Dynamic Debt Runs and the Market for Variable Rate Demand Obligations*

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Abstract

A variable rate demand obligation (VRDO) is a tax-exempt municipal bond whose interest rate resets on a periodic basis. In addition, its bondholders have a “tender” option to liquidate their positions at par, exposing the issuer to possible runs. The VRDO market experienced large-scale runs in 2008 during the financial crisis, which contributed to the largest municipal bankruptcy in history by Jefferson County in Alabama. In this paper we develop a dynamic model of VRDO and runs. In the model both the interest rate and the decision of creditors to run are endogenously determined in closed form. We show that a higher pre-specified maximum interest rate or a higher liquidity premium reduces the likelihood of run. We structurally estimate the model using the method of simulated maximum likelihood. The structural estimation allows us to compute the model-implied likelihood of run, to quantitatively assess different roles of liquidity and credit components, and to examine different contributing factors to the most recent episode of runs during the crisis.

Keywords: Variable Rate Demand Obligation, VRDO, Debt Run, Municipal Bankruptcy

JEL Classifications: G10, G20, G21

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1 Introduction

The recent financial crisis of 2007–2009 has witnessed the excessive use of short-term debt.\textsuperscript{1} Despite some main advantages that it has (e.g., mitigating agency problems), short-term debt exposes the issuers to runs, which played a central role in the financial crisis. For example, investors ran on asset-backed commercial paper (ABCP), repo, banks, and money market mutual funds during the crisis. Much attention has been paid to these runs in the literature,\textsuperscript{2} however, another important type of more “quiet” runs, those on variable rate demand obligation (VRDO), is under researched. The VRDO runs were one of the key contributors to the default of $4.6 billion municipal bonds issued by Jefferson County in Alabama on November 9, 2011, the largest municipal bankruptcy to date in the history of the United States.

A variable rate demand obligation (VRDO) is a nominally long-term tax-exempt municipal bond for which the interest rate resets on a periodic basis (typically, daily or weekly) and bondholders are able to liquidate their positions at par through a “put” or “tender” feature on reset dates with advance notice. The tender option makes a VRDO essentially a short-term instrument and exposes the issuer to possible runs. The interest rate is capped by a contractually specified maximum interest rate. On each interest reset date, a dealer, called a \textit{remarketing agent}, is required to set the interest rate at the rate necessary, in its judgment, as the lowest rate that permits the sale of the VRDO at par. The remarketing agent is also responsible for reselling to new investors the bonds that have been tendered. To ensure that existing bondholders are able to liquidate their positions in the event that the remarketing agent is unable to remarket the securities, a VRDO typically operates with a liquidity facility provided

\textsuperscript{1}See, for example, Gorton (2008), Brunnermeier (2009), Krishnamurthy (2010), and Shin (2009).

\textsuperscript{2}Several papers empirically investigate these runs; see, for instance, Covitz, Liang, and Suarez (2012) for the run on ABCP, Gorton and Metrick (2012) for the run on repo, Kacperczyk and Schnabl (2012) and Wermers (2012) for the run on money market mutual funds. In addition, Schroth, Suarez, and Taylor (2012) provide a model of ABCP runs, and Martin, Skeie, and Von Thadden (2012) provide a model of repo runs.
by a third party (or by the issuer itself in the less frequent cases).

**Figure 1: The Municipal Swap Index Rate and Repo Rates**

This figure plots the weekly Municipal Swap Index (blue solid line) and the taxable repo rate (red dotted line) in the period between May 1991 and October 2012.

The runs on VRDO are evident in the spike of 7.96% of the weekly Municipal Swap Index (MSI) on September 24, 2008, as shown by the solid line in Figure 1 above, while the taxable repo rate was only 1.75% on that day, as shown by the dotted line in Figure 1. The MSI, published by Securities Industry and Financial Markets Association (SIFMA), is a market index reflecting the average yields on hundreds of weekly resettable high-grade VRDOs. The MSI rate spike was triggered by the bankruptcy of Lehman Brothers declared on September 15, 2008 and the subsequent panic in the market of money market mutual funds (e.g., runs on the Reserve Primary Fund that “broke the buck”, and other money market mutual funds). Using the iMoneyNet daily flow data, we find that in last two weeks of September, 2008, there
were about $40 billion outflows from the tax-exempt money market mutual funds, which accounts for about 10% of the total amount of VRDO outstanding. The direct cause of the MSI rate spike is the run on VRDO amid concerns about whether banking institutions that explicitly provided liquidity facility would be able to meet their obligations.

The VRDO market also experienced smaller-scale runs in mid-February 2008 when many monoline insurance companies, such as FGIC and XL Capital Assurance, were downgraded below double-A. At the same time, about 50% of the market for auction rate securities (ARS) had auction failures, or runs. The runs on VRDO and ARS played an important role in the default of $4.6 billion municipal bonds issued by Jefferson County, AL, the largest municipal bankruptcy to date in the history of the United States. The county’s downward spiral began in earnest in mid-February 2008 when monthly auctions failed for $1.4 billion of the county’s auction rate securities and holders of $548.5 million in VRDOs exercised their option to tender them to liquidity providers. In less than four years, Jefferson County filed for Chapter 9 municipal bankruptcy on November 9, 2011.

In this paper, we develop a dynamic model of VRDO in continuous time to address the following questions: How is the VRDO interest rate determined in equilibrium? What causes VRDO runs? In addition, we conduct a structural estimation of our model to quantitatively assess how different determinants contributed to the run on VRDO in 2008 during the financial crisis. The structural estimation also allows us to decompose the observed MSI rate into liquidity-related and credit-related components and quantitatively examine how different components account for its dynamics.

Our model of VRDO, built on He and Xiong (2012, HX hereafter), captures most

3Like VRDO, ARS are also long-term bonds whose interest rates are reset periodically. However, unlike VRDO, ARS rates are reset through auctions and the bondholders do not have a tender option. These key differences led to the collapse of the ARS market in the second half of 2008. See McConnell and Saretto (2010) and Han and Li (2009) for more details.
key characteristics of VRDO, such as, the floating interest rate with periodic resets, the tender option that allows an investor to put back the bond, and the pre-specified interest rate cap. The model is very tractable: there exists a unique threshold equilibrium that can be solved in closed form. In the threshold equilibrium, investors decide to run if the fundamental falls below a certain threshold. The threshold is endogenously determined by taking into account the negative externalities of current investors' decision to run on future investors.

We obtain four main results. First, we derive the endogenous VRDO interest rate in closed form, and show that the equilibrium interest rate can be decomposed into a tax-adjusted risk-free component, a liquidity component, and a credit component. The interest rate moves inversely with the issuer’s fundamental. As the issuer’s fundamental deteriorates, the interest rate increases in order to compensate investors for the higher risk. Interestingly, when the fundamental becomes low enough and hits the run threshold, the interest rate jumps as soon as a run is triggered. The jump occurs in our model because the run increases both the bankruptcy probability and the possible loss in principal payment. Another interesting property of the model is that following a run, the interest rate may or may not jump to the maximum interest rate. This property is consistent with the data that runs do not always drive the interest rate to the maximum level.

Second, theoretically we prove that, all else equal, the likelihood of run decreases with the maximum interest rate as well as the liquidity component of the interest rate process. The intuition is straightforward: a higher maximum interest rate (or a higher liquidity-related component) allows the interest rate to increase more in a severely adverse environment; therefore, it increases the expected interest income for creditors and they will rollover the debt more frequently. In the extreme case where the maximum interest rate is sufficiently high, then the likelihood of runs is zero. These predictions are consistent with the finding in McConnell and Saretto (2010)
that ARS with a higher maximum auction rate suffers from fewer auction failures.

Our results further suggest that compared to the setting in HX where the interest rate is fixed, the issuer is subject to less frequent runs under realistic parameter values in our model where the interest rate changes over time with the issuer’s fundamental. The intuition is that the flexible pricing allows for prompt adjustment of interest rate payments to at least partially compensate investors for the increased risk, and thus induce them to rollover the debt more often. In contrast, in the environment with a fixed interest rate, when the economy is hit by a large negative shock, the interest rate can not be adjusted and would inevitably become too low, resulting in more runs.

Third, we conduct a structural estimation of the model using the method of simulated maximum likelihood (SML) and using the data of the weekly MSI rate and 1-week repo rate in the sample period between 1991 and 2012. Specifically, we estimate four parameters: the liquidity premium, the strength of the liquidity facility, the asset volatility, and the standard deviation of pricing errors, while the rest of the parameters are directly inferred from the data. Our estimation results suggest that the average liquidity component accounts for less than 1 percent of the MSI-repo rate differential, and the average time from a run to the eventual bankruptcy is about 2 years. The latter result is roughly in line with the bankruptcy of Jefferson County, AL. Based on the implied run probability, our model singles out the most recent episode in the first several weeks following Lehman’s bankruptcy in 2008.

Fourth, we further extend the model to allow for time variation in the liquidity premium, the strength of the liquidity facility, etc. We assume that there are two regimes: one is the normal regime and the other is the “crisis” regime with heightened liquidity premium and asset volatility as well as weakened liquidity support. The extended model remain very tractable and can be structurally estimated in the same way as before. Our structural estimation results suggest that the run-induced spike
in September, 2008 is largely due to the increased liquidity premium demanded by investors. By regressing the MSI-repo spread on the liquidity and credit components, we also find that the credit component account for the most of the time variation in the MSI-repo spread, about 66%, and the liquidity component explains about an additional 19%.

Our paper contributes to the theoretical debt-run literature that examine the determinants of runs. The most closely related paper is Schroth, Suarez, and Taylor (2012, hereafter SST). Both this paper and SST are built on HX, but extend the He-Xiong model and apply to different markets: SST focuses on the ABCP market while our paper focuses on the VRDO market. This paper is thus complementary to SST: the creditors in the ABCP market do not get interim interest payments, but, instead, get a single principal payment which can be adjusted to induce them to rollover the debt; in contrast, the creditors in the VRDO market get the interim interest payments that are tied to the fundamentals, but the face value is fixed. Therefore, in both papers, there are time-varying fundamental-dependent yields: the floating interest rate in this paper and the yield-to-maturity in theirs. However, the above distinctive characteristics of these two markets have very different implications. For example, in our paper, the interest rate can jump as soon as a run is triggered and the rate may or may not jump to the maximum rate, while in their paper, an hypothetical maximum yield must be imposed to obtain an equilibrium in which a run drives the average yield to the maximum one and there are no jumps. This striking difference happens because in our model the interest rate, reset at each point of time, applies to all the creditors, but in their paper only the maturing creditors get the most up-to-date yield and it is the average yield-to-maturity that summarizes the state of the economy. Therefore, in our paper the interest rate must jump to reflect the intensified risk, while, by construction, the average yield in their paper does not jump. In addition, the tractability of our model allows us to analytically prove how
the run threshold depends on the underlying determinants, and to use the simulated maximum likelihood to structurally estimate the model.

Our paper is also related to Longstaff (2011) that uses a reduced-form approach to examine different roles played by the marginal tax rate and the credit/liquidity components in the MSI-repo rate. The structural estimation approach in this paper is complementary to the reduced-form approach used in Longstaff (2011): instead of modeling the dynamics of the marginal tax rate in reduced form, we fix the marginal tax rate and use the structural estimation to separate the credit and liquidity components which are not separable in the reduced form approach based on only the data of the MSI-repo spread. Furthermore, our paper has a different focus on the runs on VRDO and analyze the contributing factors of the spike of the MSI rate in 2008, which are not studied in Longstaff (2011). Therefore, this paper also contributes to the credit risk literature that tries to evaluate the separate roles of credit and liquidity components (see, Huang and Huang (2012) for the structural estimation approach, and Longstaff, Mithal, and Neis (2005) for the reduced form approach, and the references therein). The additional contributions of this paper are the incorporation of the rollover/run risk into the structural identification of the credit/liquidity components, as well as the focus on the municipal market.

2 The Model

We develop a dynamic model of VRDO runs based on He and Xiong (2012). The model contains several key features of VRDOs: the pre-specified maximum interest rate, the endogenous floating short-term interest rate, the tender option for bondholders to put back the bonds, and the liquidity facility structured as imperfect credit lines.
2.1 Asset
At time 0, a government-related entity (referred to as a municipality, hereafter) issues VRDOs to borrow $1 to finance a long-horizon project that generates cash flow at a constant rate $r$. The VRDOs are nominally long-term debt whose maturity is assumed to coincide with the termination of the underlying asset. At a random time $\tau_\phi$ that arrives according to a Poisson process with intensity $\phi > 0$, the project is terminated with a final payoff. The final payoff is the realization of a geometric Brownian motion process $y_t$ at time $\tau_\phi$

$$dy_t = y_t (\mu dt + \sigma dZ_t)$$

where $\{Z_t\}$ is a standard Brownian motion.

The project’s fundamental value under a discount rate $\rho$ is determined as follows

$$F(y_t) = E_t \left[ \int_t^{\tau_\phi} e^{-\rho(s-t)} r ds + e^{-\rho(\tau_\phi-t)} y_{\tau_\phi} \right] = \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y_t.$$  \hspace{1cm} (2)

Due to the tax exempt status of VRDOs, the discount rate $\rho$ equals the after-tax risk-free rate, that is,

$$\rho = r_f (1 - \tau),$$  \hspace{1cm} (3)

where $r_f$ denotes the taxable Treasury rate and $\tau$ the marginal tax rate.

2.2 Agents
In the model, there are a continuum of creditors with measure 1, a remarketing agent, and a liquidity provider. All agents in this economy are risk neutral and have the same discount rate of $\rho$.

The liquidity provider, usually a large bank, provides liquidity support in the form of a direct Letter of Credit (LOC). The direct LOC designates the liquidity provider as the first source of payment of principal and interest. Assuming the direct LOC
simplifies the liquidation event, which is exogenous in the model. In practice, the liquidity facility can also be structured as a *standby* LOC under which the issuer is the first source of liquidity and the liquidity provider acts as a back-up, or as a *Standby Bond Purchase Agreement* (SBPA) under which the VRDOs are insured by an investment-grade insurer and if a bond cannot be remarketed the liquidity provider is obligated to buy the bond as long as the insurer maintains its investment grade rating.

The creditors have a tender option to put back the VRDOs to the remarketing agent at par value (plus any accrued interest) who then resell the tendered bonds to new investors. In the event that the remarketing agent is unable to remarket the VRDOs, the liquidity provider will buy the bonds, which then become the so-called “bank bonds” showing up on the liquidity provider’s balance sheet. For simplicity, we assume the bank bonds have the same debt contract as the existing bonds; in particular, they will be tendered back in the same way as the existing creditors.

The creditors may exercise the tender option well in advance before the VRDOs mature. For example, a creditor may experience a liquidity or preference shock so that he would like to liquidity his positions. Therefore, we assume that each creditor decides to put back the VRDO holdings at a random time $\tau_s$ which arrives following a Poisson process with intensity $\delta > 0$. In other words, the average duration before the tender option is exercised is equal to $1/\delta$, which is assumed to be much shorter than the VRDOs’ average maturity $1/\phi$ (i.e., $\delta \gg \phi > 0$) to capture the maturity mismatch.

The remarketing agent has two key responsibilities: remarketing the tendered bonds and resetting the interest rate $r_t$. The interest rate $r_t$ is set as the lowest rate that permits the sale of the VRDOs at par on each interest reset date. In our model, the remarketing agent resets the interest rate $r_t$ continuously to reflect the most up-to-date fundamental value of the project, but the interest rate cannot
exceed the pre-specified maximum interest rate \( r \). Throughout this paper we use the terms “maximum interest rate” and “max rate” interchangeably.

### 2.3 Rollover, Runs, Credit Lines, and Liquidation

The tender option allows for the creditors to put back the VRDOs to the remarketing agent, and thus exposes the issuing municipality to the rollover risk. A run on VRDO occurs if the remarketing agent is unable to find new investors to buy the tendered bonds. In this case, the tendered bonds have to be purchased by the funds drawing from the liquidity facility and become “bank bonds”.

Moreover, the run exposes the issuer to bankruptcy risk because the credit lines may not provide perfect protection as the liquidity providers themselves may experience certain financial distresses, and also because, in practice, the liquidity provider will typically charge a higher penalty interest rate for holding the bank bonds and ask the issuer to accelerate the principal payment. We assume that with probability \( \theta \delta dt \), the VRDO’s credit lines fails to provide liquidity and the project is forced into liquidation.

In default, the asset is sold at a fraction \( \alpha \) of its fundamental value (e.g., fire sale). That is, the liquidation value is

\[
L(y_t) = \alpha F(y_t) = \frac{\alpha r}{\rho + \phi} + \frac{\alpha \phi}{\rho + \phi - \mu} y_t \equiv L + l y_t.
\]  

### 2.4 VRDO Interest Rate Resets

We consider how the interest rate \( r_t \) is reset at time \( t \). Because VRDOs are usually less liquid than the Treasuries, the remarketing agent may not be able to immediately re-market the tendered VRDOs. For this reason he demands certain liquidity premium. We model it by assuming that there exists some holding costs \( \Delta_p > 0 \) for the remarketing agent to hold the tendered VRDOs in his inventory before the bonds
are re-marketed.

The interest rate \( r_t \) is valid for a short interval \([t, t + \Delta t]\) and is set such that it satisfies the following pricing equations:

(i) when \( y_t > y_s \),

\[
1 = E \left[ \int_t^{t+\Delta t} e^{-\rho(s-t)} (r_s - \Delta \rho) \, ds + e^{-\rho \Delta t} \mathbb{1}_{\{\tau_\phi > t + \Delta t\}} + e^{-\rho (\tau_\phi - t)} \min (1, y_{\tau_\phi}) \mathbb{1}_{\{\tau_\phi \leq t + \Delta t\}} \right];
\]

where the last term \( e^{-\rho (\tau_\phi - t)} \min (1, y_{\tau_\phi}) \mathbb{1}_{\{\tau_\phi \leq t + \Delta t\}} \) in the right-hand side reflects the possible loss in principal payment when the debt happens to mature within the interval.

Moreover, (ii) when \( y_t \leq y_s \),

\[
1 = E \left[ \int_t^{t+\Delta t} e^{-\rho(s-t)} (r_s - \Delta \rho) \, ds + e^{-\rho \Delta t} \mathbb{1}_{\{\tau_\phi \wedge \tau_\theta > t + \Delta t\}} + e^{-\rho (\tau_\phi - t)} \min (1, y_{\tau_\phi}) \mathbb{1}_{\{\tau_\phi \leq t + \Delta t\}} + e^{-\rho (\tau_\theta - t)} \min (1, L + l y_{\tau_\theta}) \mathbb{1}_{\{\tau_\theta \leq t + \Delta t\}} \right],
\]

where the last term \( e^{-\rho (\tau_\theta - t)} \min (1, L + l y_{\tau_\theta}) \mathbb{1}_{\{\tau_\theta \leq t + \Delta t\}} \) reflects the possible loss in principal payment in the event of bankruptcy.

Proposition 1 below derives the equilibrium interest rate \( r_t \). It shows that the rate \( r_t \) can be decomposed into a risk-free component, a liquidity component, and a credit component.

**Proposition 1** In the absence of a maximum interest rate imposed, the interest rate process, \( \{r_t\} \), satisfies

\[
r_t = R(y_t; y_s) \equiv \rho \underbrace{\Delta \rho}_{\tau_f(1-\tau)} + \underbrace{\phi (1 - y_t)^+ + 1_{\{y_t \leq y_s\}} \theta \delta (1 - [L + l y_t])^+}_{\text{credit component}};
\]
where \((x)^+\) denotes \(x\) if \(x > 0\), or zero otherwise.

**Proof.** See Appendix. ■

From Proposition 1, absent the maximum interest rate, the equilibrium interest rate is always strictly less than \(\rho + \Delta \rho + \phi + \theta \delta (1 - L)\).

Figure 2 below shows how the interest rate schedule looks like. Intuitively, the interest rate decreases with \(\vartheta\). Moreover, in the top two panels, there is an upward jump in \(r_t\) when \(y_t\) decreases to \(y^*_t\) from above.

**Figure 2: Interest Rate Schedule**

In reality, it is neither practical nor credible to have unlimited interest rate \(r_t\). In fact, there is usually an pre-specified max rate \(\tau\). We incorporate this important feature into our model and assume that \(r_t\) cannot exceed \(\tau\) at any time.

**Corollary 1** In the presence of the maximum interest rate \(\tau\), define

\[
y_{**} \equiv \inf \left\{ y : R(y; y_*) \leq \tau \right\}.
\]

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Then the interest rate schedule is given by

\[ r_t = \begin{cases} 
\tau, & \text{if } y_t \leq y^{**} \\
R(y_t; y_s), & \text{if } y_t > y^{**} 
\end{cases} \]  

(9)

The jumps in Figure 1 imply that the max rate \( \tau \) sometimes is not achievable, i.e., \( R(y^{**}; y_s) < \tau \). These jumps occur only in the cases where \( y^{**} = y_s \), implying that in these cases before the max rate is reached, VRDOs investors have already decided to run on them. Depending on which part of the interest rate schedule \( \tau \) intersects, there are eight possibilities. We analyze them one by one in the next section.

### 2.5 Parameter Restrictions

First, if the liquidity component \( \Delta_\rho \) is too large, then it is always profitable to hold the VRDOs, implying that the equilibrium threshold \( y_s \) is zero. To rule out this degenerate case, we impose the following condition

\[ 0 < \Delta_\rho < \frac{\eta_1 (\rho + \phi)}{\gamma_2 (\rho + \phi + \delta (1 + \theta))} (R - \tau), \]  

(10)

where \( \eta_1 \) and \( \gamma_2 \) are defined in Appendix A and

\[ R \equiv \rho + \phi + \theta \delta (1 - L). \]  

(11)

Note that the inequality above also implies \( \tau < R \).

The other restrictions are the same as in HX:

\[ \rho + \phi > \mu, \]

\[ \theta > \frac{\phi}{\delta (1 - L - l)}, \]
The first one of the above three restrictions imposes an upper bound on the growth rate of the fundamental to ensure the fundamental value is finite. The second restriction ensures that the parameter \( \theta \) is sufficiently high so that bankruptcy becomes likely when some creditors choose to run. The third restriction stipulates a sufficiently low premature recovery rate so that \( L + l < 1 \).

### 3 The Equilibrium

As in He and Xiong (2012), we solve the optimal rollover decision by taking as given that all other creditors use a monotone strategy with a rollover threshold \( y^* \).

#### 3.1 The Value Function

For each unit of debt, the creditor receives a stream of interest payments \( \{r_t\} \) until the earliest of the following three events occur

\[
\tau = \min \{\tau_\phi, \tau_\delta, \tau_\theta\}.
\]

The first event occurs at the stopping time \( \tau_\phi \) when the asset matures and the creditor gets a final payoff of \( \min(1, y_{\tau_\phi}) \). The second event occurs at the stopping time \( \tau_\delta \) when the creditor gets an option to roll over or tender back his bondholdings. Whether the creditor decides to roll over depends on whether the continuation value \( V(y_{\tau_\delta}; y^*) \) exceeds one dollar par value. The third event occurs at the stopping time \( \tau_\theta \) when the fundamental falls below the other creditors’ rollover threshold and the project is eventually forced to premature liquidation with payoff \( \min(1, L + l y_{\tau_\theta}) \). The stopping time \( \tau \) is the minimum of these three stopping times, representing the earliest time when any of these three events occur for the first time.
Since the interest rate \( r_t \) solely depends on \( y_t \), a creditor’s value function is only a function of \( y_t \) given the conjectured threshold \( y_* \) taken by other creditors. We denote the creditor’s value function as \( V (y_t; y_*) \). Due to risk neutrality, the value function is determined as follows:

\[
V (y_t; y_*) = E_t \left[ \int_t^\tau e^{-\rho(s-t)} r_s ds + e^{-\rho(\tau-t)} \min (1, y_t) 1_{\{\tau=\tau_o\}} + e^{-\rho(\tau-t)} \min (1, L + ly_t) 1_{\{\tau=\tau_o\}} + e^{-\rho(\tau-t)} \max_{\text{rollover or run}} (V (y_{t+1}; y_*), 1) 1_{\{\tau=\tau_o\}} \right].
\]

Note that the current creditor’s decision to run imposes negative externalities on future creditors because the run exposes the municipality issuer to a larger bankruptcy probability and increases bankruptcy costs. Such externalities give rise to strategic complementarity in the creditors’ rollover decision because anticipating future creditors’ decision to run makes the current creditor more likely to run.

The endogenous interest payment \( \{r_t\} \) in this paper, a key departure from HX, affects the creditors’ rollover decision in a profound way. For example, when the fundamental falls below the other creditors’ rollover threshold, the interest payment increases accordingly to reflect the weakened fundamental. The higher interest payments are used to offset the lower final payoff from premature liquidation. However, there is a limit to the offset because the interest rate is capped from the above.

The Hamilton-Jacobi-Bellman (HJB) equation is given below:

\[
\rho V (y_t; y_*) = \mu y_t V_y (y_t; y_*) + \frac{\sigma^2}{2} y_t^2 V_{yy} (y_t; y_*) + r_t + \phi \left[ \min (1, y_t) - V (y_t; y_*) \right] + \theta \delta 1_{\{y_t < y_*\}} \left[ \min (1, L + ly_t) - V (y_t; y_*) \right] + \delta \max_{\text{rollover or run}} (0, 1 - V (y_t; y_*)).
\]
It shows that the creditor’s required return, \( \rho V (y_t; y_*) \) on the left hand side, is equal to the expected increase in his continuation value as summarized by the terms on the right hand side.

### 3.2 The Unique Threshold Equilibrium

From Proposition 1, it is easy to show that if \( y_t > y_* \), \( r_t + \phi \min (1, y_t) = \rho + \Delta + \phi; \) and if \( y_t \leq y_* \), \( r_t + \phi \min (1, y_t) + \theta \delta \min (1, L + b y_t) = \rho + \Delta + \phi + \theta \delta \). Therefore, the HJB equation can be further simplified as

\[
\rho V (y_t; y_*) - \nu y_t V_y (y_t; y_*) - \frac{\sigma^2}{2} y_t^2 V_{yy} (y_t; y_*)
\]

\[
= \begin{cases} 
(\rho + \Delta + \phi) - \phi V (y_t; y_*) , & \text{if } y_t > y_*; \\
(\rho + \Delta + \phi + \delta (1 + \theta)) - (\phi + \delta (1 + \theta)) V (y_t; y_*) , & \text{if } y_t \leq y_*.
\end{cases}
\]

We can show that the only solution is \( y_* = 0 \) and \( V (y_t; y_*) = \frac{\rho + \Delta + \phi}{\rho + \phi} > 1 \). That is, the creditors will never run.

Furthermore, if \( \Delta = 0 \), then \( V (y_t; y_*) = 1 \). This result is intuitive. If the rate \( r_t \) can be set without bound, the remarketing agent can set it such that the VRDOs are at par value any time. In particular, even in the very bad state, the remarketing agent can set the rate as extremely high as possible to make the bonds priced at par. Therefore, at any time, creditors are indifferent between running or not, since either choice gives them the same value of $1.

However, in reality, it is impossible to promise an unlimitedly high interest rate. Therefore, the terms of VRDOs typically stipulate a maximum interest rate in practice. As we show in the subsequent analysis, once we incorporate this aspect of max rate into our model, the equilibrium threshold \( y_* \) will be strictly positive and finite; that is, the decision of running from VRDOs will be no longer trivial.

Following HX, we focus on symmetric monotone equilibria where all creditors in
equilibrium will choose the same threshold $y_*$. For the analysis to be meaningful, we consider $\bar{\tau} > \rho + \Delta \rho$ hereafter. We denote the set of feasible parameter values $(\Delta \rho, \bar{\tau})$ as $\Theta (\Delta \rho, \bar{\tau})$:

$$
\Theta (\Delta \rho, \bar{\tau}) \equiv \{ (\Delta \rho, \bar{\tau}) : \bar{\tau} > \rho + \Delta \rho \text{ and } \bar{\tau} \text{ satisfies (10)} \} .
$$

**Theorem 1** For any pair of feasible parameters in the space $\Theta (\Delta \rho, \bar{\tau})$, there exists a unique monotone equilibrium in which the run threshold $y_*$ is uniquely determined — each maturing creditor chooses to roll over his debt if $y_t > y_*$, and to run otherwise.

**Proof.** See Appendix C.

![Equilibrium Domain](image)

The proof of Theorem 1 also shows how different cases are associated with different regions in the parameter space. Figure 3 above depicts the breakdown of the parameter space for all eight different cases. In general, fixing the max rate $\bar{\tau}$, the run threshold decreases with $\Delta \rho$ — Cases G or H are associated with small values of $\Delta \rho$.
Cases D, E, or F are associated with intermediate values of \( \Delta \); and Cases A, B, or C are associated with large values of \( \Delta \). On the other hand, fixing the liquidity component \( \Delta \), for a large range of values of \( \Delta \), Cases B or F will result for any feasible max rate \( \tau \) where the max rate cannot be reached in equilibrium.

3.3 Model Implications

Our model is very tractable so that we derive the unique threshold equilibrium in closed form. In addition, the equilibrium interest rate has an analytical formula. The tractability allows us to rigorously investigate several implications of the model below.

**Interest Rate Jumps**

One of our main results is the possible jumps in the interest rate process \( \{r_t\} \) in our model. Recall that depending on parameter values, there are totally eight different types of interest rate schedule, as shown in Figure 2. In particular, in most cases (Cases, A, B, D, E, F), jumps occur once the fundamental value \( y_t \) hits the run threshold \( y_* \) and the interest rate jumps immediately from a lower value to the max rate \( \tau \). These jumps happen because runs triggered by the hitting of the run threshold \( y_* \) increase the bankruptcy probability, and most likely investors are unable to get full principal payments in the event of bankruptcy since \( y_* \leq \frac{1-F}{1} \) in these cases. As a result, a jump in the interest rate is needed to compensate investors for the intensified bankruptcy probability and the possible loss.

Interestingly, no jumps occur in Schroth, Suarez, and Taylor (2012). The different results stem from the following key distinctive between our model and theirs. Schroth, Suarez, and Taylor (2012) focuses on the ABCP market where the investors do not receive interim interest payments, but rather get compensated one by the higher promised face value of commercial papers they hold. Consequently, investors of bonds
that mature at a different time are offered by new bonds with a different face value, and thus have a yield to maturity that is different from those for earlier investors. In their model, it is the average yield to maturity that matters, which is a smooth process without jumps. In contrast, in our model of VRDO, investors continuously get interest payments and the time-varying interest rate applies to all the investors. Therefore, to reflect the intensified default risk, the interest rate has to jump.

Another interesting observation is the interest rate may jump to the maximum rate $\tau$ in some cases as soon as a run is triggered, but under some circumstances the interest rate may only jump to a level that is lower than $\tau$ if the maximum rate is set too high. Moreover, in our model there is an endogenous upper bound on the interest rate, $R$ given in Eq. (11). This endogenous upper bound results in the limit when the fundamental value $y_t$ tends to zero. To avoid such uninteresting case, throughout the paper we consider the maximum interest rate $\tau$ that is always strictly less than $R$.

The Likelihood of Runs

Our model is so tractable that we can rigorously examine how different factors affect the likelihood of runs. There are two additional parameters in our model, maximum interest rate $\tau$ and the liquidity premium $\Delta_p$, in addition to those in HX.

First, we show in Proposition 2 below that the equilibrium rollover threshold $y_*$ decreases with the maximum interest rate $\tau$. The intuition is straightforward: a higher maximum interest rate $\tau$ allows the interest rate to increase more in a severely adverse environment; therefore, it increases the expected interest income for creditors and they will rollover more frequently. In the extreme case where the maximum interest rate $\tau$ is higher than the endogenous upper bound $R$, then the run threshold is zero (i.e., $y_* = 0$), that is, the likelihood of runs is zero.
Proposition 2. The equilibrium rollover threshold $y_*$ decreases with the maximum interest rate $\tau$.

Proof. See Appendix C. ■

Second, Proposition 3 shows that the equilibrium rollover threshold $y_*$ also decreases with the liquidity premium $\Delta_\rho$. The intuition is similar as before: all else equal, a higher liquidity premium $\Delta_\rho$ leads to a higher interest rate payment, which discourages investors from running on the bonds.

Proposition 3. (i) The equilibrium rollover threshold $y_*$ decreases with $\Delta_\rho$.

(ii) When $\Delta_\rho$ goes to 0, the equilibrium run threshold $y_*$ tends to infinity.

Proof. See Appendix C. ■

4 Estimation

We use simulated maximum likelihood (SML) to conduct a structural estimation of the model. In the structural estimation, we allow for possible pricing errors in the MSI rate observed in the data:

$$r_t = R(y_t; y_*(\theta)) + w_t, \text{ where } w_t \sim N(0, \nu^2).$$ (13)

4.1 Data

The weekly data of 1-week tax-exempt MSI rate is obtained directly from the Securities Industry and Financial Markets Association website for the period from May
We obtain the 1-week Treasury repo rate from the Bloomberg for the same dates as the MSI data.

To calibrate some parameters (e.g., maximum interest rate), we obtain information about VRDOs from the Municipal Securities Rulemaking Board (MSRB)’s SHORT database from its inception date of April 1, 2009 to November 8, 2012. The SHORT database has been built from the Short-term Obligation Rate Transparency (SHORT) System and the Real-Time Transaction Reporting System (RTRS), which the MSRB launched in early 2009 to collect and disseminate interest rates and important descriptive information about these ARS and VRDOs. The SHORT database provides a centralized source of information about municipal ARS and VRDOs that was previously unavailable. Starting from May 2011, MSRB rules require VRDO remarketing agents to report to the MSRB the aggregate amount of par value of “bank bonds” and bonds held by investors or the remarketing agents. There are 20,547 distinct VRDOs in the SHORT database during our sample period. We focus on the VRDOs with weekly interest resets, which accounts for 90.7% of the whole sample (i.e., 18,630).

The SHORT database does not contain maturity information. Therefore, we merge it with the Mergent Municipal Bond database to collect information on maturities of the VRDOs.

### 4.2 Calibration

There are ten primitive model parameters in the model: \( \tau, \rho, \phi, \delta, \mu, \sigma, L, l, \theta, \Delta \rho \) and one auxiliary parameter \( v \) to estimate. As explained above, the auxiliary parameter \( v \) is introduced to allow for possible pricing errors in the observed MSI rate, which is needed to account for some possible factors beyond our model. Using the SHORT database and the Mergent Municipal Bond database, we directly measure parameters \( \tau, \rho, \phi, \delta, \mu, \sigma \). We then use the simulated maximum likelihood (SML) method to
estimate the remaining parameters.

The contractual maximum interest rate $\tau$ can be easily calibrated from the SHORT database. Among the 18,630 VRDOs with weekly interest resets in the SHORT database, 53.42% of them have the max rate of 12%, 26.13% of them have the max rate of 10%, and 10.37% of them have 15%. The weighted average of these three rates is 11.76%. Therefore, we set $\tau$ as 12%.

The average debt maturity of our merged VRDO sample from the SHORT and Mergent databases is 25.2 years (and the median is 25.96 years). Therefore we set $1/\phi$, the expected asset maturity, to 25 based on the assumption that the average VRDO maturity coincides with the average asset maturity; that is, $\phi = 0.04$.

The average time between two consecutive tenders in our model is $1/\delta$. Because we the constituents of the MSI index have weekly interest resets and the tender option can only be exercised on each interest reset date, $1/\delta$ should be multiples of $\Delta t$ where $\Delta t = 7/365.25$ corresponds to 0.0192 years or 7 days. Moreover, once a run occurs, the proportion of all the bonds that are tendered and subsequently become “bank banks” is $\delta \Delta t$ in our model where $\Delta t = 7/365.25$ corresponds to 0.0192 years or 7 days. Because the tender option is less frequently exercised, we set $\delta = 12$, meaning that on average the VRDOs are tendered on a monthly basis, and about 23% of VRDOs are tendered and become “bank banks”.

The tax-adjusted risk-free rate $\rho$ is set to the average value of the repo rate times $(1 - \tau)$ when $\tau$ is chosen as 40% (Longstaﬀ, 2011). Following Schroth, Suarez, and Taylor (2012), we set $\mu = \rho$.

We find that the values of $L$ and $l$ do not matter that much in our SML estimation below. Also choosing $\mu = \rho$ implies that $l = \frac{\alpha \phi}{\rho + \phi - \mu} = \alpha$. Under our parameter restrictions, $l < \frac{\alpha \phi}{2 \phi + \phi} = 75\%$ and we set $l = 70\%$, which is on the low end of the estimates in the literature which range from 80% to 97%.\textsuperscript{5} Furthermore, under our

\textsuperscript{5}See, for instance, Schroth, Suarez, and Taylor (2012) for the estimates of the recovery rate for
parameter restrictions, $L$ is bounded between $\frac{L}{\rho + \phi}$ from below and $\min (1 - l, l)$ from above. We set $L$ to the middle point of these bounds; that is, $L = 27\%$.

### 4.3 Simulated Maximum Likelihood (SML) Estimation

The joint probability of observing the interest rate $r_t$ and the project’s fundamental $y_t$, conditional on the fundamental $y_{t-1}$ in the previous period, is given by

$$f (r_t, y_t \mid y_{t-1}, \theta) = f (r_t \mid y_t, \theta) f (y_t \mid y_{t-1}, \theta),$$

where by Eq. (13)

$$f (r_t \mid y_t, \theta) = \frac{1}{\sqrt{2\pi v}} \exp \left[ -\frac{(r_t - R(y_t; y_\ast (\theta)))^2}{2v^2} \right],$$

and because $y_t \mid y_{t-1}$ follows a log-normal distribution

$$f (y_t \mid y_{t-1}, \theta) = \frac{1}{y_t \sqrt{2\pi \Delta t \sigma}} \exp \left[ -\frac{(\log (y_t) - [\log (y_{t-1}) + (\mu - \frac{1}{2} \sigma^2) \Delta t])^2}{2\sigma^2 \Delta t} \right].$$

ABCP conduits; Coval and Stafford (2007) for the estimates of the recovery rate for stocks; and Andrade and Kaplan (1998), Hennessy and Whited (2007), Ellul, Jotikashira, and Lundblad (2010), for the estimates of the recovery rate for corporate bonds.
Integrating out the unobserved fundamentals \( \{y_t\} \), we obtain the marginal log-likelihood function as

\[
\ln \mathcal{L}(\theta; r) = \ln \left[ \prod_{t=1}^{T} f \left( r_t, y_t \mid y_{1:t-1}, y_0, \theta \right) dy_{1:T} \right]
\]

where \( E^{y_{1:T}}[\cdot] \) denotes the expectation with respect to the fundamental process and \( y_{1:T} \) denotes the time series \( \{y_t\}_{t=1}^{T} \). We can evaluate the expectation in the above equation using Monte Carlo simulations. Specifically, we run \( S \) rounds of simulations, and in each round a time series \( y_{1:T}^{(s)} \) is simulated, \( s = 1, \cdots, T \). Therefore, we use the empirical analog to the log-likelihood function as follows

\[
\ln \mathcal{L}(\theta; r) = \ln \left( \frac{1}{S} \sum_{s=1}^{S} \left[ \prod_{t=1}^{T} f \left( r_t \mid y_t^{(s)}, \theta \right) \right] \right).
\]  

We find that 1,000 random draws are sufficient to render the simulation error negligible. We therefore set \( S = 1000 \) in the estimations.

The SML estimator is the maximizer of \( \ln \mathcal{L}(\theta; r) \)

\[
\hat{\theta} = \arg \max_{\theta} \ln \mathcal{L}(\theta; r).
\]  

Then the simulated estimate of the asymptotic variance is calculated as the diagonal elements of the inverse of the Hessian matrix.
5 Estimation Results

The estimation results are reported in Table 1 below. From Panel A of Table 1, the average liquidity component is estimated to be about 38 basis points, or roughly 0.8 percent of the average interest rate in excess of the tax-adjusted risk free rate. \( \theta \) is estimated to be 0.0395, implying that the average time from a run to the eventual bankruptcy is equal to \( 1/ (\theta \delta) = 2.1 \) years.

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<tr>
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<tbody>
<tr>
<td>( \Delta_p ) (b.p.)</td>
<td>38</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>( \theta )</td>
<td>0.0395</td>
<td>0.011</td>
<td></td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0.0782</td>
<td>0.0018</td>
<td></td>
</tr>
<tr>
<td>( \nu ) (b.p.)</td>
<td>50</td>
<td>1</td>
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<table>
<thead>
<tr>
<th>Panel B: Equilibrium Run Threshold</th>
</tr>
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<tbody>
<tr>
<td>This model</td>
</tr>
<tr>
<td>HX</td>
</tr>
</tbody>
</table>

From Panel B of Table 1, we can see that under the same parameter values, our model has a lower equilibrium run threshold than HX. This result suggests that compared to the setting in HX where the interest rate is fixed, the issuer is subject to less frequent runs under realistic parameter values in our model where the interest
rate changes over time with the issuer’s fundamental. The intuition is that the flexible pricing allows for prompt adjustment of interest rate payments to at least partially compensate investors for the increased risk, and thus induce them to rollover the debt more often. In contrast, in the environment with a fixed interest rate, when the economy is hit by a large negative shock, the interest rate can not be adjusted and would inevitably become too low, resulting in more runs.

Based on the observed excess interest rate and the annual estimates, we are able to estimate the underlying fundamental process \{y_t\} as follows. At each point of time \(t\), we simulate 10,000 pricing errors \(w_t^{(n)}\) and then compute 10,000 simulated interest rates, \(\hat{y}_t^{(n)}\), based on Eq. (13); next, we use the equilibrium interest rate formula in Proposition 1 to back out the implied \(y_t^{(n)}\); lastly, we estimate \(y_t\) as the sample average of the simulated fundamental \(y_t^{(n)}\), \(n = 1, \cdots, 10,000\). We use the model’s structural estimation to compute the probability that a run will happen within the next week.

\[
\Pr(\text{run within the 1 week}) = \Pr(y_{t+\Delta t} \leq y_\ast | y_t; \theta) = \Pr \left( \tilde{z} \leq - \frac{\log \left( \frac{\mu}{\sigma^2} \right) + \frac{1}{2} \Delta t}{\sigma \sqrt{\Delta t}} \equiv -Z_t^\ast | y_t; \theta \right) = \Phi(-Z_t^\ast).
\]

\(^6\)In the cases that the implied fundamental \(y_t^{(n)}\) is censored, we replace it by the conditional mean. For example, if \(\hat{y}_t^{(n)} < \rho + \Delta y\), then it implies \(y_t^{(n)} > 1\). In this case, we estimate \(y_t^{(n)}\) by the conditional mean

\[
E[y_t | y_{t-1}, y_t > 1] = y_{t-1} \exp \left( \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t \right) E \left[ \exp \left( \sigma \sqrt{\Delta t} \tilde{z} \right) \mid \tilde{z} > -Z = - \frac{\log(y_{t-1}) + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t}{\sigma \sqrt{\Delta t}} \right] = y_{t-1} \exp(\mu \Delta t) \frac{\Phi \left( \frac{Z + \sigma \sqrt{\Delta t}}{\Phi(Z)} \right)}{\Phi(Z)}.
\]

where \(\Phi\) stands for the cumulative distribution function of a standard normal distribution.
In Figure 4 below we plot in Panel A the estimated $y_t$ process as well as the estimated run threshold $y^*$ (shown in red dot-dash line), and in Panel B the implied 1-week run probability. From Figure 4 Panel A, we can see that the fundamental process becomes less volatile since 2009. The low volatility in this period is largely attributable to the low interest rate environment which narrows the MSI-repo spread. Note also that the fundamental process exceeds 1 frequently in 1990s and early 2000s as a reflection of strong economy in that long-lasting boom period.

A more striking observation is that in the past two decades, the run threshold has been approached occasionally, however, only the most recent run episode in 2008 in the first several weeks following Lehman’s bankruptcy is singled out in Panel B of Figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure4.png}
\caption{Filtered Fundamental Process and Run Probabilities}
\end{figure}

5.1 Estimation of an Extended Regime-Switching Model

In the baseline model, a few parameters, such as the liquidity premium $\Delta_\rho$, the strength of the liquidity facility $\theta$, and asset volatility $\sigma$, are time invariant. To make the model more realistic, in this subsection, we extend the baseline model to
allow these parameters to be time varying. An important benefit of this extension is that we can quantitatively assess how much of the spike in 2008 following Lehman’s bankruptcy is due to the weakening of the liquidity facility, etc. Another benefit of this extension is that, for example, with the time-varying liquidity premium, we are able to conduct time-series analysis to investigate how much of time variation in the observed MSI-repo spread is due to liquidity risk, and how much is due to credit risk, etc.

We keep most of the baseline model’s features unchanged, except that there are two regimes: one is the normal regime and the other is the “crisis” regime with heightened liquidity premium and asset volatility as well as weakened liquidity support. Across different regimes, a few parameters have different values. Specifically, in regime \( i \in \{ u, d \} \), we assume the following parameter values \( \Delta^i, \theta^i, \mu^i, \sigma^i, L^i, l^i \), and upon the arrival of a Poisson process with intensity \( \lambda_i \), regime \( i \) switches to the other regime.

It is straightforward to show that the interest rate regime derived in Proposition 1 applies to the extended model: in regime \( i \), the interest rate \( r^i = R^i (y_t; y^*_t) \) follows the same interest rate schedule as if the regime were the only single regime in the economy. The main change is reflected by the impact of regime switching on the value function, as shown by the Hamilton-Jacobi-Bellman equation below:

\[
\rho V^i (y_t; y^*_t, y^d_t) = \mu^i y_t V^i_p (y_t; y^*_t, y^d_t) + \frac{1}{2} (\sigma^i)^2 y_t^2 V^i_{yy} (y_t; y^*_t, y^d_t) + r^i \\
+ \lambda_i \left[ V^j (y_t; y^*_t, y^d_t) - V^i (y_t; y^*_t, y^d_t) \right] \\
+ \phi \left[ \min (1, y_t) - V^i (y_t; y^*_t, y^d_t) \right] \\
+ \theta \delta_{1(y_t < y^*_t)} \left[ \min (1, L + l y_t) - V^i (y_t; y^*_t, y^d_t) \right] \\
+ \delta \max_{\text{rollover or run}} \left( 0, 1 - V^i (y_t; y^*_t, y^d_t) \right)
\]

Note that there is a new term \( \lambda_i \left[ V^j (y_t; y^*_t, y^d_t) - V^i (y_t; y^*_t, y^d_t) \right] \) on the right-hand side of this equation, reflecting the change in the value function upon the switching
of the regime.

Figure 5
Time Varying Liquidity Premium $\Delta_\rho$ and Strength of Guarantee $\theta$

The extended model can be similarly solved as the baseline model, but its structural estimation is more challenging because of a much larger number of parameters to estimate. To keep our structural estimation simpler, we only allow two parameters, $\Delta_\rho$ and $\theta$, to change. To simplify it further, we conduct the annual structural estimation to the baseline model year by year. Figure 5 above reports the annual estimates of $\Delta_\rho$ and $\theta$ in Panels A and B, as well as the implied 1-week run probability in Panel C. It is evident from Panel A of Figure 5 that the liquidity premium more than doubled in 2008, while according to Panel B of Figure 5, the strength of the liquidity facility weakened to the lowest level since 2005. These findings suggest that the run-induced spike in September 2008 may well be a liquidity event in the sense that investors demanded a higher liquidity premium to hold the VRDOs following Lehman’s bankruptcy.
Table 2: Liquidity and Credit Components of the MSI-Repo Spread

This table reports OLS regression coefficients of using the MSI-Repo spread as the dependent variable, as well as t-stat in parentheses. In regression (I), the MSI-repo spread is regressed on the estimated fundamental $y_t$. In regression (II), the MSI-repo spread is regressed on the estimated fundamental $y_t$ and the liquidity premium $\Delta_{ho}(t)$. In regression (III), the MSI-repo spread is regressed on the estimated fundamental $y_t$, the liquidity premium $\Delta_{ho}(t)$, as well as the strength of the liquidity facility $\theta(t)$. The estimates of the fundamental $y_t$, the liquidity premium $\Delta_{ho}(t)$, as well as the strength of the liquidity facility $\theta(t)$ are done by the method of simulated maximum likelihood (SML). The data are the MSI rate as well as weekly repo rate from May 1991 to October 2012.

<table>
<thead>
<tr>
<th></th>
<th>(I)</th>
<th>(II)</th>
<th>(III)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y(t)$</td>
<td>-0.0781</td>
<td>-0.0781</td>
<td>-0.0789</td>
</tr>
<tr>
<td></td>
<td>(-46.8)</td>
<td>(-70.8)</td>
<td>(-72.2)</td>
</tr>
<tr>
<td>$\Delta_{</td>
<td></td>
<td></td>
<td></td>
</tr><tr>
<td>ho}(t)$</td>
<td>1.0954</td>
<td>0.8882</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(38.1)</td>
<td>(20.2)</td>
<td></td>
</tr>
<tr>
<td>$\theta(t)$</td>
<td></td>
<td>0.0210</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(6.2)</td>
<td></td>
</tr>
<tr>
<td>Const.</td>
<td>0.0775</td>
<td>0.0742</td>
<td>0.0743</td>
</tr>
<tr>
<td></td>
<td>(49.6)</td>
<td>(71.7)</td>
<td>(73.0)</td>
</tr>
<tr>
<td>$R^2(%)$</td>
<td>66.2</td>
<td>85.3</td>
<td>85.8</td>
</tr>
</tbody>
</table>

The structural estimation provides a clean identification of liquidity and credit components of the MSI-repo spread. We run the OLS regressions of the MSI-repo spread on the estimated liquidity and credit components. The regression results are reported in Table 2. From Table 2, we can see that a higher liquidity premium or a weaker liquidity support in general leads to a higher MSI-repo spread. The regression results of Column (I) suggest that the credit component alone accounts for about
66.2% of the time variation in the MSI-repo spread, while the results in Column (II) suggest that the liquidity premium has further explanatory power in explaining the time variation by an additional 19%. The time variation in the strength of the liquidity facility seems to explain a little, about 0.5% on top of what can be explained by the liquidity and credit components.

6 Conclusions

We develop a dynamic model of VRDO as well as the endogenous decision of creditors to run. The model captures VRDO’s distinctive characteristics, such as the tender option, the time-varying fundamental-driven interest rate, and the pre-specified maximum interest rate. In a regime-switching setting, the model also allows for time-varying liquidity premium, asset volatility, as well as the strength of the liquidity facility, etc. Using the weekly data of the Municipal Swap Index and the 1-week repo rate, we structural estimate the model using the method of simulated maximum likelihood. Based on the structural estimation results, we show that the model singles out the most recent episode in the first several weeks following Lehman’s bankruptcy in 2008. Furthermore, we show that the run-induced spike in September, 2008 is largely due to the increased liquidity premium demanded by investors. By regressing the MSI-repo spread on the liquidity and credit components, we also find that the credit component account for the most of the time variation in the MSI-repo spread, about 66%, and the liquidity component explains about an additional 19%.

References


Han, Song, and Dan Li, 2009, Liquidity crisis, runs, and security design — Lessons from the collapse of the auction rate securities market, Federal Reserve Board Working Paper.


Appendix A: Notation

We denote by $-\gamma_1$ and $\eta_1$ two real roots of the quadratic equation $\frac{1}{2}\sigma^2x(x - 1) + \mu x - (\rho + \phi + \delta (1 + \theta)) = 0$

$$-\gamma_1 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 [\rho + \phi + \delta (1 + \theta)]}}{\sigma^2} < 0,$$

$$\eta_1 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 [\rho + \phi + \delta (1 + \theta)]}}{\sigma^2} > 0,$$

and denote by $-\gamma_2$ and $\eta_2$ two real roots of the quadratic equation $\frac{1}{2}\sigma^2x(x - 1) + \mu x - (\rho + \phi) = 0$

$$-\gamma_2 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 (\rho + \phi)}}{\sigma^2} < 0,$$

$$\eta_2 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 (\rho + \phi)}}{\sigma^2} > 0.$$

The following notation is used in determining equilibrium threshold $y_*$:

$$K_1 = \frac{\tau + \delta (1 + \theta L)}{\rho + \phi + \delta (1 + \theta)}$$

$$K_2 = \frac{\phi + \delta \theta l}{\rho + \phi + \delta (1 + \theta) - \mu}$$

$$K_3 = \frac{\tau + \phi + \delta (1 + \theta L)}{\rho + \phi + \delta (1 + \theta)}$$

$$K_4 = \frac{\theta \delta l}{\rho + \phi + \delta (1 + \theta) - \mu}$$

$$K_5 = \frac{\tau}{\rho + \phi}$$

$$K_6 = \frac{\phi}{\rho + \phi - \mu}$$

$$K_7 = \frac{\overline{\rho} + \phi + \delta (1 + \theta)}{\rho + \phi + \delta (1 + \theta)}$$

$$K_8 = \frac{\overline{\rho} + \phi}{\rho + \phi}$$

where

$$\overline{\rho} = \rho + \Delta_\rho.$$
Thus, where the unknown coefficients can be solved by the value-matching and smooth-
pasting conditions as

\[
A_1 = \frac{-\gamma_1 A_2 y_*^{\gamma_1 - 1} + \eta_1 A_3 y_*^{\gamma_1 - 1} - K_2}{\eta_1 y_*^{\gamma_1 - 1}}
\]

\[
A_2 = \frac{\eta_1 (K_1 - K_7) + (\eta_1 - 1) K_2 y_*^{\gamma_1}}{(\eta_1 + \gamma_1) y_*^{\gamma_1}}
\]

\[
A_3 = \frac{\gamma_2 (K_8 - K_7) + (\gamma_2 - \gamma_1) A_2 y_*^{\gamma_1}}{(\eta_1 + \gamma_2) y_*^{\gamma_2}}
\]

\[
A_4 = A_2 y_*^{\gamma_1} + A_3 y_*^{\gamma_1 + \gamma_2} + (K_7 - K_8) y_*^{\gamma_2}
\]

**Case B:** \(y_* \leq 1\) and \(\overline{p} + \phi (1 - y_*) \leq \tau \leq \overline{p} + \phi (1 - y_*) + \theta \delta (1 - [L + ly_*])\)

In this case,

\[
r_t = \begin{cases} 
\tau, & \text{if } 0 < y_t \leq y_*; \\ 
\overline{p} + \phi (1 - y), & \text{if } y_* < y_t \leq 1; \\ 
\overline{p}, & \text{if } y_t > 1. 
\end{cases}
\]

It implies

\[
\rho V - \mu y V_y - \frac{\sigma^2}{2} y^2 V_{yy}
\]

\[
= \begin{cases} 
\tau + \phi (y - V) + \theta \delta (L + ly - V) + \delta (1 - V), & \text{if } y \in (0, y_*] \\
\overline{p} + \phi (1 - y) + \phi (y - V), & \text{if } y \in (y_*, 1] \\
\overline{p} + \phi (1 - V), & \text{if } y \in (1, \infty)
\end{cases}
\]

\[
= \begin{cases} 
\tau + \delta (1 + \theta L) + (\phi + \theta \delta \theta) y - (\phi + \delta (1 + \theta)) V, & \text{if } y \in (0, y_*] \\
\overline{p} + \phi (1 - V), & \text{if } y \in (y_*, \infty)
\end{cases}
\]

Thus

\[
V = \begin{cases} 
\frac{\tau + \delta (1 + \theta L)}{\rho + \phi + \delta (1 + \theta)} + \frac{\phi + \delta \theta \overline{p} + B_1 y^* + B_1 y^*}{\rho + \phi + \delta (1 + \theta) - \mu} y + B_1 y^*, & \text{if } y \in (0, y_*] \\
\frac{\overline{p} + \phi}{\rho + \phi + \delta (1 + \theta) - \mu} + B_2 y^* - \gamma_2, & \text{if } y \in (y_*, \infty)
\end{cases}
\]

\[
= \begin{cases} 
K_1 + K_2 y + B_1 y^* + B_1 y^*, & \text{if } y \in (0, y_*] \\
K_8 + B_2 y^* - \gamma_2, & \text{if } y \in (y_*, \infty)
\end{cases}
\]

where the unknown coefficients can be solved by the value-matching and smooth-pasting conditions as

\[
B_1 = -\frac{\gamma_2 (K_1 - K_8) + (\gamma_2 + 1) K_2 y_*}{(\eta_1 + \gamma_2) y_*^{\gamma_1}}
\]

\[
B_2 = -\frac{K_2 + \eta_1 B_1 y_*^{\gamma_1 - 1}}{\gamma_2 y_*^{\gamma_2 - 1}}
\]
Case C: \( y_s \leq 1 \) and \( \tau < \bar{\tau} + \phi (1 - y_s) \)

Denote

\[
y_{**} = \frac{\bar{\tau} + \phi - \tau}{\phi} \in (y_*, 1)
\]

In this case,

\[
r_t = \begin{cases} 
\tau, & \text{if } 0 < y_t \leq y_s; \\
\bar{\tau}, & \text{if } y_s < y_t \leq y_{**}; \\
\bar{\tau} + \phi (1 - y), & \text{if } y^* < y_t \leq 1; \\
\bar{\tau}, & \text{if } y_t > 1.
\end{cases}
\]

It implies

\[
\rho V - \mu y V_y - \frac{\sigma^2}{2} y^2 V_{yy} = \begin{cases} 
\tau + \phi (y - V) + \theta \delta (L + l y - V) + \delta (1 - V), & \text{if } y \in (0, y_s] \\
\bar{\tau} + \phi (y - V), & \text{if } y \in (y_s, y_{**}] \\
\bar{\tau} + \phi (1 - y) + \phi (y - V), & \text{if } y \in (y_{**}, 1] \\
\bar{\tau} + \phi (1 - V), & \text{if } y \in (1, \infty)
\end{cases}
\]

Thus

\[
V = \begin{cases} 
\frac{\tau + \phi (1 + \theta L)}{\rho + \phi + \delta (1 + \theta)} + \frac{\phi + \theta \delta l}{\rho + \phi + \delta (1 + \theta) - \mu} y + C_1 y^\gamma, & \text{if } y \in (0, y_s] \\
\frac{\tau + \phi}{\rho + \phi} y + C_2 y^{-\gamma_2} + C_3 y^{\gamma_2}, & \text{if } y \in (y_s, y_{**}] \\
\frac{\tau + \phi}{\rho + \phi} + C_4 y^{-\gamma_2}, & \text{if } y \in (y_{**}, \infty)
\end{cases}
\]

where the unknown coefficients can be solved by the value-matching and smooth-
pasting conditions as

\[
C_1 = \frac{\gamma_2 (K_5 - K_1) + (\gamma_2 + 1) (K_6 - K_2) y_* + (\eta_2 + \gamma_2) C_3 y_*^{\eta_2}}{(\eta_1 + \gamma_2) y_1^{\eta_1}}
\]

\[
C_2 = \frac{(K_6 - K_2) + \eta_2 C_3 y_*^{\eta_2 - 1} - \eta_1 C_1 y_*^{\eta_1 - 1}}{\gamma_2 y_*^{\gamma_2 - 1}}
\]

\[
C_3 = -\frac{\gamma_2 (K_5 - K_8) + (\gamma_2 + 1) K_6 y_*}{(\eta_2 + \gamma_2) y_*^{\eta_2}}
\]

\[
C_4 = -\frac{K_6 - \gamma_2 C_2 y_*^{\eta_2 - 1} + \eta_2 C_3 y_*^{\eta_2 - 1}}{\gamma_2 y_*^{\gamma_2 - 1}}
\]

**Case D:** \( 1 < y_* < \frac{1 - L}{l} \) and \( \tau \geq \bar{\tau} + \theta \delta (1 - L - l) \)

Define \( y_* \) as in Case A:

\[
y_* = \frac{\bar{\tau} + \phi + \theta \delta (1 - L) - \bar{\tau}}{\phi + \theta \delta l}
\]

In this case

\[
r_t = \begin{cases} 
\tau, & \text{if } 0 < y_t \leq y_*; \\
\bar{\tau} + \phi (1 - y) + \theta \delta (1 - [L + l y]), & \text{if } y_* < y_t \leq 1; \\
\bar{\tau} + \theta \delta (1 - [L + l y]), & \text{if } 1 < y_t \leq y_*; \\
\bar{\tau}, & \text{if } y_* < y_t \leq 1 - \frac{L}{l}; \\
\bar{\tau}, & \text{if } y_t > 1 - \frac{L}{l}.
\end{cases}
\]

It implies

\[
\rho V - \mu y V_y - \frac{c^2}{2} y^2 V_{yy} = \begin{cases} 
\tau + \phi (y - V) + \theta \delta (L + l y - V) + \delta (1 - V), & \text{if } y \in (0, y_*] \\
\bar{\tau} + \phi (1 - y) + \theta \delta (1 - [L + l y]), & \text{if } y \in (y_*, 1] \\
\bar{\tau} + \theta \delta (1 - [L + l y]) + \phi (1 - V) + \delta (1 - V), & \text{if } y \in (1, y_*] \\
\bar{\tau} + \phi (1 - V), & \text{if } y \in (y_*, \frac{1 - L}{l}] \\
\bar{\tau} + \phi (1 - V), & \text{if } y \in (\frac{1 - L}{l}, \infty)
\end{cases}
\]

\[
= \begin{cases} 
\tau + \delta (1 + \theta L) + (\phi + \theta \delta l) y - (\phi + \delta (1 + \theta)) V, & \text{if } y \in (0, y_*] \\
\bar{\tau} + (\phi + \delta (1 + \theta)) (1 - V), & \text{if } y \in (y_*, y_*] \\
\bar{\tau} + \phi (1 - V), & \text{if } y \in (y_*, \infty)
\end{cases}
\]
Thus,

\[
V = \begin{cases}
\frac{\phi + \delta l}{\rho + \phi + \delta (1 + \rho)} + \frac{\phi + \delta l}{\rho + \phi + \delta (1 + \rho)} y + D_1 y^m, & \text{if } y \in (0, y_{**}] \\
\frac{\phi + \delta l}{\rho + \phi + \delta (1 + \rho)} + D_2 y^{-\gamma_1} + D_3 y^m, & \text{if } y \in (y_{**}, y_*) \\
\frac{\rho}{\rho + \phi} + D_4 y^{-\gamma_2}, & \text{if } y \in (y_*, \infty)
\end{cases}
\]

= \begin{cases}
K_1 + K_2 y + D_1 y^m, & \text{if } y \in (0, y_{**}] \\
K_7 + D_2 y^{-\gamma_1} + D_3 y^m, & \text{if } y \in (y_{**}, y_*) \\
K_8 + D_4 y^{-\gamma_2}, & \text{if } y \in (y_*, \infty)
\end{cases}

where the unknown coefficients can be solved by the value-matching and smooth-pasting conditions as

\[
\begin{align*}
D_1 &= \frac{-\gamma_1 D_2 y_{**}^{-\gamma_1} + \eta_1 D_3 y_{**}^{-1} - K_2}{\eta_1 y_{**}^{-1}} \\
D_2 &= \frac{\eta_1 (K_1 - K_7) + (\eta_1 - 1) K_2 y_{**}}{(\eta_1 + \gamma_1) y_{**}^{-\gamma_1}} \\
D_3 &= \gamma_2 (K_8 - K_7) + (\gamma_1 - \gamma_2) D_2 y_{**}^{-\gamma_1} \\
D_4 &= D_2 y_{**}^{-\gamma_2} + D_3 y_{**}^{-1} + (K_7 - K_8) y_{**}^{-\gamma_2}
\end{align*}
\]

**Case E:** \(1 < y_* < \frac{1-l}{l}\) and \(\rho + \theta \delta (1 - L - ly_*) < \rho < \rho + \theta \delta (1 - L - l)\)

Define

\[
y_{**} = \frac{\rho + \theta \delta (1 - L - \rho)}{\theta \delta l} \in (1, y_*)
\]

In this case,

\[
r_t = \begin{cases}
\rho, & \text{if } 0 < y_t \leq 1; \\
\rho, & \text{if } 1 < y_t \leq y_{**}; \\
\rho + \theta \delta (1 - [L + ly_t]), & \text{if } y_{**} < y_t \leq y_*; \\
\bar{\rho}, & \text{if } y_* < y_t \leq \frac{1-L}{l}; \\
\bar{\rho}, & \text{if } y_t > \frac{1-L}{l}.
\end{cases}
\]
It implies

$$\rho V - \mu y V'_y - \frac{\sigma^2}{2} y^2 V''_y = \begin{cases} 
\mathcal{P} + \phi (y - V) + \theta \delta (L + ly - V) + \delta (1 - V), & \text{if } y \in (0, 1] \\
\mathcal{P} + \phi (1 - V) + \theta \delta (L + ly - V) + \delta (1 - V), & \text{if } y \in (1, y_*) \\
\mathcal{P} + \theta \delta (1 - [L + ly]) + \phi (1 - V) + \theta \delta (L + ly - V) + \delta (1 - V), & \text{if } y \in (y_*, y_*) \\
\mathcal{P} + \phi (1 - V), & \text{if } y \in (y_*, \infty) 
\end{cases}$$

Thus,

$$V = \begin{cases} 
\frac{\tau + \delta (1 + \theta L)}{\rho \phi + \delta (1 + \theta)} + \frac{\phi + \theta \delta}{\rho \phi + \delta (1 + \theta) - \mu} y + E_1 y^n, & \text{if } y \in (0, 1] \\
\frac{\tau + \delta (1 + \theta L)}{\rho \phi + \delta (1 + \theta) - \mu} y + E_2 y^{-\gamma_1} + E_3 y^{n_1}, & \text{if } y \in (1, y_*) \\
\frac{\tau + \delta (1 + \theta L)}{\rho \phi + \delta (1 + \theta)} + E_4 y^{-\gamma_1} + E_5 y^{n_1}, & \text{if } y \in (y_*, y_*) \\
\frac{\tau + \delta (1 + \theta L)}{\rho \phi + \delta (1 + \theta)} + E_6 y^{-\gamma_2}, & \text{if } y \in (y_*, \infty) 
\end{cases}$$

where the unknown coefficients can be solved by the value-matching and smooth-pasting conditions as

$$\begin{align*}
E_1 &= (K_3 + K_4) - (K_1 + K_2) + (E_2 + E_3) \\
E_2 &= -\frac{(K_4 - K_2) (\eta_1 - 1) + \eta_1 (K_3 - K_1)}{\eta_1 + \gamma_1} \\
E_3 &= -\frac{K_4 + \gamma_1 (E_4 - E_2) y_0^{\gamma_1-1} - \eta_1 E_5 y_0^{n_1-1}}{(\eta_1 + \gamma_1) y_0^{n_1-1}} \\
E_4 &= E_2 + \frac{\eta_1 (K_3 - K_7) + (\eta_1 - 1) K_4 y_*}{(\eta_1 + \gamma_1) y_*^{\gamma_1}} \\
E_5 &= \frac{\gamma_2 (K_8 - K_7) + (\gamma_1 - \gamma_2) E_4 y_*^{-\gamma_1}}{(\eta_1 + \gamma_2) y_*^{n_1}} \\
E_6 &= \frac{(K_7 - K_8) + E_4 y_*^{-\gamma_1} + E_5 y_*^{n_1}}{y_*^{-\gamma_2}}
\end{align*}$$
Case F: $1 < y_s < \frac{1-L}{\lambda}$ and $\tau \leq \rho + \theta \delta (1 - L - l y_s)$

In this case, 

$$r_1 = \begin{cases} 
\tau, & \text{if } 0 < y_t \leq 1; \\
\tau, & \text{if } 1 < y_t \leq y_s; \\
\bar{\rho}, & \text{if } y_s < y_t \leq \frac{1-L}{\lambda}; \\
\bar{\rho}, & \text{if } y_t > \frac{1-L}{\lambda}.
\end{cases}$$

It implies 

$$\rho V - \mu y V_y - \frac{\sigma^2}{2} y^2 V_{yy}$$

$$= \begin{cases} 
\tau + \phi (y - V) + \theta \delta (L + Ly - V) + \delta (1 - V), & \text{if } y \in (0, 1] \\
\tau + \phi (1 - V) + \theta \delta (L + Ly - V) + \delta (1 - V), & \text{if } y \in (1, y_s] \\
\bar{\rho} + \phi (1 - V), & \text{if } y \in (y_s, \infty)
\end{cases}$$

Thus, 

$$V = \begin{cases} 
\frac{\tau + \phi (1 + \theta L)}{\rho + \phi + \delta (1 + \theta)} + \frac{\phi + \theta \delta l}{\rho + \phi + \delta (1 + \theta)} y + F_1 y^{n_1}, & \text{if } y \in (0, 1] \\
\frac{\tau + \phi (1 + \theta L)}{\rho + \phi + \delta (1 + \theta)} + \frac{\phi + \theta \delta l}{\rho + \phi + \delta (1 + \theta)} y + F_2 y^{-\gamma_1} + F_3 y^{n_1}, & \text{if } y \in (1, y_s] \\
F_4 y^{-\gamma_2}, & \text{if } y \in (y_s, \infty)
\end{cases}$$

where the unknown coefficients can be solved by the value-matching and smooth-pasting conditions as 

$$F_1 = (K_3 + K_4) - (K_1 + K_2) + F_2 + F_3$$

$$F_2 = -\frac{\eta_1 (K_3 - K_1) + (\gamma_1 - 1) (K_4 - K_2)}{\eta_1 + \gamma_1}$$

$$F_3 = -\frac{\gamma_2 (K_3 - K_8) - (\gamma_2 + 1) K_4 y_s + (\gamma_1 - \gamma_2) F_2 y_s^{-\gamma_1}}{(\eta_1 + \gamma_2) y_s^{n_1}}$$

$$F_4 = -\frac{K_4 - \gamma_1 F_2 y_s^{-\gamma_1 - 1} + \eta_1 F_3 y_s^{n_1 - 1}}{\gamma_2 y_s^{\gamma_2 - 1}}$$

Case G: $y_s \geq \frac{1-L}{\lambda}$ and $\tau \geq \rho + \theta \delta (1 - L - l)$
Define $y_{**}$ as in Case A:

$$y_{**} = \frac{\bar{p} + \phi + \theta \delta (1 - L) - \bar{r}}{\phi + \theta \delta l}$$

In this case

$$r_t = \begin{cases} 
    \tau, & \text{if } 0 < y_t \leq y_{**}; \\
    \bar{p} + \phi (1 - y) + \theta \delta (1 - [L + ly]) , & \text{if } y_{**} < y_t \leq 1; \\
    \bar{p} + \theta \delta (1 - [L + ly]) , & \text{if } 1 < y_t \leq \frac{1-L}{L}; \\
    \bar{p}, & \text{if } \frac{1-L}{L} < y_t \leq y_{**}; \\
    \bar{p}, & \text{if } y_t > y_{**}.
\end{cases}$$

It implies

$$\rho V - \mu yV_y - \frac{\sigma^2}{2} y^2 V_{yy} = \begin{cases} 
    \tau + \phi (y - V) + \theta \delta (L + ly - V) + \delta (1 - V) , & \text{if } y \in (0, y_{**}] \\
    \bar{p} + \phi (1 - y) + \theta \delta (1 - [L + ly]) + \phi (y - V) + \theta \delta (L + ly - V) + \delta (1 - V) , & \text{if } y \in (y_{**}, 1] \\
    \bar{p} + \phi (1 - V) + \theta \delta (1 - V) + \delta (1 - V) , & \text{if } y \in (1, \frac{1-L}{L}] \\
    \bar{p} + \phi (1 - V) , & \text{if } y \in (y_{**}, y_{*}] \\
    \bar{p} + \phi (1 - V) , & \text{if } y \in (y_{*}, \infty)
\end{cases}$$

Thus,

$$V = \begin{cases} 
    \frac{\tau + \delta (1 + \theta L)}{\rho + \rho + \delta (1 + \theta)} + \frac{\phi + \delta \theta l}{\rho + \rho + \delta (1 + \theta)} - \mu y + G_1 y^n , & \text{if } y \in (0, y_{**}] \\
    \bar{p} + \phi + \delta (1 + \theta) y + G_2 y^{-\gamma_1} + G_3 y^n , & \text{if } y \in (y_{**}, y_{*}] \\
    \bar{p} + \phi + \delta (1 + \theta) y - \frac{\phi + \delta (1 + \theta) y}{\rho + \rho + \delta (1 + \theta)} - \mu y + G_4 y^{-\gamma_2} , & \text{if } y \in (y_{*}, \infty)
\end{cases}$$

$$= \begin{cases} 
    K_1 + K_2 y + G_1 y^n , & \text{if } y \in (0, y_{**}] \\
    K_7 + G_2 y^{-\gamma_1} + G_3 y^n , & \text{if } y \in (y_{**}, y_{*}] \\
    K_8 + G_4 y^{-\gamma_2} , & \text{if } y \in (y_{*}, \infty)
\end{cases}$$

where the unknown coefficients can be solved by the value-matching and smooth-
pasting conditions as

\[

g_1 = \frac{-\gamma_1 g_2 y^{* \gamma_1 - 1} + \eta_1 g_3 y^{* \gamma_1 - 1} - K_2}{\eta_1 y^{* \gamma_1 - 1}}
\]

\[

g_2 = \frac{\eta_1 (K_1 - K_7) + (\eta_1 - 1) K_2 y^{* \gamma_1}}{(\eta_1 + \gamma_1) y^{* \gamma_1}}
\]

\[

g_3 = \frac{\gamma_2 (K_8 - K_7) + (\gamma_1 - \gamma_2) g_2 y^{* \gamma_1}}{(\eta_1 + \gamma_2) y^{\eta_1}}
\]

\[

g_4 = G_2 y^{* \gamma_1 + \gamma_2} + G_3 y^{\eta_1 + \gamma_2} + (K_7 - K_8) y^{2}
\]

**Case H:** \( y_* \geq \frac{1-L}{L} \) and \( \tau < \bar{\tau} + \theta \delta (1 - L - l) \)

Define

\[
y^{* \gamma_1} = \frac{\bar{\tau} + \theta \delta (1 - L) - \tau}{\theta \delta l} \in \left[ 1, \frac{1-L}{L} \right]
\]

In this case,

\[
r_l = \left\{\begin{array}{ll}
\tau, & \text{if } y_t < 1 \\
\bar{\tau}, & \text{if } 1 \leq y_t \leq y_{**} \\
\bar{\tau} + \theta \delta (1 - [L + l y_t]), & \text{if } y_{**} < y_t \leq \frac{1-L}{L} \\
\bar{\tau}, & \text{if } \frac{1-L}{L} < y_t \leq y_*
\end{array}\right.
\]

It implies

\[
pV - \mu y V_y - \frac{\sigma^2}{2} y^2 V_{yy} = \left\{\begin{array}{ll}
\tau + \phi (y - V) + \theta \delta (L + ly - V) + \delta (1 - V), & \text{if } y \in (0, 1] \\
\tau + \phi (1 - V) + \theta \delta (L + ly - V) + \delta (1 - V), & \text{if } y \in (1, y_{**}] \\
\bar{\tau} + \phi (1 - V) + \theta \delta (1 - V) + \delta (1 - V), & \text{if } y \in (\frac{1-L}{L}, y_{*}] \\
\bar{\tau} + \phi (1 - V), & \text{if } y \in (y_*, \infty)
\end{array}\right.
\]

\[
\tau + \delta (1 + \theta L) + (\phi + \theta \delta l) y - (\phi + \delta (1 + \theta)) V, & \text{if } y \in (0, 1] \\
\tau + \phi + \delta (1 + \theta L) + \theta \delta ly - (\phi + \delta (1 + \theta)) V, & \text{if } y \in (1, y_{**}] \\
\bar{\tau} + (\phi + \delta (1 + \theta)) (1 - V), & \text{if } y \in (y_{**}, y_{*}] \\
\bar{\tau} + \phi (1 - V), & \text{if } y \in (y_*, \infty)
\]

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Thus,

\[
V = \begin{cases} 
\frac{\tau + \delta(1+\theta)}{\rho + \phi + \delta(1+\theta)} y + H_1 y^{\eta_1}, & \text{if } y \in (0, 1) \\
\frac{\tau + \phi + \delta(1+\theta)}{\rho + \phi + \delta(1+\theta)} y + H_2 y^{\gamma_1} + H_3 y^{\eta_1}, & \text{if } y \in (1, y^*) \\
\frac{\tau + \phi + \delta(1+\theta)}{\rho + \phi + \delta(1+\theta)} y^{\gamma_1} + H_5 y^{\eta_1}, & \text{if } y \in (y^*, y_*] \\
K_1 + K_2 y + H_1 y^{\eta_1}, & \text{if } y \in (0, 1] \\
K_3 + K_4 y + H_2 y^{\gamma_1} + H_3 y^{\eta_1}, & \text{if } y \in (1, y^*) \\
K_7 + H_4 y^{\gamma_1} + H_5 y^{\eta_1}, & \text{if } y \in (y^*, y_*] \\
K_8 + H_6 y^{\gamma_2}, & \text{if } y \in (y_*, \infty) 
\end{cases}
\]

where the unknown coefficients can be solved by the value-matching and smooth-pasting conditions as

\[
H_1 = (K_3 + K_4) - (K_1 + K_2) + (H_2 + H_3) \\
H_2 = -\frac{(K_4 - K_2)(\eta_1 - 1) + \eta_1 (K_3 - K_1)}{\eta_1 + \gamma_1} \\
H_3 = -\frac{K_4 + \gamma_1 (H_4 - H_2) y_*^{\gamma_1 - 1} - \eta_1 y_*^{\eta_1 - 1}}{\eta_1 y_*^{\eta_1 - 1}} \\
H_4 = H_2 + \eta_1 (K_3 - K_7) + (\eta_1 - 1) K_4 y_*^{\gamma_1} \\
H_5 = \frac{\gamma_2 (K_8 - K_7) + (\gamma_1 - \gamma_2) H_4 y_*^{\gamma_1}}{(\eta_1 + \gamma_1) y_*^{\eta_1}} \\
H_6 = \frac{(K_7 - K_8) + H_4 y_*^{\gamma_1} + H_5 y_*^{\eta_1}}{y_*^{\gamma_2}}
\]
Appendix C: Proofs

Lemma 1 is used in determining the equilibrium VRDO rate in Proposition 1.

**Lemma 1** Suppose \( \{y_t\} \) is a Geometric brownian motion process satisfying Eq. (1), then given positive constants \( L, l > 0 \), the following is true

\[
\lim_{\Delta t \to 0} E \left[ \min \left( 1, L + ly_{t+\Delta t} \right) \right] = \min \left( 1, L + ly_t \right).
\]

**Proof of Lemma 1.** First, note that

\[
E \left[ \min \left( 1, L + ly_{t+\Delta t} \right) \right] = P_r \left( L + ly_{t+\Delta t} > 1 \right) + E \left[ \left( L + ly_{t+\Delta t} \right) 1_{\{L + ly_{t+\Delta t} \leq 1\}} \right].
\]

Because

\[
y_{t+\Delta t} = y_t \exp \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (Z_{t+\Delta t} - Z_t) \right]
\]

we have

\[
P_r \left( L + ly_{t+\Delta t} > 1 \right)
= P_r \left( y_t \exp \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (Z_{t+\Delta t} - Z_t) \right] > \frac{1 - L}{l} \right)
= P_r \left( \sigma (Z_{t+\Delta t} - Z_t) > \log \left( \frac{1 - L}{l} \right) - \log (y_t) - \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t \right)
= \Phi \left( \frac{\log (y_t) - \log \left( \frac{1 - L}{l} \right) + \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t}{\sigma \sqrt{\Delta t}} \right)
\]

and

\[
E \left[ y_{t+\Delta t} 1_{\{L + ly_{t+\Delta t} \leq 1\}} \right]
= E \left[ y_t \exp \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (Z_{t+\Delta t} - Z_t) \right] \right] 1_{\{\sigma (Z_{t+\Delta t} - Z_t) \leq \log \left( \frac{1 - L}{l} \right) - \log (y_t) - \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t \}}
= y_t \exp \left[ \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t \right] \int_{-\infty}^{\log \left( \frac{1 - L}{l} \right) - \log (y_t) - \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{z^2}{2} \right] \, dz
= y_t \exp (\mu \Delta t) \Phi \left( \frac{\log \left( \frac{1 - L}{l} \right) - \log (y_t) - \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t}{\sigma \sqrt{\Delta t}} - \frac{\sigma \sqrt{\Delta t}}{2} \right)
\]
Therefore,

\[
E \left[ \min \left( 1, L + l y_t + \Delta t \right) \right] = \Phi \left( \frac{\log (y_t) - \log \left( \frac{1-L}{L} \right) + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t}{\sigma \sqrt{\Delta t}} \right) + \frac{\log (y_t) - \log \left( \frac{1-L}{L} \right) + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t}{\sigma \sqrt{\Delta t}} \right] + l y_t \exp (\mu \Delta t) \Phi \left( \frac{\log \left( \frac{1-L}{L} \right) - \log (y_t) - \left( \mu + \frac{\sigma^2}{2} \right) \Delta t}{\sigma \sqrt{\Delta t}} \right).
\]

Because

\[
\lim_{\Delta t \to 0} \Phi \left( \frac{\log (y_t) - \log \left( \frac{1-L}{L} \right) + \left( \mu - \frac{\sigma^2}{2} \right) \Delta t}{\sigma \sqrt{\Delta t}} \right) = \begin{cases} 
0, & \text{if } y_t < \frac{1-L}{L} \\
\frac{1}{2}, & \text{if } y_t = \frac{1-L}{L} \\
1, & \text{if } y_t > \frac{1-L}{L}
\end{cases},
\]

\[
\lim_{\Delta t \to 0} \Phi \left( \frac{\log \left( \frac{1-L}{L} \right) - \log (y_t) - \left( \mu + \frac{\sigma^2}{2} \right) \Delta t}{\sigma \sqrt{\Delta t}} \right) = \begin{cases} 
1, & \text{if } y_t < \frac{1-L}{L} \\
\frac{1}{2}, & \text{if } y_t = \frac{1-L}{L} \\
0, & \text{if } y_t > \frac{1-L}{L}
\end{cases},
\]

therefore, we have

\[
\lim_{\Delta t \to 0} E \left[ \min \left( 1, L + l y_t + \Delta t \right) \right] = \begin{cases} 
\frac{1}{2} + L \frac{1}{2} + l y_t \frac{1}{2} = 1, & \text{if } y_t = \frac{1-L}{L} \\
1, & \text{if } y_t > \frac{1-L}{L}
\end{cases}.
\]

ProofofProposition1. Consider the case \( y_t > y_* \) first. For small enough \( \Delta t \), with probability \( \phi \Delta t \), the debt will mature with face value \( y_r \); and with probability \( 1 - \phi \Delta t \), the debt will not mature and is paid off at par value $1$. From Lemma 1 and Eqs. (5)-(6), and letting \( \Delta t \) be small enough, we have if \( y_t > y_* \)

\[
1 \approx (r_t - \Delta \rho) \Delta t + (1 - \phi \Delta t) (1 - \rho \Delta t) + \phi \Delta t \left( \lim_{\Delta t \to 0} E \left[ \min \left( 1, y_t + \Delta t \right) \right] \right),
\]
or \( r_t = \rho + \Delta \rho + \phi (1 - y_t)^+ \), if \( y_t > y_* \), where \((x)^+\) denotes \(x\), if \(x > 0\), or zero, otherwise.

The case \( y_t \leq y_* \) is similar, and

\[
1 \approx (r_t - \Delta \rho) \Delta t + (1 - (\phi + \theta \delta) \Delta t) (1 - \rho \Delta t) + \phi \Delta t \left( \lim_{\Delta t \to 0} E \left[ \min (1, y_{r_t}) \right] \right) \\
+ \theta \delta \Delta t \left( \lim_{\Delta t \to 0} E \left[ \min (1, L + l y_{r_t}) \right] \right),
\]

implying \( \rho + \Delta \rho + \phi (1 - y_t)^+ + \theta \delta (1 - \left| L + l y_t \right|)^+ \), if \( y_t \leq y_* \). 

Before we prove Theorem 1, we prove the following lemma first, which is needed in proving the theorem.

We want to prove that under the restriction in (10), \( W_B (0) < 1 \).

Note

\[
(W_B (0) - 1) (\eta_1 + \gamma_2) \\
= \frac{\eta_1}{\rho + \phi + \delta (1 + \theta)} + \gamma_2 \rho - \rho \\
= \frac{\eta_1}{\rho + \phi + \delta (1 + \theta)} + \gamma_2 \rho - \rho \\
= \left( \rho - \rho \right) \left( \frac{\eta_1}{\rho + \phi + \delta (1 + \theta)} + \gamma_2 \rho - \rho \right) - \frac{\eta_1}{\rho + \phi + \delta (1 + \theta)} - \frac{\eta_1}{\rho + \phi + \delta (1 + \theta)}
\]

which is strictly negative under the restriction in Eq. (10).

Furthermore, if \( \rho < \rho + \phi \), then \( W_C (0) < W_B (0) < 1 \).

The following lemmas are also needed.

**Lemma 2** For \( \eta_1 \), defined in Appendix A, the following statement is true under the restrictions on parameter values:

\[
\frac{\eta_1}{\rho + \phi + \delta (1 + \theta)} - \frac{ \eta_1 - 1}{\rho + \phi + \delta (1 + \theta) - \mu} > 0
\]

**Proof of Lemma 2.** It is equivalent to proving the following inequality

\[
\frac{\rho + \phi + \delta (1 + \theta)}{\mu} > \eta_1 = \sqrt{\frac{\frac{1}{2} \sigma^2 - \mu}{\sigma^2}} + 2 \sigma^2 \left[ \rho + \phi + \delta (1 + \theta) \right] - \left( \mu - \frac{1}{2} \sigma^2 \right)
\]

which holds as long as \( \rho + \phi + \delta (1 + \theta) > \mu \). Therefore, under the restriction \( \rho + \phi > \mu \), the statement indeed holds. 

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Lemma 3 For \( \eta_i \) and \( \gamma_i, \ i = 1, 2, \) defined in Appendix A, the following statement is true under the restrictions on parameter values:

\[
\frac{\eta_1 \gamma_1}{\rho + \phi + \delta (1 + \theta)} = \frac{(\eta_1 - 1)(\gamma_1 + 1)}{\rho + \phi + \delta (1 + \theta) - \mu} = \frac{\eta_2 \gamma_2}{\rho + \phi} = \frac{(\eta_2 - 1)(\gamma_2 + 1)}{\rho + \phi - \mu} = \frac{2}{\sigma^2}
\]

Proof of Lemma 3. Note that \( \eta_1 \gamma_1 = \frac{2}{\sigma^2}(\rho + \phi + \delta (1 + \theta)) \) and \( \eta_1 - \gamma_1 = \frac{2}{\sigma^2}(\mu - \frac{1}{2}\sigma^2) \). Similarly, we can prove \( \frac{\eta_2 \gamma_2}{\rho + \phi} = \frac{(\eta_2 - 1)(\gamma_2 + 1)}{\rho + \phi - \mu} = \frac{2}{\sigma^2} \). Thus, we have \( \frac{\eta_1 \gamma_1}{\rho + \phi + \delta (1 + \theta)} \).

Lemma 4 Under the restrictions on parameter values, the following statements hold:

(i) \( W_A(y) \) is strictly monotonically increasing. Furthermore, \( W_A(0) = \infty \).

(ii) \( W_B(y) \) is strictly monotonically increasing. Furthermore, \( W_B(0) < 1 \).

(iii) If \( \tau < \overline{\tau} + \phi \), \( W_C(y) \) is strictly monotonically increasing for \( y < y_{c*} \). Furthermore, \( W_C(0) < 1 \).

(iv) If \( \tau < \overline{\tau} + \theta \delta (1 - L - l) \), then \( W_E(y) \) is strictly monotonically increasing. Furthermore, \( W_E(\infty) > 1 \).

(v) \( W_F(y) \) is strictly monotonically increasing. Furthermore, \( W_F(0) = \infty \) and \( W_F(\infty) = \infty \).

Proof of Lemma 4. (i) To prove this statement, we only need to prove \( \eta_1 (K_1 - K_\tau) + (\eta_1 - 1) K_2 y_{c*}^4 < 0 \). This indeed holds because

\[
\eta_1 (K_1 - K_\tau) + (\eta_1 - 1) K_2 y_{c*}^4 = - (\overline{\tau} + \phi + \theta \delta (1 - L) - \tau) \left( \frac{\eta_1}{\rho + \phi + \delta (1 + \theta)} - \frac{\eta_1 - 1}{\rho + \phi + \delta (1 + \theta) - \mu} \right) < 0.
\]

(ii) The increasing monotonicity of \( W_B(0) \) is obvious because \( \eta_1 > 1 \) and \( K_2 > 0 \). Furthermore, \( W_B(0) < 1 \) results directly from the restriction in (10).

(iii) First, note that when \( \tau < \overline{\tau} + \phi \), \( \gamma_2 (K_8 - K_3) - (\gamma_2 + 1) K_3 y_{c*}^6 < 0 \), because \( \frac{\gamma_2 (\rho + \phi - \mu)}{(\gamma_2 + 1)(\rho + \phi)} < 1 \), and

\[
\gamma_2 (K_8 - K_3) - (\gamma_2 + 1) K_3 y_{c*}^6 = (\overline{\tau} + \phi - \tau) \frac{\gamma_2 + 1}{\rho + \phi - \mu} \left( \frac{\gamma_2 (\rho + \phi - \mu)}{(\gamma_2 + 1)(\rho + \phi)} - 1 \right) < 0.
\]
Furthermore, when $\tau < \tau + \phi$, for $y < y_*^C$

$$W'_C (y) = \frac{(\eta_1 - 1) K_2 + (\gamma_2 + 1) K_6}{\eta_1 + \gamma_2} + \frac{\gamma_2 (K_8 - K_5) - (\gamma_2 + 1) K_6 y_*^C}{(\eta_1 + \gamma_2) (y_*^C)^{\eta_2}} \eta_2 y_{2-1}$$

$$> \frac{(\eta_1 - 1) K_2 + (\gamma_2 + 1) K_6}{\eta_1 + \gamma_2} + \frac{\gamma_2 (K_8 - K_5) - (\gamma_2 + 1) K_6 y_*^C}{(\eta_1 + \gamma_2) (y_*^C)^{\eta_2}} \eta_2 (y_*^C)^{\eta_2 - 1}$$

$$= \frac{(\eta_1 - 1) K_2 + (\gamma_2 + 1) K_6}{\eta_1 + \gamma_2} + \frac{\phi \eta_2}{\eta_1 + \gamma_2} \gamma_2 - \eta_2 (\gamma_2 + 1) K_6_{\eta_1 + \gamma_2}$$

$$> \frac{(\eta_1 - 1) K_2 + (\gamma_2 + 1) K_6}{\eta_1 + \gamma_2} + \frac{\phi \eta_2}{\eta_1 + \gamma_2} \gamma_2 - \eta_1 (\gamma_2 + 1) K_6_{\eta_1 + \gamma_2}$$

$$= \frac{(\eta_1 - 1) K_2}{\eta_1 + \gamma_2} + \frac{\phi \eta_2}{\eta_1 + \gamma_2} \gamma_2 + \frac{\gamma_2 (\gamma_2 + 1) K_6}{\eta_1 + \gamma_2}$$

$$> 0.$$  

(iv) First, note that $E_2 < 0$ because

$$E_2 = (K_4 - K_2) (\eta_1 - 1) + \eta_1 (K_3 - K_1)$$

$$= \frac{\phi}{\rho + \phi + \delta (1 + \theta)} - \frac{\eta_1 - 1}{\rho + \phi + \delta (1 + \theta) - \mu}$$

$$> 0.$$  

Next, if $\tau < \tau + \delta (1 - L - l)$, then

$$\eta_1 (K_3 - K_7) + (\eta_1 - 1) K_4 y_*^E$$

$$= -\theta \delta y_*^E \left( \frac{\eta_1}{\rho + \phi + \delta (1 + \theta)} - \frac{\eta_1 - 1}{\rho + \phi + \delta (1 + \theta) - \mu} \right)$$

$$< 0.$$  

Therefore, the coefficient of $y^{-\gamma_1}$ in $W_E (y)$ is strictly negative and thus $W_E (y)$ is strictly monotonically increasing. Lastly, $W_E (\infty) = \frac{\eta_1 K_7 + \gamma_2 K_8}{\eta_1 + \gamma_2} > 1$.

(v) Because $W_F (y) = \frac{\eta_1 K_7 + \gamma_2 K_8}{\eta_1 + \gamma_2} + \frac{(\eta_1 - 1) K_4}{\eta_1 + \gamma_2} y + \frac{\eta_1 + \gamma_2}{\eta_1 + \gamma_2} E_2 y^{-\gamma_1}$ and $E_2 < 0$ as proved in (iv), $W_F (y)$ is strictly monotonically increasing. It also implies $W_F (0) = -\infty$ and $W_F (\infty) = \infty$.  

---

**Proof of Theorem 1.** The equilibrium threshold $y_*$ is determined by the condition $V (y_*; y_*) = 1$. Note that depending on parameter specification, the value function $V (y; y_*)$ takes a different form in any one of the eight different cases as shown in Appendix B. Define $W (y_*) \equiv V (y_*; y_*)$. Here we prove that there always exists a unique $y_*$ such that $W (y_*) = 1$. To simplify notation, we replace $y_*$ by $y$ and express
$W(y_*)$ as $W(y)$ throughout the proof.

Recall that in each of eight different cases, the function $W(y)$ takes a different form. Define

$$y^A_* = \frac{\bar{\nu} + \phi + \theta \delta (1 - L) - \nu}{\phi + \theta \delta l},$$

$$y^C_* = \frac{\bar{\nu} + \phi - \nu}{\phi} < 1,$$

$$y^E_* = \frac{\bar{\nu} + \theta \delta (1 - L) - \nu}{\theta \delta l} < \frac{1 - L}{l}.$$

Below we list all eight different functional forms $W(y)$ may take.

If $y \leq 1$, there are three cases:

- **Case A:** $y > y^A_*$$W_A(y) = \eta_1 K_7 + \gamma_2 K_8 \frac{\eta_1 (K_1 - K_7) + (\eta_1 - 1) K_2 y^A_*}{\eta_1 + \gamma_2} y^{-\gamma_1} + (\eta_1 - 1) K_2 y^A_* y^{-\gamma_1}$

- **Case B:** $y^C_* \leq y \leq y^A_*$$W_B(y) = \eta_1 K_1 + \gamma_2 K_8 \frac{\eta_1 (K_1 - K_7) + (\eta_1 - 1) K_2 y^A_*}{\eta_1 + \gamma_2} y^{-\gamma_1}$

- **Case C:** $y < y^C_*$$W_C(y) = \eta_1 K_1 + \gamma_2 K_8 \frac{\eta_1 (K_1 - K_7) + (\eta_1 - 1) K_2 y^A_*}{\eta_1 + \gamma_2} y^{-\gamma_1}$

If $1 < y \leq \frac{L - L}{L}$, there are three cases:

- **Case D:** $\bar{\nu} \leq \bar{\nu} + \theta \delta (1 - L - l)$

$$W_D(y) = W_A(y)$$

- **Case E:** $\bar{\nu} < \bar{\nu} + \theta \delta (1 - L - l)$ and $y > y^E_*$$W_E(y) = \eta_1 K_1 + \gamma_2 K_8 \frac{\eta_1 (K_1 - K_7) + (\eta_1 - 1) K_2 y^A_*}{\eta_1 + \gamma_2} y^{-\gamma_1} + \left[ \eta_1 + \gamma_1 E_2 + \frac{\eta_1 (K_3 - K_7) + (\eta_1 - 1) K_4 y^E_*}{(\eta_1 + \gamma_2) (y^E_*)^{-\gamma_1}} \right] y^{-\gamma_1}$
• Case F: $\tau < \bar{\tau} + \theta \delta (1 - L - l)$ and $y \leq y_{es}^E$

$$W_F(y) = \frac{\eta_1 K_3 + \gamma_2 K_8}{\eta_1 + \gamma_2} + \frac{(\eta_1 - 1) K_4}{\eta_1 + \gamma_2} y - \frac{\eta_1 (K_3 - K_1) + (\eta_1 - 1)(K_4 - K_2)}{\eta_1 + \gamma_2} y^{-\gamma_1}$$

If $y > \frac{1-L}{\tau}$, there are two cases:

• Case G: $\tau \geq \bar{\tau} + \theta \delta (1 - L - l)$

$$W_G(y) = W_A(y)$$

• Case H: $\tau < \bar{\tau} + \theta \delta (1 - L - l)$,

$$W_H(y) = W_E(y)$$

Note that $y_{es}^A > y_{es}^C$ as long as $\tau > \bar{\tau}$. Furthermore, $y_{es}^C > 0$ is equivalent to $\tau < \bar{\tau} + \phi$, and $y_{es}^A > 1$ is equivalent to $\tau < \bar{\tau} + \theta \delta (1 - L - l)$. Therefore, we prove the existence of a solution $W(y) = 1$ by considering three different possibilities: $\tau \geq \bar{\tau} + \theta \delta (1 - L - l)$, $\bar{\tau} + \phi \leq \tau < \bar{\tau} + \theta \delta (1 - L - l)$, and $\tau < \bar{\tau} + \phi$.

We first consider the first possibility $\tau \geq \bar{\tau} + \theta \delta (1 - L - l)$ where $y_{es}^A \leq 0 < y_{es}^C \leq 1$. Note that the restrictions on parameter values imply that $W_B(0) < 1$ and $W_A(\infty) > 1$. In addition, it is straightforward to check that $W_A(y_{es}^A) = W_B(y_{es}^A)$. Since $W_A(y)$ is strictly monotonically increasing and $y_{es}^A \leq 1$, we know

$$W_B(0) < W_A(y_{es}^A) = W_B(y_{es}^A) \leq W_A(1) < W_A(\infty).$$

Therefore, if $W_B(y_{es}^A) \geq 1$, then there exists a unique solution $y \leq y_{es}^A$ such that $W_B(y) = 1$ because $W_B(0) < 1$ and $W_B(y)$ is strictly monotonically increasing. If $W_A(1) < 1$, then there exists a unique solution $y \geq 1$ such that $W_A(y) = 1$ because $W_A(\infty) > 1$ and $W_A(y)$ is strictly monotonically increasing. Otherwise, if $W_A(y_{es}^A) = W_B(y_{es}^A) < 1 \leq W_A(1)$, then from the monotonicity of $W_A(y)$, there exists a unique solution $y_{es}^A < y \leq 1$ such that $W_A(y) = 1$.

Next, we consider the second possibility $\bar{\tau} + \phi < \tau < \bar{\tau} + \theta \delta (1 - L - l)$ where $y_{es}^A > 1$ and $1 < y_{es}^C < \frac{1-L}{\tau}$. Note that the restrictions on parameter values imply $W_E(\infty) > 1$. It is straightforward to check that $W_F(y_{es}^E) = W_E(y_{es}^E)$ and $W_B(1) = W_F(1)$. Since $W_i(y)$, $i = B, E, F$ are strictly monotonically increasing and $y_{es}^E > 1$, we know

$$W_B(1) = W_F(1) < W_F(y_{es}^E) = W_E(y_{es}^E).$$

Therefore, if $W_B(1) \geq 1$, then there exists a unique solution $y \leq 1$ such that $W_B(y) = 1$ because $W_B(0) < 1$ and $W_B(y)$ is strictly monotonically increasing. If $W_E(y_{es}^E) < 1$, then there exists a unique solution $y > y_{es}^E$ such that $W_E(y) = 1$ because $W_E(\infty) >$
1 and \( W_E (y) \) is strictly monotonically increasing. Otherwise, if \( W_B (1) = W_F (1) < 1 \leq W_F (y_{**}^E) = W_E (y_{**}^E) \), then from the monotonicity of \( W_F (y) \), there exists a unique solution \( 1 < y \leq y_{**}^E \) such that \( W_F (y) = 1 \).

Lastly, we consider the last possibility \( \tau < \bar{\rho} + \phi \) where \( y_{**}^A > 1 > y_{**}^C > 0 \) and \( 1 < y_{**}^E = \frac{\rho + \delta (1 - L) - \tau}{\rho \bar{\rho}} < \frac{1 - \bar{L}}{\bar{L}}. \) Note that the restrictions on parameter values imply that \( W_C (0) < W_B (0) < 1 \). In addition, it is straightforward to check that \( W_B (y_{**}^C) = W_C (y_{**}^C) \). Since \( W_A (y) \) is strictly monotonically increasing and \( y_{**}^A \leq 1 \), we know

\[
W_C (0) < W_B (y_{**}^C) = W_C (y_{**}^C) < W_B (1) = W_F (1).
\]

Therefore, if \( W_C (y_{**}^C) \geq 1 \), then there exists a unique solution \( y \leq y_{**}^C \) such that \( W_C (y) = 1 \) because \( W_C (0) < 1 \) and \( W_C (y) \) is strictly monotonically increasing. Otherwise, if \( W_C (y_{**}^C) < 1 \), then by the same argument used for the second possibility, we can prove that there exists a unique solution \( y_{**}^C < y \leq 1 \) if \( W_B (1) \geq 1 \), or \( y > y_{**}^E \) if \( W_B (y_{**}^E) < 1 \), or \( 1 < y \leq y_{**}^E \) if \( W_B (1) = W_F (1) < 1 \leq W_F (y_{**}^E) = W_E (y_{**}^E) \).

**Proof of Proposition 2.** By the implicit function theorem, \( \frac{dy}{\frac{\partial W}{\partial \tau}} = -\frac{\partial W/\partial y}{\partial W/\partial \tau} \). We have shown in Lemma 4 that \( \frac{\partial W}{\partial y} > 0 \). Therefore, we only need to show that \( \frac{\partial W}{\partial \tau} > 0 \) for each of functions \( W_A (y), \cdots, W_H (y) \) in order to prove the claim. From Lemma 2, we have

\[
\frac{\partial W_A (y)}{\partial \tau} = \frac{y^{-\gamma_1}}{(\eta_1 + \gamma_2)} \frac{\partial}{\partial \tau} \left[ \frac{\eta_1 (K_1 - K_7) (y_{**}^A)^{\gamma_1} + (\eta_1 - 1) K_2 (y_{**}^A)^{\gamma_1 + 1}}{\rho + \phi + \delta (1 + \theta) - \rho + \phi + \delta (1 + \theta) - \mu} \right] > 0.
\]

When \( \tau < \bar{\rho} + \phi \), for \( y < y_{**}^C \), from Lemma 3, we have

\[
\frac{\partial W_C (y)}{\partial \tau} = \frac{1}{(\eta_1 + \gamma_2)} \frac{\partial}{\partial \tau} \left[ \frac{\frac{\eta_1 K_1 + \gamma_2 K_8}{\eta_1 + \gamma_2} + \frac{\gamma_2 (K_8 - K_7) - (\gamma_2 + 1) K_8 C^2}{(\eta_1 + \gamma_2) y_{**}^C} y_{**}^2}{\rho + \phi + \delta (1 + \theta)} \right] > 0. \]
When \( r < \rho + \theta \delta (1 - L - l) \), from Lemma 3, we have

\[
\frac{\partial W_E (y)}{\partial \rho} = y^{-\gamma_1} \frac{\partial}{\partial \rho} \left[ \eta_1 (K_3 - K \gamma) (y_{e*})^{\gamma_1} + (\eta_1 - 1) K \eta (y_{e*})^{1+\gamma_1} \right]
\]
\[
= \left( \frac{\gamma_1 + 1}{\eta_1 + \gamma_2} \right) y^{-\gamma_1} (y_{e*})^{\gamma_1} \left[ \frac{\eta_1}{\rho + \phi + \delta (1 + \theta)} \frac{1}{\eta_1 + \gamma_2} \right]
\]
\[
> 0.
\]

Lastly, \( \frac{\partial W_D (y)}{\partial \rho} = \frac{\partial W_E (y)}{\partial \rho} = \frac{\eta_1}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta (1 + \theta)} > 0 \).

**Proof of Proposition 3.** (i) By the implicit function theorem, \( \frac{dy}{\delta} = -\frac{\partial W/\partial y}{\partial W/\partial \rho} \).

We have shown in Lemma 4 that \( \partial W/\partial y > 0 \). Therefore, we only need to show that \( \partial W/\partial \Delta \rho > 0 \) for each of functions \( W_A (y), \cdots, W_H (y) \) in order to prove the claim. From Lemma 3, we have

\[
\frac{\partial W_A (y)}{\partial \Delta \rho} = \frac{1}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta (1 + \theta)} + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta (1 + \theta)}
\]
\[
- \frac{\left( \frac{\gamma_1 + 1}{\eta_1 + \gamma_2} \right)}{\left( \frac{1}{\eta_1 + \gamma_2} \right)} \frac{\eta_1}{\rho + \phi + \delta (1 + \theta)} \frac{1}{\eta_1 + \gamma_2} = \frac{\eta_1 - 1}{\rho + \phi + \delta (1 + \theta) - \mu}
\]
\[
> \frac{\gamma_1 + 1}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta (1 + \theta)} + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta (1 + \theta) - \mu}
\]
\[
= \frac{\eta_1}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta (1 + \theta)} + \frac{\eta_1 - 1}{\rho + \phi + \delta (1 + \theta) - \mu}
\]
\[
= \frac{\gamma_1 + 1}{\eta_1 + \gamma_2} \frac{1}{\rho + \phi + \delta (1 + \theta)} + \frac{\eta_1 - 1}{\rho + \phi + \delta (1 + \theta) - \mu}
\]
\[
> 0;
\]
and when \( \tau < \bar{p} + \phi \), for \( y < y_{**} \), \( \frac{\partial W_C(y)}{\partial \Delta_{\rho}} = \frac{(\eta_2 - 1)\rho^{\gamma_2}(y_{**})^{-\gamma_2}}{\eta_1 + \gamma_2} \left( \frac{\gamma_2 + 1}{\rho + \phi - \mu} - \frac{\gamma_2}{\rho + \phi} \right) > 0 \); and furthermore, when \( \tau < \bar{p} + \theta \delta (1 - L - l) \), we have

\[
\frac{\partial W_E(y)}{\partial \Delta_{\rho}} = \frac{1}{\eta_1 + \gamma_2 \rho + \phi + \delta (1 + \theta)} + \frac{\gamma_2 + 1}{\eta_1 + \gamma_2 \rho + \phi} \\
- \frac{(\gamma_1 + 1)^2 \gamma_1}{\eta_1 + \gamma_2} \left( \frac{\eta_1}{\rho + \phi + \delta (1 + \theta)} - \frac{(\eta_1 - 1)}{\rho + \phi + \delta (1 + \theta) - \mu} \right) > 0.
\]

Lastly, \( \frac{\partial W_E(y)}{\partial \Delta_{\rho}} = \frac{\partial W_F(y)}{\partial \Delta_{\rho}} = \frac{\gamma_2 + 1}{\eta_1 + \gamma_2 \rho + \phi} > 0 \).

(ii) Next we prove by contradiction the statement by considering two cases: \( \tau > \rho + \theta \delta (1 - L - l) \) and \( \tau \leq \rho + \theta \delta (1 - L - l) \).

(ii-A) Suppose \( \tau > \rho + \theta \delta (1 - L - l) \). Suppose \( \bar{p} < y_0 < \infty \). Then \( \lim_{\tau \to \bar{p}} W_A(y_*) = 1 = \lim_{\tau \to \bar{p}} W_A(y_0) \). However, note that when \( \bar{p} \) tends to \( \rho \), \( K_7 \) and \( K_8 \) tend to 1. Therefore, we have

\[
\lim_{\tau \to \bar{p}} W_A(y_0) = 1 + \lim_{\tau \to \bar{p}} \frac{\eta_1 (K_7 - 1) + \gamma_2 (K_8 - 1)}{\eta_1 + \gamma_2} + \frac{\eta_1 (K_1 - K_7) + (\eta_1 - 1) K_2 y_{**}^{-\gamma_1}}{\eta_1 + \gamma_2} (y_{**})^{-\gamma_1} y_0^{-\gamma_1} < 1,
\]

which is a contradiction.
(ii-B) If $\tau \leq \rho + \theta \delta (1 - L - l)$, then $\lim_{\tau \to \rho} W_B(1) < 1$ because

$$W_B(1) - 1 = \frac{\eta_1 (K_1 - 1) + \gamma_2 (K_3 - 1) + (\eta_1 - 1) K_2}{\eta_1 + \gamma_2}$$

$$\to \frac{\eta_1}{\eta_1 + \gamma_2} \left( \frac{\tau + \delta (1 + \theta L)}{\rho + \phi + \delta (1 + \theta)} - 1 \right) + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\phi + \theta \delta l}{\rho + \phi + \delta (1 + \theta) - \mu}$$

$$= \frac{-\eta_1}{\eta_1 + \gamma_2} \left( \frac{(\rho + \theta \delta (1 - L - l)) - \tau}{\rho + \phi + \delta (1 + \theta)} \right)$$

$$< 0.$$ Similarly as before, we can prove that in this case, $\lim_{\tau \to \rho} y_* = \infty$. We can prove it by contradiction. Suppose $\lim_{\tau \to \rho} y_* = y_0 < \infty$. Then $\lim_{\tau \to \rho} W_E(y_*) = 1 = \lim_{\tau \to \rho} W_E(y_0)$. However,

$$\lim_{\tau \to \rho} W_E(y_0) = 1 + \left( \frac{\eta_1 + \gamma_1}{\eta_1 + \gamma_2} E_2 + \frac{\eta_1 (K_3 - K_7) + (\eta_1 - 1) K_4 y_*^E}{(\eta_1 + \gamma_2) (y_*^E - \gamma_1)} \right) y_0^{-\gamma_1} < 1,$$

which is a contradiction. ■