left(\Gamma_{\lambda,\lambda}+\Gamma_{k,k}-\mu_2\right)\Gamma_{\lambda,\lambda}\Gamma_{k,k}}{\Gamma_{k,\lambda}\mu_2\left(\mu_2-\Gamma_{k,k}\right)}b_k+\frac{b_\lambda}{\mu_2}<0\\
\Leftrightarrow\quad & \frac{\mu_2\left(\mu_2-\Gamma_{\lambda,\lambda}\right)-\Gamma_{k,k}\left(\mu_2-\Gamma_{\lambda,\lambda}\right)}{\Gamma_{k,\lambda}\mu_2\left(\mu_2-\Gamma_{k,k}\right)}b_k+\frac{b_\lambda}{\mu_2}<0\\
\Leftrightarrow\quad & \frac{\left(\mu_2-\Gamma_{\lambda,\lambda}\right)b_k}{\Gamma_{k,\lambda}\mu_2}+\frac{b_\lambda}{\mu_2}<0\\
\Leftrightarrow\quad & \frac{\Gamma_{k,\lambda}b_\lambda+\left(\mu_2-\Gamma_{\lambda,\lambda}\right)b_k}{\Gamma_{k,\lambda}\mu_2}<0\\
\Leftrightarrow\quad & \Gamma_{k,\lambda}b_\lambda+\left(\mu_2-\Gamma_{\lambda,\lambda}\right)b_k<0\\
\Leftrightarrow\quad & \mu_2<\frac{\left(\rho+\left(1-\alpha\right)\delta\right)\upsilon}{\gamma_I+\sigma}
\end{aligned}
$$

$\square$