Abstract

We study bilateral exchange, both direct trade, and indirect trade that happens through intermediaries, or middlemen. We develop a model of this activity and present a sequence of applications. We illustrate how, and how many, intermediaries might get involved, and how the terms of trade are determined. We have something to say about the roles of buyers and sellers in bilateral exchange, and how to interpret prices. We develop a particular bargaining solution and discuss how it relates to other solutions. We also illustrate how bubbles can emerge in the value of inventories.
1 Introduction

We study bilateral exchange, both direct trade, and indirect trade that happens through intermediaries, or middlemen. We develop a model of this activity and present a sequence of examples and applications. The framework illustrates how, and how many, middlemen might get involved. Although there is much good economic research on intermediation, in general, a neglected aspect that seems to be important to business practitioners is that there are often multiple middlemen engaged in getting goods from the originator to end user – e.g., from farmer to broker to distributor to retailer to consumer.\footnote{This example is from Cooke (2000). As another example, Ellis (2009) describes the internet like this: “If a majority of the wholesale companies being advertised are not true wholesale companies, then what are they and where are they getting their products? They are likely just middleman operating within a chain of middleman. A middleman-chain occurs when a business purchases its resale products from one wholesale company, who in turn purchases the products from another wholesale company, which may also purchase the products from yet another wholesale company, and so on.”} A feature we emphasize is that the terms of trade one might negotiate with a second-party intermediary depend on upcoming negotiations with third, fourth and other downstream intermediaries. This is what we call bargaining with bargainers. We also have something to say about the roles of buyers and sellers – in particular, which is which – in theories of bilateral exchange. We similarly have something to say about the interpretation of prices. We develop a simple bargaining solution and discuss how it relates to other solutions. Additionally, we illustrate how bubbles can emerge in the value of inventories as they get traded and retraded across intermediaries.

In terms of related work, it was not so long ago that Rubinstein and Wolinsky (1987) motivated their paper as follows:

Despite the important role played by intermediation in most markets, it is largely ignored by the standard theoretical literature. This is because a study of intermediation requires a basic model that describes explicitly the trade frictions that give rise to the function of intermediation. But this is missing from the standard market models, where the actual process of trading is left unmodeled.

Subsequently, a variety of studies attempted to rectify the situation by explicitly analyzing
the roles of middlemen and how they affect the quality of matches, the time required to conduct transactions (liquidity), the variety of goods on the market, bid-ask spreads, and other interesting phenomena.

Rubinstein and Wolinsky (1987) themselves focus on search frictions, and for them, middlemen are agents who have an advantage over the original suppliers in the rate at which they meet buyers (see also Bose and Sengupta 2010). Focusing instead on information frictions, Biglaiser (1993) and Li (1998) present models where middlemen are agents with expertise that allows them to distinguish high- from low-quality goods. The presence of informed middlemen can give the original suppliers greater incentive to produce high-quality output, helping to ameliorate lemons problems. In other papers (e.g., Johri and Leach 2002, Shevchenko 2004, Smith 2004 and Watanabe 2010a,b), middlemen hold inventories of either more, or more types of, commodities that help buyers obtain their preferred goods more easily (see also Dong 2009). In Tse (2009) the cost of trade between individuals is increasing in the physical distance between them, so that spatial considerations give rise to intermediation (see also Kalai, Postlewaite and Roberts 1978). In Masters (2007,2008), middlemen may have comparative advantage in intermediation because they have a disadvantage in production; they can also have an advantage in terms of bargaining.

Generally speaking, middlemen may hold inventories or they may act as market makers that get buyers and sellers together (Yavas 1992,1994,1996; Gehrig 1993). Models of this activity in financial markets include Miao (2006), Duffie, Garleanu and Pedersen (2005), and Lagos and Rocheteau (2009), who study how changes in frictions affect traders’ bargaining positions and bid–ask spreads. In Weill (2007) and Lagos, Rocheteau and Weill (2009), market makers can “lean against the wind” by adjusting inventories. Many of these applications are part of the New Monetarist economics surveyed by Nosal and Rocheteau (2010) and Williamson and Wright (2010a,b), defined by an endeavor to explicitly model the exchange process, and institutions that facilitate this process, like money, banks, intermediaries, etc. We say more later about the relationship between intermediation and money. For now, we
mention that early search-based models of monetary exchange, such as Kiyotaki and Wright (1989), not only make predictions about which goods might emerge as media of exchange, as a function of goods’ properties and agents’ beliefs, they also make (perhaps less well known) predictions about which agents might emerge as middlemen.

Search theory is the right tool for analyzing intermediaries, and related institutions, for the reason articulated by Rubinstein and Wolinsky: it models exchange explicitly. In a sense, this study is as much about the basic methods of search theory as it is about the substantive topic of middlemen. We set up the environment differently, in several ways, from previous studies. This is because we are less interested in why middlemen have a role, in that much of the analysis revolves around environments where trade must be intermediated. Instead, we focus on equilibrium patterns of trade, with potentially long chains of intermediation, and the determination of the terms of trade. Still, for comparison, we present a version that generalizes standard models like Rubinstein and Wolinsky (1987). We also strive to bridge some gaps between different branches of search theory by showing how several different models can be interpreted in light of our framework. This leads us to a discussion of questions such as, who is the buyer and who is the seller, and what is the price, in bilateral trade. We focus on a particular bargaining solution that we argue is attractive for equilibrium search theory, but we also present and compare results for other bargaining solutions.

The rest of the paper is organized as follows. Section 2 presents the basic model and some examples to illustrate our methods. Section 3 presents the version that can be considered an extension of the standard model. Section 4 goes into more detail concerning the dynamics of exchange and pricing in intermediated markets. Section 5 presents a discussion of bigger issues, in terms of interpretation, in this class of models. Section 6 discusses how intermediated trade may be further intermediated – by money – helping to make sense of much informal discussion in the related literature. Section 7 takes up the possibility of bubbles, defined as equilibrium trajectories where the prices of inventories can exceed their fundamental value. Section 8 concludes.
2 The Model

2.1 Basic Assumptions

Consider a set of agents $A = \{A_1, A_2, \ldots A_N\}$, where $N \leq \infty$. They are spatially separated with the following connections: $A_n$ can meet, and hence trade, with $A_{n-1}$ and $A_{n+1}$ but no one else. We can represent the population as a graph with the set of nodes $A$ connected as shown in Figure 1. There are search frictions, which means it can take time and other resources for $A_n$ to meet $A_{n+1}$. There is an indivisible object $x$ in fixed supply, and a divisible object $y$ that anyone can produce at unit cost (i.e., the utility of producing $y$ units of output is $-y$). Only $A_1$ is endowed with $x$, and he can either try to trade it to $A_2$ in exchange for $y_1$, or consume it himself for utility $\gamma_1$. Hence, $\gamma_1$ is an opportunity cost of trading $x$, although for many purposes one can alternatively interpret it as a production cost.

![Population graph](image)

Figure 1: Population graph

More generally, if any agent $A_n$ acquires $x$ from $A_{n-1}$, he can either consume it for payoff $\gamma_n$, or try to trade it to $A_{n+1}$ for payoff $u(y_n) = y_n$. If $A_1$ trades $x$ to $A_2$ and $A_2$ trades it to ... before some $A_n$ eventually consumes it, we say trade is intermediated and call $A_2, \ldots A_{n-1}$ intermediaries or middlemen (in principle $A_n$ could also try to trade $x$ back to $A_{n-1}$ but this never happens in equilibrium). Also, for most of what we do it is assumed that $A_n$ exits the market after trading $x$ to $A_{n+1}$. If one wanted to keep the economy going forever, it is a simple matter to replace every $A_n$ with a “clone” of himself after he leaves the market. As an alternative to “cloning,” in Section 3 we “recycle” agents by allowing them to continue in, rather than exit from, the market after each trade.\(^2\)

\(^2\)Both what we call “cloning” and “recycling” can be found in the search literature, and the former is usually easier (e.g., Burdett and Coles 1998). In our baseline model, we simply “terminate” them.
2.2 Example: $N = 2$

Consider an economy with $N = 2$ - or, equivalently, for this exercise, $N = \infty$ with $\gamma_n > 0$ for $n \leq 2$ and $\gamma_n = 0$ for all $n > 2$, since this implies $x$ will never be traded beyond $A_2$ (see below). In this case there can be no middlemen, but it is still useful as a vehicle to illustrate our trading protocol and as an input into the more interesting cases to follow. We begin by ignoring search, and asking what happens if $A_1$ with $x$ happens to meet $A_2$. If $\gamma_2 \leq \gamma_1$, there are no gains from trade, and $A_1$ simply consumes $x$. If $\gamma_2 > \gamma_1$, they play the following game:

Stage 1: $A_1$ moves by making an offer “give me $y_1$ for $x$.”

Stage 2: $A_2$ moves by accepting or rejecting, where:

- Accept means the game ends;
- Reject means we go to stage 3.

Stage 3: Nature moves (a coin toss) with the property that:

- With probability $\theta_1$, $A_1$ makes $A_2$ a take-it-or-leave-it offer;
- With probability $1 - \theta_1$, $A_2$ makes $A_1$ a take-it-or-leave-it offer.

Figure 2 shows the game tree.\(^3\) If the initial offer $y_1$ is accepted, $A_1$ gets payoff $y_1$ and $A_2$ gets $\gamma_2 - y_1$. If $y_1$ is rejected, with probability $\theta_1$, $A_1$ gets the whole surplus leaving $A_2$ with his outside option 0, and with probability $1 - \theta_1$, $A_1$ gets his outside option $\gamma_1$ while $A_2$ gets the surplus $\gamma_2 - \gamma_1$. The unique subgame perfect equilibrium is: at stage 1, $A_1$ makes

\(^3\)We are not sure of the original use of this particular extensive form, but it is obviously related to Stahl (1972), Rubinstein (1982), Binmore (1987) and McCleod and Malcomson (1993), to name a few. The exact specification, with just two rounds of bargaining where the second has a coin flip to determine who makes the final offer, appeared in early versions of Cahuc, Postel-Vinay and Robin (2006), but they ultimately switched to a more standard game, which gives the same answer in their linear model. We say more about this below, when we introduce nonlinear utility. With linear utility, one can actually ignore the first round and simply use a coin flip to determine who makes a take-it-or-leave-it offer, as in several previous search models (e.g., Gale 1990 or Mortensen and Wright 2002). However, since coin flips induce risk, with nonlinear utility, this mechanism is not bilaterally efficient, and is dominated by our bargaining game.
Figure 2: Game tree

$A_2$ his reservation offer, which means $A_2$ indifferent between accepting and rejecting, and he accepts. The indifference condition is $\gamma_2 - y_1 = (1 - \theta_1)(\gamma_2 - \gamma_1)$, or

$$y_1 = (1 - \theta_1)\gamma_1 + \theta_1\gamma_2.$$  \hspace{1cm} (1)

Payoffs are $V_1 = \gamma_1 + \theta_1(\gamma_2 - \gamma_1)$ and $V_2 = (1 - \theta_1)(\gamma_2 - \gamma_1)$. Of course, agents are not compelled to participate, but as long as $\gamma_2 > \gamma_1$ we have $V_1 \geq \gamma_1$ and $V_2 \geq 0$, so the payoffs beat the outside options. Equivalently, defining the total surplus as the sum of payoffs minus outside options, $S_{12} = V_1 - \gamma_1 + V_2 = \gamma_2 - \gamma_1$, the agents trade as long as $S_{12} \geq 0$.

For comparison, consider the standard generalization of Nash (1950) bargaining, where threat points are given by the outside options:

$$y_1 = \arg \max_y (y - \gamma_1)^{\theta_1} (\gamma_2 - y)^{1-\theta_1}$$  \hspace{1cm} (2)

It is easy to see that this is equivalent to (1). Hence, our game implements the Nash solution. It also implements Kalai’s (1977) proportional bargaining solution, which has become popular in search theory recently (see Lester, Postlewaite and Wright 2010 for references), since it is the same as Nash in this example, giving $A_1$ a a fraction $\theta_1$ of $S_{12}$. We call the probability $\theta_n$ the bargaining power of $A_n$ when playing with $A_{n+1}$, and below we allow $\theta_n$
to vary across agents.\footnote{Rubinstein and Wolinsky (1987) use a simple surplus-splitting rule, corresponding to $\theta_n = 1/2$ when we have linear utility (but see below, where we have nonlinear utility). They say “The reason that we abandon the strategic approach [in their 1985 paper] here is that it would greatly complicate the exposition without adding insights.” Binmore, Rubinstein and Wolinsky (1986), provide a strong argument in favor of the strategic approach – it makes the timing, threat points, etc. less ambiguous – and we find this clarifies rather than complicates the analysis. Also, whatever approach one takes, in models of middlemen it is clearly desirable to go beyond the symmetric case $\theta = 1/2$ (e.g., one reason athletes, musicians etc. employ agents may be that agents are better at bargaining).}

If it takes time and effort for $A_n$ to meet $A_{n+1}$, the value of search is

$$rV_n = \alpha_n(y_n - V_n) - c_n,$$

where $r$ is the rate of time preference, $\alpha_n$ a Poisson arrival rate, $c_n$ a flow search cost, and we distinguish $V_n$, the value of looking for a game, from $V_n$, the value of playing one. Since $c_n$ is only paid when $A_n$ has $x$ and is looking for $A_{n+1}$, not when $A_{n-1}$ is looking for $A_n$, we can also interpret it as an inventory carrying or storage cost. In any case, we have

$$V_n = \frac{\alpha_n y_n - c_n}{r + \alpha_n}, \quad (3)$$

and $A_n$ is willing to search for $A_{n+1}$ only if this exceeds his opportunity cost $\gamma_n$, or

$$(r + \alpha_n) \gamma_n \leq \alpha_n y_n - c_n. \quad (4)$$

Using (4) and (1), we see that search by $A_1$ is viable iff

$$c_1 + (r + \alpha_1) \gamma_1 \leq \alpha_1 [(1 - \theta_1) \gamma_1 + \theta_1 \gamma_2], \quad (5)$$

which says the expected payoff covers the direct search cost and opportunity cost, appropriately capitalized. Since (5) implies $S_{12} \geq 0$, the binding constraint for trade is the viability of search, not the outside options. We can let the search frictions vanish either by letting $r \to 0$ and $c_1 \to 0$, or letting $\alpha_n \to \infty$, since all that matters is $r/\alpha_n$ and $c_n/\alpha_n$. When the search frictions vanish (5) holds iff $S_{12} \geq 0$.

### 2.3 Example: $N = 3$

Now consider $N = 3$: an originator $A_1$; a potential end user $A_3$; and a potential middleman $A_2$ – or equivalently, for this exercise, $N = \infty$ with $\gamma_n = 0$ for all $n > 3$. Note that $A_3$ is
an end user here in the sense that if he acquires \( x \) he consumes it, since there is no one left to take it off his hands, but it is possible that \( A_1 \) prefers consuming \( x \) rather than searching for \( A_2 \), or \( A_2 \) prefers consuming it rather than searching for \( A_3 \). Different from some related models, here \( A_1 \) and \( A_3 \) cannot meet directly (this is relaxed in Section 3), so the only way to get \( x \) from \( A_1 \) to \( A_3 \) is via intermediary \( A_2 \).\(^5\) Given these assumptions, we ask which trades occur, and at what terms.

Working backwards, if \( A_2 \) with \( x \) meets \( A_3 \), as in the case \( N = 2 \), we have

\[
y_2 = (1 - \theta_2)\gamma_2 + \theta_2\gamma_3. \tag{6}
\]

Payoffs from this trade are \( V_2 = y_2 \) and \( V_3 = (1 - \theta_2)(\gamma_3 - \gamma_2) \), and the total surplus is \( S_{23} = \gamma_3 - \gamma_2 \), so they trade as long as \( \gamma_3 \geq \gamma_2 \). More stringently, for search by \( A_2 \) to be viable we require \( V_2 \geq \gamma_2 \), or

\[
c_2 + (r + \alpha_2)\gamma_2 \leq \alpha_2 [(1 - \theta_2)\gamma_2 + \theta_2\gamma_3]. \tag{7}
\]

If (7) holds then, upon acquiring \( x \), \( A_2 \) looks to trade it to \( A_3 \); if (7) fails then \( A_2 \) consumes \( x \) himself. In the latter case, \( A_3 \) is irrelevant, and effectively we are back to \( N = 2 \).

So, suppose (7) holds, and back up to where \( A_1 \) meets \( A_2 \). When \( A_1 \) makes the initial offer \( y_1 \), \( A_2 \)'s indifference condition is \(-y_1 + V_2 = (1 - \theta_1) (V_2 - \gamma_1) \). Inserting \( V_2 \), we have

\[
y_1 = (1 - \theta_1)\gamma_1 + \theta_1 \frac{\alpha_2 y_2 - c_2}{r + \alpha_2}. \tag{8}
\]

The payoffs are \( V_1 = y_1 \) and \( V_2 = (1 - \theta_1) \left( \frac{\alpha_2 y_2 - c_2}{r + \alpha_2} - \gamma_1 \right) \), and \( S_{12} \geq 0 \) iff

\[
c_2 \leq -\gamma_1 (r + \alpha_2) + \alpha_2 [\theta_2\gamma_3 + (1 - \theta_2)\gamma_2].
\]

More stringently, for search by \( A_1 \) to be viable we require \( V_1 \geq \gamma_1 \), or using (6) and (8)

\[
c_1 \leq -(r + \alpha_1)\gamma_1 + \alpha_1 \left\{ (1 - \theta_1)\gamma_1 + \theta_1 \frac{\alpha_2 [(1 - \theta_2)\gamma_2 + \theta_2\gamma_3] - c_2}{r + \alpha_2} \right\}.
\]

\(^5\) Thus, we cannot ask here why the market doesn’t cut out the middlemen — or, in more modern jargon, why there isn’t disintermediation. On that issue, practitioners put it this way: “why doesn’t every wholesaler just buy from the manufacturer and get the deepest discount? The answer is simple — not all wholesalers (or companies claiming to be wholesalers) can afford to purchase the minimum bulk-order requirements that a manufacture requires. Secondly, many manufactures only do business with companies that are established” (Ellis 2009). We do not model this explicitly, but it might be worth pursuing in future work.
Summarizing, after some algebra, for \( x \) to pass from \( A_1 \) to \( A_2 \) to \( A_3 \) we require

\[
(r + \alpha_2) c_1 + \alpha_1 \theta_1 c_2 \leq -(r + \alpha_1 \theta_1) (r + \alpha_2) \gamma_1 + \alpha_1 \theta_1 \alpha_2 (1 - \theta_2) \gamma_2 + \alpha_1 \theta_1 \alpha_2 \theta_2 \gamma_3 \tag{9}
\]

\[
c_2 \leq -(r + \alpha_2 \theta_2) \gamma_2 + \alpha_2 \theta_2 \gamma_3. \tag{10}
\]

If the inequality in (10) is reversed then \( A_2 \) consumes \( x \) if he gets it, and he gets it if

\[
c_1 \leq -(r + \alpha_1 \theta_1) \gamma_1 + \alpha_1 \theta_1 \gamma_2 \tag{11}
\]

since this makes search by \( A_1 \) viable when \( A_2 \) consumes \( x \). Note that reversing the inequality in (11) means \( A_1 \) will not search for \( A_2 \) given \( A_2 \) consumes \( x \), but search by \( A_1 \) may still be viable if \( A_2 \), instead of consuming \( x \), flips it to \( A_3 \). As a special case, when search frictions vanish \((r \to 0 \text{ and } c_n \to 0)\), search by \( A_2 \) is viable iff \( \gamma_2 \geq \gamma_2 \). Given \( A_2 \) searches, in this case, search by \( A_1 \) is viable iff \( \gamma_1 \leq (1 - \theta_2) \gamma_2 + \theta_2 \gamma_3 \). Alternatively, if \( A_2 \) does not search because \( \gamma_3 < \gamma_2 \), then search by \( A_1 \) is viable iff \( \gamma_2 \geq \gamma_1 \).

To develop some more economic intuition, return to the case there are search frictions, \( r > 0 \) and \( c_n > 0 \), but now suppose \( \gamma_2 = 0 \) so that \( A_2 \) is a pure middleman, with no desire to consume \( x \) himself. If \( A_2 \) obtains \( x \) he searches for \( A_3 \) if the expected payoff exceeds the pure search cost, \( \alpha_2 \theta_2 \gamma_3 \geq c_2 \). If this inequality is reversed \( A_2 \) does not want \( x \), and the market shuts down with \( A_1 \) consuming it. But if \( \alpha_2 \theta_2 \gamma_3 \geq c_2 \), so that \( A_2 \) would search for \( A_3 \), then \( A_1 \) searches for \( A_2 \) iff

\[
(r + \alpha_2) c_1 + \alpha_1 \theta_1 c_2 \leq -(r + \alpha_1 \theta_1) (r + \alpha_2) \gamma_1 + \alpha_1 \theta_1 \alpha_2 \gamma_2 \gamma_3.
\]

In words, the RHS is \( A_1 \)'s expected share of \( A_2 \)'s expected share of the end user’s payoff, net of his opportunity cost, while the LHS is \( A_1 \)'s direct search cost and the amount he has to compensate \( A_2 \) for \( A_2 \)'s search costs, all appropriately capitalized.

When \( \gamma_2 = 0 \), \( r \to 0 \) and \( c_n \to 0 \), \( A_1 \) searches for \( A_2 \) who searches for \( A_3 \) iff \( \theta_2 \gamma_3 \geq \gamma_1 \). The salient point is that \( \gamma_3 > \gamma_1 \) is not enough to get \( x \) from \( A_1 \) to \( A_3 \), even when \( \gamma_2 \), \( r \) and \( c_n \) are negligible. This is a typical holdup problem. Potential middleman \( A_2 \) knows that \( A_3 \) is willing to give anything up to \( \gamma_3 \) to get \( x \), and \( A_1 \) would be willing to let it go.
for as little as $\gamma_1$, which sounds like there is a deal to be done. But when $A_2$ meets $A_3$ he only gets $y_2 = \theta_2 \gamma_3$. He may protest he needs more than this just to cover his cost, which in this case is $y_1 = (1 - \theta_1) \gamma_1 + \theta_1 \theta_2 \gamma_3$. Being educated in economics, however, $A_3$ would (implicitly) counter that this cost is sunk and irrelevant for the negotiations. And he would be right. Hence $A_2$ will not intermediate a deal unless $y_2 \geq y_1$, or $\theta_2 \gamma_3 \geq \gamma_1$ in this case. This market failure is due to a lack of commitment. If $A_3$ and $A_2$ could sign a binding ex ante contract, the former could commit paying the latter at least enough to cover his cost, but such commitments are not allowed here.\textsuperscript{6}

More generally, without the assumption $r \to 0$, $c_n \to 0$ and $\gamma_2 = 0$, the substantive results are similar but richer (e.g., there is an additional aspect of holdup because the cost $c_2$ is also sunk when $A_2$ meets $A_3$). One conclusion to draw from this is that whether trade even gets off the ground, as well as the terms of trade and payoffs when it does, depend on not only fundamentals and bargaining power in any bilateral trading opportunity, but also on these parameters in downstream opportunities. Thus, gains from trade between $A_1$ and $A_2$ depend on $A_2$’s bargaining power when he later meets $A_3$. This is what we call bargaining with bargainers. To close the Section, we remark that, as when $N = 2$, our game implements the following generalized Nash or equivalently proportional bargaining outcomes for $n = 1, 2$:

$$
y_n = \arg \max_y (y - \gamma_n)^{\theta_n} \left(\gamma_{n+1} - y\right)^{1-\theta_n}
$$

(12)

### 3 Competing Risks

Here we present a modified formulation where, as in many of the other models of intermediated trade discussed in the Introduction, there are large numbers of agents of each type, and any agent can meet any other. Following most of the literature, assume there are exactly $N = 3$ types: originators $A_1$; potential middlemen $A_2$; and potential end users $A_3$. As in competing risk models in duration analysis, from whence we take the name, there is e.g. a

\textsuperscript{6}As in many search models, it seems a reasonable assumption that you cannot make a contract with someone before you make a contact with someone.
risk of $A_1$ meeting either $A_2$ or $A_3$. Also, to compare with some other middleman papers, and to reduce the algebra, we assume $\gamma_1 = \gamma_2 = 0$, but as we mention below, this does not affect the main results. Also, different from our baseline model, here after trade agents continue in, rather than exit from, the market.

With more types of meetings we need more notation. Let $\sigma_n$ be the probability $A_n$ searches, $n = 1, 2$; let $\mu = 1$ if $A_1$ trades with $A_2$ and $\mu = 0$ otherwise; and let $m$ be the probability that $A_2$ has $x$ in steady state. To keep track of $A_2$’s inventory, write $V_{2i}$ where $i \in \{0, 1\}$. Also, let $\alpha_{nn'}$ be the rate at which $A_n$ meets $A_{n'}$, and $y_{nn'}$ the equilibrium outcome in their bargaining game. The value functions satisfy:

$$
\begin{align*}
\rho V_1 &= \sigma_1 [\alpha_{12}(1-m)\mu y_{12} + \alpha_{13}y_{13} - c_1] \\
\rho V_3 &= \sigma_1\alpha_{31}(\gamma_3 - y_{13}) + \sigma_2\alpha_{32}(\gamma_3 - y_{23}) \\
\rho V_{20} &= \sigma_1\alpha_{21}\mu(V_{21} - V_{20} - y_{12}) \\
\rho V_{21} &= \sigma_2 [\alpha_{23}(y_{23} + V_{20} - V_{21}) - c_2]
\end{align*}
$$

The indifference conditions in bargaining are:

$$
\begin{align*}
\gamma_3 - y_{13} &= (1 - \theta_{13})\gamma_3 \\
\gamma_3 - y_{23} &= (1 - \theta_{23})(\gamma_3 + V_{20} - V_{21}) \\
V_{21} - V_{20} - y_{12} &= (1 - \theta_{12})(V_{21} - V_{20})
\end{align*}
$$

Let the population be given by the vector $(\pi_1, \pi_2, \pi_3)$, where $\pi_n$ is the measure of type $A_n$ agents, with $\pi_1 + \pi_2 + \pi_3 = 1$. Since the number of meetings between $n$ and $n'$ must be the same as the number of meetings between $n'$ and $n$, there are three identities: $\pi_1\alpha_{12} = \pi_2\alpha_{21}$, $\pi_2\alpha_{23} = \pi_3\alpha_{32}$, and $\pi_3\alpha_{31} = \pi_1\alpha_{13}$. The steady state value of $m$ is given by the equating the inflow and outflow of $A_2$ inventories, $m\alpha_{23} = (1-m)\alpha_{21}\mu$, which solves for

$$
m = \frac{\pi_1\alpha_{12}\mu}{\pi_2\alpha_{23} + \pi_1\alpha_{12}\mu}.
$$

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7Notice that, due to “recycling” there is no capital gain (change in continuation value) from trading for $A_1$ and $A_3$ – e.g., the surplus for $A_1$ from trading with $A_2$ is the transfer $y_{12}$ plus continuation value $V_1$, minus the outside option $V_1$, which nets to $y_{12}$.
A steady state equilibrium is described by the trade decision $\mu$ between $A_1$ and $A_3$, the
search decision $\sigma_n$ for $n = 1, 2$, a solution $m$ to (20), and terms of trade and value functions
solving (13)-(19).

Depending on parameters, different types of equilibrium can arise. In type 0 equilibrium, $A_1$
does not even search and the market shuts down; in type 1 equilibrium, $A_1$ searches and trades $x$
to $A_3$ but not $A_2$; and in type 2 equilibrium, $A_1$ searches and trades $x$ to $A_3$ or $A_2$, whoever
he meets first. It is automatic that $A_1$ and $A_2$ always trade with $A_3$ if they meet, and
that $A_2$ must search for $A_3$ if he trades with $A_1$, which keeps manageable the number of
cases. However, even if $\sigma_1 = 0$, we must specify as part of the equilibrium whether $A_1$ and
$A_2$ would trade, if they were to meet, off the equilibrium path. To construct the equilibrium
set we must check two conditions. The first is the viability condition for $A_1$ to search: $\sigma_1 = 1$
if $\nabla_1 > 0$ and $\sigma_1 = 0$ if $\nabla_1 < 0$. The second is the condition that determines whether $A_1$
and $A_2$ trade: $\mu = 1$ if $S_{12} > 0$ and $\mu = 0$ if $S_{12} < 0$, where $S_{12} = \nabla_{21} - \nabla_{20}$. The viability
for $A_2$ to search is: $\sigma_2 = 1$ if $\nabla_{21} - \nabla_{20} > \gamma_2$, but since $\gamma_2 = 0$, this condition is the same as
the condition for $\mu = 1, S_{12} > 0$.

Here we focus on economics, relegating algebra to the Appendix. We begin with type 0
equilibria, in which $A_1$ does not search and the market shuts down. There are two versions
of this equilibrium: one where $A_1$ and $A_2$ would trade if they met, and one where they would
not. For latter the case, we need $\alpha_{23}\theta_{23}\gamma_3 \leq c_2$ to guarantee $\mu = 0$, and $\alpha_{13}\theta_{13}\gamma_3 \leq c_1$
to guarantee $\sigma_1 = 0$, defining a region by the black lines in the upper right of Figure 3 (ignore
the red lines for now). For the former case, we need $\alpha_{23}\theta_{23}\gamma_3 \geq c_2$ to guarantee $\mu = 1$ and

$$0 \geq (\pi_2\alpha_{23} + \pi_1\alpha_{12}) [\pi_2r + \pi_2\alpha_{23}\theta_{23} + \pi_1\alpha_{12}(1 - \theta_{12})] (\alpha_{13}\theta_{13}\gamma_3 - c_1)$$
$$+ \alpha_{12}\alpha_{23}\pi_2^2\theta_{12} (\alpha_{23}\theta_{23}\gamma_3 - c_2)$$

(21)
to guarantee $\sigma_1 = 1$, giving a region in the upper left of Figure 3. It is no surprise that
search or storage costs above certain thresholds eliminate trade; the point of the analysis is
to determine exactly how these thresholds depend on the search and bargaining parameters,
the $\alpha$’s and $\theta$’s, as well as $x$’s fundamental value $\gamma_3$.  

12
Figure 3: Equilibria with competing risks

Type 1 equilibrium with $\mu = 0$ and $\sigma_1 = 1$ exists iff $\alpha_{23}\theta_{23}\gamma_3 \leq c_2$ and $\alpha_{13}\theta_{13}\gamma_3 \geq c_1$, in the lower right of Figure 3. In this case, there is trade, but $A_2$ plays no role because they have high search costs ($c_2/\gamma_3$ is big), they meet $A_3$ infrequently ($\alpha_{23}$ is small), or they do not bargain very well ($\theta_{23}$ is low). Alternatively, type 2 equilibrium where middlemen play a role exists iff $\alpha_{23}\theta_{23}\gamma_3 \geq c_2$ and

$$0 \leq (\pi_2\alpha_{23} + \pi_1\alpha_{12}) [\pi_2r + \pi_2\alpha_{23}\theta_{23} + \pi_1\alpha_{12}(1 - \theta_{12})] (\alpha_{13}\theta_{13}\gamma_3 - c_1)$$

$$+ \alpha_{12}\alpha_{23}\pi_2^2\theta_{12} (\alpha_{23}\theta_{23}\gamma_3 - c_2),$$

in the lower left of Figure 3. For intermediaries to have a role, $A_2$ must have a comparative advantage over $A_1$ in terms of storage, search or bargaining. Perhaps the most interesting case is this: starting in the upper right of Figure 3, in type 0 equilibrium, suppose we lower $c_2$, until we cross the diagonal line. We are then in type 2 equilibrium, with active trade. This shows how an active market may require intermediation.\(^8\)

In type 2 equilibrium, one can check that the profit margin of a middleman is

$$y_{23} - y_{12} \propto \theta_{23} (\gamma_3r + c_2) + (1 - \theta_{12})[\theta_{23}\gamma_3 (\alpha_{23} + \alpha_{21}) - c_2] > 0,$$

\(^8\)Recall that these results are for $\gamma_1 = \gamma_2 = 0$. Relaxing this, the results are similar except the regions active trade shrink, as shown by the red lines in Figure 3.
which is increasing in $\alpha_{23}$.\footnote{This follows from calculating $y_{23} = \frac{[r + \alpha_{23} + \sigma_{1}\alpha_{21}(1 - \theta_{12})] \theta_{23} \gamma_{3} - (1 - \theta_{23}) c_{2}}{r + \sigma_{1}\alpha_{21}\mu(1 - \theta_{12})}$ and $y_{12} = \frac{\theta_{12}(\alpha_{23}\theta_{23}\gamma_{3} - c_{2})}{r + \sigma_{1}\alpha_{21}\mu(1 - \theta_{12})}$, and noting that the second term on the LHS of (23) is positive by the equilibrium condition for $\mu = 1$. One can also show $y_{23} - y_{13} \propto \gamma_{3} \left[ r + \alpha_{21}(1 - \theta_{12}) \right] (\theta_{23} - \theta_{13}) + (\gamma_{3}\alpha_{23}\theta_{23} - c_{2}) - \theta_{23}(\gamma_{3}\alpha_{23}\theta_{13} - c_{2})$ $y_{13} - y_{12} \propto \gamma_{3} \left[ \theta_{13} \left[ r + \alpha_{21}(1 - \theta_{12}) \right] + \theta_{23}\alpha_{23} (\theta_{13} - \theta_{12}) \right] + \theta_{12} c_{2}].$These may be ambiguous, in general, but $\theta_{23} \geq \theta_{13}$ implies $y_{23} > y_{13}$, and $\theta_{13} \geq \theta_{12}$ implies $y_{13} > y_{12}$.} Also, note that in the limit as $r \to 0$ the market shuts down. The reason is that when either $A_{1}$ or $A_{2}$ trade with $A_{3}$, if $r \approx 0$ they only get their outside option, which is not enough for them to search in the first place – again, a holdup problem. In any case, having solved this version, we now return to our baseline model where $A_{n}$ can only trade $x$ to $A_{n+1}$, and investigate longer intermediation chains.

4 Multiple Middlemen

When $A_{n+1}$ receives offer $y_{n}$ from $A_{n}$, given that he plans to trade $x$ again, his indifference condition is

$$\frac{\alpha_{n+1}y_{n+1} - c_{n+1}}{r + \alpha_{n+1}} - y_{n} = (1 - \theta_{n}) \left( -\gamma_{n} + \frac{\alpha_{n+1}y_{n+1} - c_{n+1}}{r + \alpha_{n+1}} \right).$$

(24)

Solving for $y_{n}$, we have

$$y_{n} = (1 - \theta_{n}) \gamma_{n} + \theta_{n} \frac{\alpha_{n+1}y_{n+1} - c_{n+1}}{r + \alpha_{n+1}} \equiv \rho_{n}(y_{n+1}).$$

(25)

We interpret $y_{n} = \rho_{n}(y_{n+1})$ as a best response condition for $A_{n}$: it gives $y_{n}$, his initial offer strategy when he meets $A_{n+1}$, as a function of others’ strategies, as summarized by $y_{n+1}$.

Given $A_{n}$ gets $x$, the conditions for $A_{n}$ and $A_{n+1}$ to trade in equilibrium are:

1. $V_{n} = \frac{\alpha_{n}y_{n} - c_{n}}{r + \alpha_{n}} \geq \gamma_{n}$, so $A_{n}$ wants to search;

2. $V_{n,n+1} = y_{n} \geq \gamma_{n}$, so $A_{n}$ wants to trade;

3. $V_{n+1,n} = -y_{n} + V_{n+1} \geq 0$, so $A_{n+1}$ wants to trade.
The second condition is not binding given the first, while the third reduces to\(^{10}\)

\[
y_{n+1} \geq \frac{\gamma_n(r + \alpha_{n+1}) + c_{n+1}}{\alpha_{n+1}}. \tag{26}
\]

Now, to investigate how long intermediation chains can be, consider a quasi-stationary environment, where \(\alpha_n, c_n\) and \(\theta_n\) are the same for all \(n\), while \(\gamma_n = \gamma\) for \(n \leq N\), \(\gamma_{N+1} = \hat{\gamma} > \gamma\), and \(\gamma_n = 0\) for \(n > N + 1\). We call \(A_{N+1}\) the end user because, if he gets \(x\), he consumes it, since \(A_n\) does not value it for \(n > N + 1\).\(^{11}\)

Now (25) can be written \(y_n = \rho(y_{n+1})\) for \(n < N\), where \(\rho(y) = (1-\theta)\gamma + \theta(\alpha y - c) / (r + \alpha)\), while \(y_N = (1 - \theta)\gamma + \theta \hat{\gamma}\). Clearly, \(\rho(y)\) has a unique fixed point,

\[
y^* = \frac{(1 - \theta)\gamma (r + \alpha) - \theta c}{r + \alpha (1 - \theta)},
\]

where we assume \(c < (1 - \theta)\gamma (r + \alpha) / \theta\) so \(y^* > 0\). It is easy to see \(y_N > \gamma > y^* > \rho(0)\), as in Figure 4. The way to read in Figure 4 is: given \(A_{n+1}\) correctly anticipates getting \(y_{n+1}\) from \(A_{n+2}\), he ends up giving \(y_n\) to \(A_n\). Now, to find equilibrium, begin by working backwards: set \(y_N = (1 - \theta)\gamma + \theta \hat{\gamma}\) and iterate on \(y_n = \rho(y_{n+1})\) to construct a sequence \(\{y_n^*\}\), where it is obvious that \(y_n \to y^*\) as \(n \to -\infty\). Then, since we are actually interested in what happens as \(n\) increases, moving forward in real time, pick a point in this sequence and reiterate forward. This generates a candidate equilibrium.

The sequence \(\{y_n^*\}\) is only a candidate equilibrium because we still have to check if search is viable for all agents in the chain. Clearly we cannot have arbitrarily long chains, since, going backwards in time, this would involve starting arbitrarily close to \(y^* < \gamma\), and if \(y_n < \gamma\) the holder of \(x\) would rather consume it than search. Consider e.g. starting with \(A_{N-2}\) holding \(x\). If there is to be trade, \(A_{N-2}\) searches for and trades \(x\) to \(A_{N-1}\), who then searches for and trades \(x\) to \(A_N\), who finally \(A_N\) searches for and trades \(x\) to the end user. To see if this is viable, solve for \(y_{N-2} = \rho^2(y_N) = \rho^2[(1 - \theta)\gamma + \theta \hat{\gamma}]\) and check

\[
V_{N-2} = \frac{\alpha \rho^2(y_N) - c}{r + \alpha} \geq \gamma.
\]

\(^{10}\)By inserting \(y_{n+1} = \rho_n^{-1}(y_n)\), one can see (26) holds if \(y_n \geq \gamma_n\).

\(^{11}\)As above, we are asserting here that \(x\) cannot pass from \(A + 1\) to \(A_n\) for \(n > N + 1\), given \(A_n\) derives no utility from it. The proof is a special case of the no-bubble result given below.
Hence, if \( c \) is not too big, we can support trade with two middlemen between the originator and end user. For any \( c > 0 \) we cannot support trade with an arbitrary number of middlemen, since \( y_{N-j} \rightarrow y^* < \gamma \), so there is a maximum viable chain. However, if \( \gamma = c = 0 \), then \( y^* = 0 \) and there are arbitrarily long chains, starting near 0 and growing to \( y_N \).

As an aside, let \( T_n \) be the random date when \( A_n \) trades with \( A_{n+1} \). There are two interesting properties of the trading process, one economic and one statistical. First, notice from Figure 4 that \( \Delta y \) increases over time: \( y_N - y_{N-j} > y_{N-j} - y_{N-j-1} \). Thus, as \( x \) gets closer to the end user, not only \( y \) but the increments in \( y \) go up. Second, since the underlying arrival rates are Poisson, as is well known, the interarrival times \( T_n - T_{n-1} \) are distributed exponentially. This means a high probability of short and a low probability of long interarrival times, and so typical realizations of the stochastic process have exchange \emph{bunched}, with many trades occurring in short periods, separated by long periods of inactivity. This gives an appearance of market frenzies interspersed by lulls, even though Poisson arrivals are memoryless, so there are no frenzies or lulls in any meaningful economic sense.\footnote{This is explained by Çinlar (1975, 79-80), e.g., as follows: “the interarrival times \( T_1, T_2 - T_1, T_3 - T_2, \ldots \) are independent and identically distributed random variables, with the \ldots exponential distribution \ldots Note that this density is monotone decreasing. As a result, an interarrival time is more likely to have a length in \([0, s]\) than in a length in \([t, t + s]\) for any \( t \). Thus, a Poisson process has more short intervals than long ones.}

Figure 4: Path of \( y_n \)
illustrates these properties: the statistical property, that periods of rapid trading are followed by lulls; and the economic property, that $y$ grows at an increasing rate as we get closer to a final $y_N = (1 - \theta)\gamma + \theta\bar{\gamma}$ that is pinned down by fundamentals.

5 Discussion

Since the Introduction, we have refrained from using the words buyer, seller and price. This is intentional, as we want to raise some issues associated with such usage. First, we contend that in the analog to our model found in much of the search literature, in our notation, $x$ represents a good and $y$ money, and with this interpretation $y$ is the price, the agent who trades $x$ for $y$ is a seller, and the one who trades $y$ for $x$ is a buyer. Noteworthy papers that we interpret in this way include, in addition those on middlemen discussed earlier, Diamond (1971,1987), Butters (1977), Burdett-Judd (1983) and Rubinstein-Wolinsky (1985), all of which have an indivisible object corresponding to $x$ called a consumption good (or in some applications a production good like labor), and a divisible object $y$ interpreted as the price (or wage). Of course, although they may think of $y$ as dollars, and very often say so explicitly,

Therefore, a plot of the time series of arrivals on a line looks, to the naive eye, as if the arrivals occur in clusters.” But since process is memoryless, actually, “knowing that an interarrival time has already lasted $t$ units does not alter the probability of its lasting another $s$ units.”
these models do not literally have money – what they have is transferrable utility.13

Identifying money with (more accurately, confusing money with) transferrable utility is standard practice by even the best economic theorists. Consider Binmore (1992): “Sometimes it is assumed that contracts can be written that specify that some utils are to be transferred from one player to another ... Alert readers will be suspicious about such transfers ... Utils are not real objects and so cannot really be transferred; only physical commodities can actually be exchanged. Transferable utility therefore only makes proper sense in special cases. The leading case is that in which both players are risk-neutral and their von Neumann and Morgenstern utility scales have been chosen so that their utility from a sum of money $x$ is simply $U(x) = x$. Transferring one util from one player to another is then just the same as transferring one dollar.” Unfortunately, it ain’t neccessarily so – and this is about much more than an abhorrence for the practice of putting money in the utility function.

In fact, in models that try to be serious about money, it is not at all trivial to transfer dollars bilaterally across agents, because they tend to run out; and in any case, payoffs are usually not linear in dollars. Examples using search-based monetary theory include Shi (1995), Trejos-Wright (1995), Kocherlakota (1998), and Wallace (2001). The key point to emphasize here is that all those models take a diametric position to the above-mentioned applications outside of monetary economics: they assume $γ$ is a consumption good and $x$ is

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13Submitted in evidence, from middlemen papers, consider the following. Rubinstein and Wolinsky (1987, p.582) describe payoffs as the “consumption values (in monetary terms) of a unit.” Biglaiser (1993, p.213) starts with “Each buyer is endowed with money.” Yavas (1994) describes a standard model of middleman by “The sellers and the middlemen value the good (in monetary terms) at zero, while the buyers value the good at one.” Yavas (1992) is more careful, saying “In order to avoid the additional questions associated with having money in the economy, this endowment has not been labeled as money.” Johri and Leach (2002) are also careful to say “units of a divisible numeraire good are exchanged for units of an indivisible heterogeneous good,” although they have no problem assuming payoffs are linear in the former. Even those less cavalier about money are quick to decide who is the buyer and seller, and what is the price.

Considering search models outside the middleman literature, Butters (1977, p.466) says “A single homogeneous good is being traded for money.” Burdett and Judd (1983, pp.955,960) say consumers search “to lower the expected costs of acquiring, a desired commodity, balancing the monetary cost of search against its monetary benefit,” and firms want to “make more money.” Diamond (1971,1987) does not mention money explicitly, although he does refer, in our notation, to $γ$ as the price, and buyers are those with payoff $γ − y$ while sellers are those with payoff $p$. In many places in their book, Osborne and Rubinstein (1990) describe models as “A single indivisible good is traded for some quantity of a divisible good (‘money”).” Gale (1987, p.20) more accurately says “A single, indivisible commodity is traded. Buyers and sellers have transferable utility.” One can go on, but we think this makes the point.
money. The most apparent difference is that, under one interpretation, money is divisible and consumption goods are indivisible, while under the other, money is indivisible and goods indivisible. Superficially, this favors the first interpretation, of the nonmonetary economists, since divisibility is one of the properties (along with storability, portability and recognizability) commonly associated with currency. On reflection, however, we do think this detail should be given much weight. 14

A better discriminating criterion stems from the functional definitions of money: it is a unit of account, a store of value, and a medium of exchange. The unit of account function – which means that American prices tend to be quoted in dollars, European prices in euros, etc. – seems relatively uninteresting, because for anything of consequence it cannot matter whether we measure prices in dollars or euros any more than whether we measure distance in feet or meters. Moving to the store of value function, it seems clear that in the baseline search models it is actually $x$ and not $y$ that constitutes a store of value: $x$ is a durable good that, when acquired by $A_n$ some date, enables him to enjoy a payoff $y_n$ at some future date. The more natural interpretation of $y$ is that it is a perishable good, or a service rather than a good, that is not carried across time but produced for immediate consumption. It seems hard to imagine a service as money, while $x$ is rather obviously a storable asset in the model.

Indeed, $x$ satisfies the standard definition of a medium of exchange: an object that is accepted in trade not to be consumed or used in production by everyone that accepts it, but is instead traded again later. Now, $x$ happens to be commodity money in the above examples, since some end user ultimately does consume it for a direct payoff – as opposed to fiat money, which does not generate a direct payoff for anyone but is still used as a medium of exchange. We have more to say about these issues below; for now, we emphasize that $y$ is evidently not a medium of exchange in the above specifications: it is accepted by everyone

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14 One reason is that earlier contributions to the search-based monetary literature actually have both $x$ and $y$ indivisible, while more recent ones have them both divisible (see the surveys cited in the Introduction). Another is that, as a matter of historical fact, objects used as money including coins were often less-than-perfectly divisible, with significant economic consequences (e.g., Sargent and Velde 2000).
for its direct payoff, and never to be traded again later. Moreover, one can argue that it is exactly the classic double-coincidence problem that makes $x$ useful here: when $A_n$ wants $y$ from $A_{n+1}$, he has nothing to offer in trade absent asset $x$. Indeed, $x$ can facilitate this exchange even if $A_{n+1}$ does not especially enjoy $x$, as in the extreme case of $\gamma_{n+1} = 0$, or, for that matter, $\gamma_{n+1} < 0$, since $A_{n+1}$ is only going to trade it again. So, based on all of this, in terms of calling either $x$ or $y$ money, it looks like the monetary search theorists may have got this one right.\(^{15}\)

Does it matter? While at some level one might say the issue is purely semantic (as if that were reason to not be interested), we think it actually may matter for how one uses the models. For instance, it determines who we call the buyer or seller and what we mean by the price. To make this point, we first argue that in nonmonetary exchange – say, when $A$ gives $B$ apples for bananas – it is \textit{not} meaningful to call either agent a buyer or seller. Of course, one can them whatever one likes, but then the labels buyer and seller convey nothing more than calling them $A$ and $B$. However, when $A$ gives $B$ apples for genuine money, for dollars, we think everyone should agree that we should call $A$ the seller and $B$ the buyer (it is less clear what to call them when, say, $A$ gives $B$ Euros for dollars, but that is beside the point). Again, one can label objects anything one likes, and be on firm ground logically, if not aesthetically. But why would anyone want to reverse the labels in, say, the Mortensen-Pissarides (1994) labor-market model for the agents we usually call workers and firms? One could prove the same theorems, but it would make a difference when ones considers applied questions, such as should we tax/subsidize search by workers or firms?

Using the interpretation in previous papers on middlemen, where $y$ is money and $x$ is a good, when $A_n$ and $A_{n+1}$ trade the former is the seller and the latter is the buyer. Using the interpretation in the monetary papers, where $x$ is money and $y$ is a good, $A_n$ is the buyer and $A_{n+1}$ is the seller. If one agrees that it makes a substantive difference who we call

\(^{15}\)One can also argue that $x$ plays much the same role here that money plays in non-search models, such as overlapping-generations models (Wallace 1980). If this is not clear, imagine $A_n$ being born at some random date $n$ and living two periods of random length: in the first he gives $y_{n-1}$ to the previous generation for $x$, which he later trades to the next generation for $y_n$.\]
workers and firms in the labor market, it similarly makes a difference in the goods markets, especially when we ask substantive questions, such as, should we tax/subsidize shoppers or retailers? Moreover, the two interpretations give opposite predictions for price behavior. If \( y \) is money and \( x \) is a good, normalizing the size of the indivisible \( x \) to unity, \( y \) is the dollar price. But if \( y \) is the good and \( x \) is money, and normalizing \( x \) the same way, \( 1/y \) is the price, since now a normalized unit of money buys \( y \) units of the good. To see how this matters, look back at Figure 5. Using the first interpretation, theory predicts prices are rising over time as more and more money \( y \) is required to buy the same amount of good \( x \). Using the second interpretation, prices are falling as more and more good \( y \) can be had for the same amount of money \( x \). For empirical work, at least, one has to make a choice here.

We do not mean to argue for a “correct” interpretation, since this depends on the issues at hand, although for us the above discussion about \( x \) as a store of value and medium of exchange seems to favor of calling \( x \) money and \( 1/y \) the price. However, we volunteer below a quick fix for users of these models, and search theory more generally, who seem to want \( x \) to be a good and \( y \) the dollar price. For this we take seriously the idea that it only makes sense to talk about buyers and sellers in monetary exchange. Second, taking monetary exchange seriously, as some papers do, typically does not lead to payoffs being linear in dollars, or to agents always being able to transfer utility one-for-one to others. Is there a simple, natural and reasonable way to embed models like Rubinstein-Wolinsky into a monetary economy? And can we make simple and reasonable assumptions that make payoffs linear in dollars, effectively making utility transferable? The answer depends on what one accepts as natural and reasonable, but for sure what we offer is simple.

6 A Monetary Model

The idea is to use the framework in Lagos-Wright. As illustrated in Figure ??, \( A_2 \) sells his labor services in CM (in hours) \( h = y/w \) for \( y \) dollars (normalizing \( w = 1 \)). Then \( A_2 \) trades with \( A_1 \), pays him \( y \). Then \( A_1 \) spends \( y \) dollars in CM, consumes \( x/p \) units of goods, and
gives $x$ to $A_2$ (normalizing $p = 1$). A caveat is that unlike Lagos-Wright where CM and DM opens alternately, CM always open here (like the proverbial 7-Eleven). Notice that there is no liquidity (cash) constraint – $A_{N+1}$ can go to the CM to get $y$ after he meets and negotiates with $A_N$. Would it be interesting to force $A_{N+1}$ to hold the money in advance? Absolutely. But we will not do that here. However, now that we opened the door to discuss monetary issues, the economics gets a lot more interesting. Example: can we have $\gamma_n = 0$ for $\forall n$ – so that $x$ is intrinsically useless (a fiat object) – but $x$ is still valued in equilibrium? This would be a pure bubble: a fundamental value of $x$ is 0 but its price is positive. Can this happen?

TBC

7 Bubbles

In the above analysis we could get chains of trade, and with $c_n = 0$, these chains can be arbitrarily long, but the equilibrium is tied down by the fact that there is a determinate end user who, if he gets $x$, definitely consumes it. Now we consider the possibility that no one ever consumes $x$ – it gets traded forever. The indifference condition and best response for $y_n$ to $y_{n+1}$ are again given by (24) and (25). As in the analysis of the quasi-stationary environment, we assume here that $\alpha_n = \alpha$, $c_n = c$ and $\theta_n = \theta$ for all $n$, but now we also assume $\gamma_n = \gamma$ for all $n$. In this stationary environment, the best response function becomes

$$y_n = (1 - \theta)\gamma + \theta \frac{\alpha y_{n+1} - c}{r + \alpha} = \rho(y_{n+1}),$$

as shown in Figure 6. The viability condition for $A_n$ to search is $\gamma \leq (\alpha y_{n+1} - c) / (r + \alpha)$.

As compared to the quasi-stationary model with an obvious end user, which was agent $N+1$, since $\gamma_n = 0$ for $n > N+1$, here we have no terminal condition like $y_n = \theta \gamma + (1 - \theta)\gamma$. Since $y_n = \rho(y_{n+1})$ implies $\partial y_{n+1} / \partial y_n = (r + \alpha)/\alpha \theta$, it is clear that there is only one solution to the difference equation that remains nonnegative and bounded: $y_n = y^*$ for all $n$. Any path that starts at $y > y^*$ looks like a bubble: agent $A_n$ is willing to search for $A_{n+1}$ because he correctly expects a high $y_n$, $A_{n+1}$ is willing to give a high $y_n$ because he expects an even higher
$y_{n+1}$, and so on. But clearly such explosive bubbles are not consistent with equilibria as long as we make the standard assumption that there is some upper bound $\mathcal{F}$ (e.g., $\mathcal{F}$ could be the total output of known universe). One cannot rationally believe that $y_n$ will grow beyond $\bar{y}$, and hence this bubble-like path for $y_n$ cannot be an equilibrium. The only equilibrium is, therefore, $y_n = y^*$ for all $n$, and since $\gamma > y^*$, we have $\nabla_n = (\alpha y^* - c) / (r + \alpha) < \gamma$. Hence, search is not viable, naturally, since there can be no gains to trade (let alone search) when all agents get a common payoff $\gamma$ from consuming $x$. Thus, whoever starts with $x$ eats it.

If $A_n$ tried to trade $x$ to $A_{n+1}$ (off the equilibrium path), he could get exactly $y_n = \gamma$ for it. We call $y_n = \gamma$ the fundamental price of $x$ since it is exactly what $x$ is worth in terms of intrinsic properties – it yields utility $\gamma$ when consumed. Note that we are in this case interpreting $y$ as money loosely speaking – and transferable utility strictly speaking – when we call $y_n$ the price; as discussed above, this is a possible but not the only possible interpretation. Going with this interpretation, we say that whenever $y_n > \gamma$ we say the price has a bubble component, using as our definition of a bubble for this analysis a price above the fundamental price. What we have shown above is that, for this specification, as in most standard economic models, bubbles are not possible. In the quasi-stationary environment above, where $\gamma_n = \gamma$ for $n \leq N$, $\gamma_{N+1} = \hat{\gamma} > \gamma$, and $\gamma_n = 0$ for $n > N + 1$, paths for $y_n$ like the one shown in 5 resemble an explosive bubble in the sense that the price $y_n$ increases by larger and larger increments with each trade, but it is not a bubble by our definition since $y_n$ is no more than the fundamental price $\hat{\gamma}$, and indeed $A_{N+1}$ gets it for less than that as long as he has some bargaining power ($\theta_n < 1$). Now, normally we cannot have prices below fundamental values – it would violate arbitrage – but here arbitrage is limited by the thinness of the market, or by search and bargaining.

What we have established for the base-line specification is a classic no-bubble result.\footnote{This no-bubble result confirms our assertion for the quasi-stationary environment that if $A_{N+1}$ gets $x$ he will consume it, by ruling out the possibility that $x$ continues to be traded when $\gamma_n = \gamma$ for all $n$. The quasi-stationary environment does not quite fit into the specification here since $\gamma_n$ is only constant for $n > N + 1$, but this is obviously not important; it is also a special case in that $\gamma_n = 0$ for $n > N + 1$, but the analysis here covers that case.}
To investigate its robustness, let us make utility of consuming $y$ more generally $U(y)$, with $U(0) = 0$, $U' > 0$ and $U'' < 0$, but still keep the cost of producing $y$ equal to $y$ (this is wlog given we can choose units of $y$ appropriately). Denote by $y_\gamma = U^{-1}(\gamma)$ the cost to $A_{n+1}$ of covering $A_n$’s outside option $\gamma$. In general the indifference condition in bargaining becomes

$$-y_n + \frac{\alpha_{n+1}U(y_{n+1}) - c_{n+1}}{r + \alpha_{n+1}} = (1 - \theta_n) \left[ y_\gamma + \frac{\alpha_{n+1}U(y_{n+1}) - c_{n+1}}{r + \alpha_{n+1}} \right],$$

which gives the best response condition

$$y_n = (1 - \theta_n)y_\gamma + \theta_n \frac{\alpha_{n+1}U(y_{n+1}) - c_{n+1}}{r + \alpha_{n+1}} = \rho_n(y_{n+1}).$$

In a stationary environment,

$$y_n = (1 - \theta)y_\gamma + \theta \frac{\alpha U(y_{n+1}) - c}{r + \alpha} = \rho_n(y_{n+1}).$$

(27)

Figure 7 shows a case where $y^* > y_\gamma$, or equivalently, $U(y^*) > \gamma$, which we could not get in the case $U(y) = y$. Since $U(y^*) > \gamma$ is necessary, but not sufficient, for satisfying the search viability condition $\gamma \leq [rU(y^*) - c] / (r + \alpha)$, we at least have a chance here which we did not have with $U(y) = y$. We will show below that the search viability condition holds when
\( \gamma \) is not too big. Also, as drawn, the Figure shows a unique positive solution to \( y^* = \rho(y^*) \), which is true iff \( 0 < \rho(0) = (1 - \theta)y_\gamma - \theta c / (r + \alpha). \)

Consider an example with \( U(y) = \sqrt{y} \), which means \( y_\gamma = \gamma^2 \). Setting \( c = 0 \), for now, for simplicity, we have

\[
\rho(y) = (1 - \theta)\gamma^2 + \frac{\theta\alpha}{r + \alpha}\sqrt{y}.
\]

To find a steady state, rewrite \( y = \rho(y) \) in terms of \( U = \sqrt{y} \):

\[
U^2 - \frac{\theta\alpha}{r + \alpha}U - (1 - \theta)\gamma^2 = 0.
\]

This is a quadratic in \( U \), and the correct (i.e., positive) solution is the higher root

\[
U = \frac{1}{2} \left\{ \frac{\theta\alpha}{r + \alpha} + \sqrt{\left(\frac{\theta\alpha}{r + \alpha}\right)^2 + 4(1 - \theta)\gamma^2} \right\}.
\]

The search viability condition \( \gamma \leq \frac{\alpha}{r + \alpha}U \) then becomes

\[
2\gamma \frac{r + \alpha}{\alpha} - \frac{\theta\alpha}{r + \alpha} \leq \sqrt{\left(\frac{\theta\alpha}{r + \alpha}\right)^2 + 4(1 - \theta)\gamma^2}.
\]

\(^{17}\)Given \( \gamma > 0 \), this obviously holds if \( c \) or \( \theta \) is small, or if \( \alpha \) is big.
Squaring and simplifying, this reduces to

\[ \gamma \leq \frac{\theta \alpha^2}{r^2 + 2r\alpha + \theta \alpha^2} \equiv \bar{\gamma}. \tag{28} \]

Notice \( \bar{\gamma} > 0 \), so search is viable for some \( \gamma > 0 \); although since \( \bar{\gamma} < 1 \) it is not viable for \( \gamma \geq 1 \).

Summarizing, we have proved for this example with \( \gamma < \bar{\gamma} \) and \( c = 0 \), there is an equilibrium such that, for all \( n, A_n \) searches and trades \( x \) it to \( A_{n+1} \) for \( y_n = y^* \). Appealing to continuity, similar results also hold with \( c > 0 \), as long as \( c \) is not too big. In this equilibrium no one ever consumes \( x \), and it is an asset that circulates forever – as a medium of exchange – even though it does have consumption value. For search to be viable we require

\[ [\alpha U(y^*) - c] / (r + \alpha) \geq \gamma, \text{ and } a \text{ fortiori } y^* > y_\gamma. \]

This says that the price of \( x \) in terms of \( y \) is above the fundamental price: it has a bubble component, \( y^* - y_\gamma > 0 \). This may be surprising to some, although probably not to anyone versed in monetary economics, where it is commonly understood that objects can be valued for their liquidity provision over and above their value as consumption goods, the way, say, cigarettes are valued in some prisons, or in other historical cases, where they not only could be consumption goods but media of exchange and stores of value.\(^{19}\)

\(^{18}\)We know \( y^* > y_\gamma = \gamma^2 \) is necessary for search to be viable, since otherwise we cannot have \( \alpha U(y^*)/(r + \alpha) \geq \gamma \). To check this, notice that

\[ y^* = (1 - \theta) \gamma^2 + \theta \frac{\alpha \sqrt{y^*}}{r + \alpha} \geq (1 - \theta) \gamma^2 + \theta \gamma \]

using \( \alpha U(y^*)/(r + \alpha) \geq \gamma \). Since \( \bar{\gamma} \leq 1 \), \( y^* \geq \gamma^2 \) holds whenever \( \gamma \leq \bar{\gamma} \). Also, without the assumption \( c = 0 \), one can show search is viable iff \( Q(\gamma) \geq 0 \), where \( Q(\cdot) \) is the quadratic

\[ Q(\gamma) = -\gamma^2 [r^2 + 2r\alpha + \alpha^2 \theta] + \gamma [\alpha^2 \theta - 2 (r + \alpha) c] - c^2. \]

Hence, \( \exists \bar{c} > 0 \) such that \( c < \bar{c} \) implies search is viable for \( \gamma \in [\gamma_1, \gamma_2] \), with \( 0 < \gamma_1 < \gamma_2 \); and for \( c > \bar{c} \) search is not viable for any \( \gamma \geq 0 \). Also, \( [\gamma_1, \gamma_2] \rightarrow [0, \bar{\gamma}] \) as \( c \rightarrow 0 \), consistent with (28).

\(^{19}\)To those not familiar with this, a classic reference is Radford’s (1945) description of a POW camp in WWII: “Between individuals there was active trading in all consumer goods and in some services. Most trading was for food against cigarettes or other food stuffs, but cigarettes rose from the status of a normal commodity to that of currency. ... With this development everyone, including nonsmokers, was willing to sell for cigarettes, using them to buy at another time and place. Cigarettes became the normal currency.” (190-1). Similarly, Friedman (1992) reports: “After World War II [in Germany] the Allied occupational authorities exercised sufficiently rigid control over monetary matters, in the course of trying to enforce price and wage controls, that it was difficult to use foreign currency. Nonetheless, the pressure for a substitute
So far we have constructed an equilibrium with a stationary bubble, \( y_n = y^* > y_\gamma \) for all \( n \). Can we have nonstationary bubbles? When \( \rho(0) > 0 \), as is clear from Figure 7, the answer is no. With \( \rho(0) > 0 \), all paths except \( y_n = y^* \) for all \( n \) either lead to \( y_n < 0 \), or explode so that eventually \( y_n > \bar{y} \). But suppose \( \rho(0) < 0 \), as shown in Figure 8, which occurs when \( c > y_\gamma (r + \alpha) (1 - \theta)/\theta \). As long as \( c \) is not too big, there are multiple steady states, \( y_1^* \) and \( y_2^* \). As in the Figure, suppose \( \alpha [U(y_1^*) - c] / (r + \alpha) > \gamma \), so that at \( y_1^* \), or by continuity near \( y_1^* \), search is viable. Then, as the Figure clearly shows, there are nonconstant paths for \( y_n \) satisfying all the equilibrium conditions, even though fundamentals are time invariant. From the left, \( y_n \) rises over time, in progressively smaller increments, until settling at \( y_1^* \); and from the right, \( y_n \) falls in progressively smaller increments, again settling at \( y_1^* \). These nonstationary bubbles are self-fulfilling prophecies: as \( A_{n+1} \) correctly anticipates that \( y_{n+1} \) will be above (below) \( y_n \), he is willing to pay more (less) than \( A_n \), but as the increments vanish so does this premium (discount).

Some of these equilibria might look somehow “wrong” to some observers, because agents are buying high and selling low. This is certainly the case in equilibria that start to the right of, and converge to \( y_1^* \), under the interpretation that \( y_n \) is the price of the asset \( x \). How can this, one may naively ask, be a good strategy? It is obviously a good strategy because although one buys \( x \) at \( y_n \) and sells for \( y_{n+1} < y_n \), the payoff from selling \( U(y_{n+1}) \) exceeds the cost of buying \( y_n \). Indeed, even the ex ante payoff (at the point of purchase) from selling in the future taking into account the search costs, both in terms of pure time costs, \( r/(r + \alpha) > 0 \), and in terms of direct costs \( c > 0 \), exceeds the cost of buying – this is exactly our viability condition for search. One cannot get anything like this in a model where \( U(y) = y \), as might have seemed natural if one though of \( y \) as money. It is quite natural, however, to assume \( U(y) \) is nonlinear if we interpret \( y \) as a divisible good or service

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currency was so great that cigarettes and cognac emerged as substitute currencies and attained an economic value far in excess of their value purely as goods to be consumed. ... Foreigners often expressed surprise that Germans were so addicted to American cigarettes that they would pay a fantastic price for them. The usual reply was “Those aren’t for smoking; they’re for trading”.' (12-3). See Burdett, Trejos and Wright (1991) for more on cigarette money.
To close this part of the discussion we revisit our particular strategic bargaining game. It is not necessary to use this game to generate interesting dynamics such as nonstationary bubbles in equilibrium. Suppose one uses the generalized Nash solution – with some trepidation for reasons discussed later. Generalizing (12) to nonlinear utility, we have:

\[
y_n = \arg \max_y [U(y) - \gamma]^\theta (\nabla_{n+1} - y)^{1-\theta}
\]

The FOC is \( \theta (\nabla_{n+1} - y_n) U''(y) = (1 - \theta) [U(y_n) - \gamma] \), which has a unique solution that does not depend on \( n \) for any given \( \nabla_{n+1} \), say \( y_n = \hat{y}(\nabla_{n+1}) \). Inserting \( \nabla_{n+1} \) we get a dynamical system analogous to (27):

\[
y_n = \hat{y} \left[ \frac{\alpha U(y_{n+1}) - c}{r + \alpha} \right] = \hat{\rho}(y_{n+1}).
\]

This has qualitatively similar properties to the system \( y_n = \rho(y_{n+1}) \) derived using our
strategic game. One can do the same for proportional bargaining, which satisfies

\[ \theta (\nabla_{n+1} - y) = (1 - \theta) [U(y) - \gamma], \]

defining uniquely \( y_n = \tilde{y}(\nabla_{n+1}) \), and hence \( y_n = \tilde{\rho}(y_{n+1}) \) after inserting \( \nabla_{n+1} \).

The different dynamical systems defined by these three bargaining solutions behave similarly, and in particular all three can generate multiple stationary equilibria as well as dynamic bubble-like equilibria where \( y_n \) increases or decreases with \( n \). There is nothing particularly special about our game. Yet we think our game is nice because of the explicit strategic foundations. A property that is frequently described as desirable of Nash bargaining is that, at least in some environments, it can be derived as the limit of a strategic game, and Nash himself argued that it was important to be able to write down some explicit environment where his bargaining solution would arise as an equilibrium (see e.g. Osborne and Rubinstein 1990). As demonstrated in Coles and Wright (1998), in nonstationary environments with nonlinear utility, this logic does not work: when one writes down the standard strategic model, say with randomly alternating counteroffers, and then takes the limit as the time between counteroffers goes to 0, one does not get the Nash solution unless the environment the bargainers take as given, which includes continuation values that are endogenous in general equilibrium, is stationary or, in some cases, unless utility is linear, or unless one agent happens to have all the bargaining power (e.g., \( \theta = 1 \)).

What one gets is a differential equation for, in our notation, \( y_n \), the steady state of which is the Nash solution. But the limit of the strategic game is not the Nash solution out of steady state. Moreover, it is demonstrated in Coles and Wright (1998) that the set of dynamic equilibria can be qualitatively different when one used the correct limit of the alternating-offer game than if one alternatively simply sticks in the Nash solution out of steady state. Moreover still, it is demonstrated that simply sticking the Nash solution into the model out of steady state is equivalent to inserting the equilibrium outcome of the alternating-offer game under the assumption that the bargaining agents have myopic expectations – they believe the continuation value will not change over time even though in
equilibrium it does change over time—hardly an attractive assumption.

Studies that show solutions from the strategic approach converge to the Nash bargaining solutions often hinge on utility being linear. Some studies such as Rubinstein (1982) and Binmore et al (1986) use preference orderings, thereby bypassing the discussion of utility. As Rubinstein (1982, p.101) notes that "this assumption precludes discussion of some interesting bargaining situation such as ... player i has a fixed bargaining cost and his utility is not linear." A study that shows the strategic bargaining solution converges to the Nash bargaining solution in the presence of nonlinear utility is Hoel (1986).

While the relationship between Nash and the strategic approach has received much attention, little is known about the strategic foundation of proportional bargaining. We have shown earlier that when utility is linear, proportional and Nash bargaining are equivalent, and they predict the same outcomes as using the strategic bargaining approach. In what follows, we compare the three bargaining solutions using an example.

Take the example with $U(y) = \sqrt{y}$. Rearrange terms in the FOC from the genarlized Nash solution, we have \[ \theta y = \theta \nabla - (1 - \theta)[U'(y) - \gamma]/U'(y), \] where $U'(y) = 1/(2\sqrt{y})$. Substituting $\nabla$ and $U'(y)$ and rearranging terms, the steady state solution of $y$ is the solution to the quadratic equation

\[
(2 - \theta)(r + \alpha)y - [2\gamma(1 - \theta)(r + \alpha) + \alpha\theta]\sqrt{y} + c\theta = 0.
\]

Using a change of variable $U = \sqrt{y}$, the solution is

\[
U = \frac{1}{2(2 - \theta)(r + \alpha)}\left\{[2\gamma(1 - \theta)(r + \alpha) + \alpha\theta] + \sqrt{[2\gamma(1 - \theta)(r + \alpha) + \alpha\theta]^2 - 4(2 - \theta)(r + \alpha)c\theta}\right\}.
\]

Inserting $U$ into the viability condition, $\gamma \leq \frac{\alpha U - c}{r + \alpha}$, and simplifying, we have

\[
\gamma^2[r^2(2 - \theta) + 2r\alpha + \alpha^2\theta] + \gamma\{r(2 - \theta) + \alpha\}2c - \alpha^2\theta + (2 - \theta)c^2 \leq 0.
\]

There exists $\tilde{c} > 0$ such that $c < \tilde{c}$ implies search is viable for $\gamma \in [\gamma_1, \gamma_2]$, with $0 < \gamma_1 < \gamma_2$; and for $c > \tilde{c}$ search is not viable for any $\gamma \geq 0$. 

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One can do the same for proportional bargaining. At steady state, we have

$$\theta y + \frac{[(1-\theta)r + \alpha]}{r + \alpha} \sqrt{y} + \left[\frac{c\theta}{r + \alpha} - \gamma(1-\theta)\right] = 0.$$ 

Setting $U = \sqrt{y}$, the quadratic solution becomes

$$U = \frac{1}{2\theta} \left\{-\frac{[(1-\theta)r + \alpha]}{r + \alpha} \sqrt{y} + \sqrt{[(1-\theta)r + \alpha]^2 / (r + \alpha)^2 - 4\theta [c\theta / (r + \alpha) - \gamma(1-\theta)]}\right\},$$

and the viability condition is

$$\gamma^2 (r + \alpha)^2 \theta + \gamma \{2\theta c (r + \alpha) + \alpha[r(1-\theta) - \theta\alpha]\} + c[\theta c + \alpha(1-\theta)] \leq 0.$$ 

As in the case of Nash, search is viable only with a sufficiently small $c > 0$.

When $c = 0$, the upper bounds for $\gamma$ are

$$\overline{\gamma}_s = \frac{\theta^2}{r^2 + 2r\alpha + \theta^2},$$

$$\overline{\gamma}_n = \frac{\theta^2}{r^2(2-\theta) + 2r\alpha + \theta^2},$$

$$\overline{\gamma}_p = \frac{\alpha[\theta\alpha - r(1-\theta)]}{\theta (r + \alpha)^2},$$

where the subscripts $s$, $n$, and $p$ represent respectively strategic bargaining, Nash bargaining, and proportional bargaining, and $\theta \alpha > r(1-\theta)$. After some algebra, one can show that $\overline{\gamma}_n \leq \overline{\gamma}_s < 1$ and $\overline{\gamma}_p < 1$. The relationship between $\overline{\gamma}_p$ and the other two is unclear in general, but $\overline{\gamma}_i = \overline{\gamma}$ when $\theta = 1$.

When $c = 0$ and $r \to 0$, $\overline{\gamma}_i = 1$, independent of $\theta$, and $U$ becomes

$$U_s = \frac{1}{2} [\theta + \sqrt{\theta^2 + 4(1-\theta)\gamma^2}]$$

$$U_n = \frac{1}{2 - \theta} [2\gamma(1-\theta) + \theta]$$

$$U_p = \frac{1}{2\theta} [2\theta - 1 + \sqrt{1 - 4\theta(1-\gamma)(1-\theta)}].$$

After some algebra, we show that $y^*_n < y^*_s < 1$, and $y^*_n = y^*_s = 1$ when $\theta = 1$. The relationship between $y^*_p$ and $y^*_s$ is less clear. Figure 9 shows that when $\gamma = 0.6$, $y^*_p > y^*_s$, the three solutions
Figure 9: Model comparison when $\gamma = 0.6$

converge when $\theta = 1$. But when $\gamma < 0.5$, part of $y_p^*$ can go below $y_s^*$. Figure 10 shows that when $\gamma = 0.2$, $y_p^* = y_s^*$ at $\theta = 0.79$ and 1.00.

In sum, with non-linear utility, even when offer arrives instantaneously, the Nash bargaining solution converges to the strategic bargaining solution only when $\theta = 1$, otherwise the price of $x$ in terms of $y$ is always higher using the strategic approach. Depending on the value of $\gamma$, the proportional bargaining solution can meet the strategic bargaining solution more than once. This last result and the result on $\gamma_i = 1$, $i = s, n, p$, indicate that slightly less restriction is needed in a frictionless environment to support the convergence among bargaining solutions than that given in the non-stationary environment in Coles and Wright (1998).

8 Conclusion

TBC

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$^{20}$Note that the intercept corresponds to $y_\gamma = \gamma^2$.
Appendix

Here we give the details behind the results in Section 3. Consider first a type 2 equilibrium, 
\( \mu = 1 \) and \( \sigma_1 = 1 \). To find the equilibrium conditions, first subtract (15) from (16) to get

\[
\nabla_{21} - \nabla_{20} = \frac{\alpha_2 \gamma_3 + \sigma_1 \alpha_2 \mu y_{12} - c_2}{r + \alpha_2 + \sigma_1 \alpha_2 \mu}. \tag{29}
\]

Inserting this into (18)-(19) and solving, we get

\[
y_{23} = \frac{[r + \alpha_2 + \sigma_1 \alpha_2 \mu(1 - \theta_{12})] \theta_{23} \gamma_3 - (1 - \theta_{23}) c_2}{r + \theta_{23} \sigma_2 + \sigma_1 \alpha_2 \mu(1 - \theta_{12})},
\]

\[
y_{12} = \frac{\theta_{12} (\alpha_2 \theta_{23} \gamma_3 - c_2)}{r + \theta_{23} \sigma_2 + \sigma_1 \alpha_2 \mu(1 - \theta_{12})}.
\]

Substituting these back into (29), we have

\[
\nabla_{21} - \nabla_{20} = \frac{\alpha_2 \theta_{23} \gamma_3 - c_2}{r + \theta_{23} \sigma_2 + \sigma_1 \alpha_2 \mu(1 - \theta_{12})}.
\]

Thus, \( \mu = 1 \) iff \( \alpha_2 \theta_{23} \gamma_3 \geq c_2 \). Now substituting \( y_{12} \) and \( (1 - m) = \pi_2 \alpha_{23}/(\pi_2 \alpha_{23} + \pi_1 \alpha_{12} \mu) \) into (13), setting \( \sigma_1 = \mu = 1 \), and rearranging terms, \( \sigma_1 = 1 \) iff (22) in the text holds.

Now consider a type 1 equilibrium, \( \mu = 0 \) and \( \sigma_1 = 1 \), which means \( \nabla_{20} = 0 \) and

\[
\nabla_{21} = (\alpha_2 y_{23} - c_2)/(r + \alpha_2).
\]

Then (18) implies

\[
y_{23} = \frac{(r + \alpha_2) \theta_{23} \gamma_3 - (1 - \theta_{23}) c_2}{r + \theta_{23} \sigma_2}.
\]
Substituting this into $\nabla_{21}$ yields $\nabla_{21} = \alpha_{23}\theta_{23}\gamma_3 - c_2$. The condition for $\mu = 0$ is thus $\alpha_{23}\theta_{23}\gamma_3 \leq c_2$. Now, $\nabla_1 = \frac{\alpha_1}{r_1}(\alpha_{13}\theta_{13}\gamma_3 - c_1)$. So $\sigma_1 = 1$ iff $\alpha_{13}\theta_{13}\gamma_3 \geq c_1$.

Finally, consider a type 0 equilibrium, $\sigma_1 = 0$. Consider first the subcase $\mu = 0$, which holds iff $\alpha_{23}\theta_{23}\gamma_3 \leq c_2$. Then $\sigma_1 = 0$ reduces to $\alpha_{13}\theta_{13}\gamma_3 \leq c_1$. Now consider the subcase $\mu = 1$, which holds iff $\alpha_{23}\theta_{23}\gamma_3 \geq c_2$. Then $\sigma_1 = 0$ iff condition (21) in the text holds. This completes the argument.
References


