Equilibrium Bank Runs Revisited*

David Andolfatto
Federal Reserve Bank of St. Louis

Ed Nosal
Federal Reserve Bank of Chicago

Bruno Sultanum
The Pennsylvania State University

February 14, 2014

Abstract

Peck and Shell (2003) show that equilibrium bank runs are possible under the optimal mechanism in the Diamond and Dybvig (1983) model. We show that their result is an artifact of their restriction to direct mechanisms. An optimal indirect mechanism eliminates the possibility of bank-run equilibria.

Keywords: Bank runs; optimal deposit contract; financial fragility.

1 Introduction

All financial intermediaries transform assets into liabilities. For example, pension funds transform assets into time-dependent liabilities. Insurance companies transform assets into state-contingent liabilities. Banks transform assets into demandable liabilities. Like any business, a shock that renders assets less valuable than outstanding liabilities can lead to financial insolvency.

Banks, however, are commonly viewed as subject to an additional vulnerability — a “defect” that stems from the peculiar structure of their liabilities. The American put option embedded in bank liabilities means that a mass redemption — triggered for any reason — is legally permitted prior to the maturation (or orderly disposal) of the underlying assets. Because the value of what is recouped in a “firesale” of illiquid assets is expected to fall short of the value of existing obligations, even the mere expectation of a mass redemption can trigger an insolvency event. Banks, unlike other intermediaries, are potentially subject to self-fulfilling bank runs.

If the demandable nature of bank liabilities render the enterprise susceptible to runs, then what explains the extensive use of this contractual structure? Bryant (1980) suggests that the American put option is a way to provide insurance against unobservable idiosyncratic liquidity risk. If this is so, then it appears that demandable debt may constitute both a solution and a problem. In a seminal paper, Diamond and Dybvig (1983) combine this idea with a sequential

*I would like to thank Marco Bassetto, Todd Keister, Ali Shourideh, Nico Trachter, and Neil Wallace for helpful discussions, and for comments on earlier versions of the paper.
service constraint to explore the possibility of welfare-improving demandable debt as a source of indeterminacy and financial instability.\footnote{Bryant (1980) does not define a bank run as a self-fulfilling prophecy; in his model, a bank run is triggered by a shock to fundamentals.}

Although Diamond and Dybvig (1983) is widely cited as providing a rationale for the existence of bank run equilibria, their justly celebrated contribution in fact falls short of explaining the phenomenon as the outcome of an optimal arrangement. For example, when there is no aggregate risk, they demonstrate that a bank run equilibrium cannot exist when the deposit contract is appropriately designed. The optimal contract in this case combines a “standard” deposit contract augmented by a clause that suspends convertibility in the event of a run. The second part of their paper is devoted to the idea that aggregate uncertainty may overturn this latter result, but they never formally characterize the optimal contractual arrangement in this more complicated environment. Thus, Diamond and Dybvig (1983) leaves us in an uncomfortable spot. The fundamental cause of bank runs in their model does not appear to hinge on properties of the environment, like private information, sequential service, or aggregate uncertainty. Instead, it appears to be a by-product of \textit{ad hoc} restrictions placed by the theorist on the nature of the contract (for example, by eliminating the suspension clause in a standard deposit contract in their model without aggregate risk — a practice, incidentally, observed frequently in history).

Indeed, this suspicion was verified twenty years later by Green and Lin (2003) who were the first to characterize an optimal bank contract in an environment with private information, sequential service, and aggregate uncertainty. In the Green and Lin (2003) version of the Diamond and Dybvig (1983) model, the first-best allocation is implementable as an unique Bayes-Nash equilibrium of a direct revelation game. A critical property of the Green-Lin contract is that it permits early withdrawal amounts in the sequential service queue to depend on the history of withdrawals to that point. The maximum withdrawal amount faced by a latecomer in the service queue is lowered following an unusually large withdrawal demand. This “partial suspension scheme” is in stark contrast to Diamond and Dybvig (1983), who restrict the maximum withdrawal amount to be insensitive to realized withdrawal demand, so that resources are necessarily exhausted in the event of a run.\footnote{Wallace (1990) reports that partial suspensions were prevalent in the banking panic of 1907, and that in one form or another must have been a feature of other suspension episodes as well.}

One property of the Green-Lin model is that incentive-compatibility constraints do not bind at the optimum. Peck and Shell (2003) modify the Green-Lin model so that incentive-compatibility constraints do bind at the optimum. They also assume that depositors do not know their position in the service queue. Although Peck and Shell (2003) just impose this latter assumption, it turns out that not revealing queue positions is part of an optimal mechanism when incentive-compatibility constraints bind.\footnote{Essentially, multiple constraints are replaced by a single constraint in expectational form, when queue positions are not revealed by the mechanism. In any case, Peck and Shell (2003) follow Green and Lin (2003) in using optimal direct revelation mechanism under these additional restrictions and demonstrate the existence of a bank run equilibrium.

As it turns out, a bank run equilibrium also exists in the Green-Lin environment when depositor types are correlated and allocations are implemented by a direct revelation mechanism; see Ennis and Keister (2009b).\footnote{GL assume that depositor types are identically and independently distributed.} But while a direct mechanism turns out to be optimal for the original Green-Lin model, it is not by any means clear that a broader class of mechanisms might not do better in a different environment. Indeed, Cavalcanti and Monteiri (2011) examine indirect mechanisms in the Ennis and Keister (2009b) environment and demonstrate that the best allocation can be uniquely implemented in dominant strategies. Unfortunately, their backward induction argu-
ment will not work in the more general Peck-Shell environment, where depositors do not know their positions in the queue.

And so, the question remains: have Peck and Shell (2003) really identified a bank-run equilibrium under an optimal contract, or is their result an artefact of their restriction to direct mechanisms? In this paper, we argue that the latter is true. In what follows, we construct indirect mechanisms that implement constrained-efficient allocations in dominant strategies. The basic intuition is as follows.

As pointed out by Diamond and Dybvig (1983), the threat of full suspension works to eliminate a run equilibrium in the absence of aggregate uncertainty, but not in its presence. This is because when there is no aggregate uncertainty, the aggregate early withdrawal demand is perfectly forecastable (with a fully diversified portfolio of depositor types). Consequently, an unusually large number of early redemptions signals perfectly that a run is on, and the bank can condition suspension on this information. Under aggregate uncertainty, on the other hand, there is a probability that an unusually large withdrawal demand is driven by fundamentals. If all the bank has to work with is a flow of messages regarding agent type (impatient or not), then it is difficult to distinguish whether fundamentals or psychology is driving a mass of early redemptions.

Our proposed solution to this problem is to exploit the idea that while the bank may not know whether a run is on, the agents in the model do know. That is, as a part of most equilibrium solution concepts, agents’ beliefs are assumed to be consistent with the reality unfolding around them — and all agents know that a run is happening when it is in progress. Can the bank somehow solicit this information, which is known to exist in the population of depositors? If it could, then the threat of suspension, conditional on soliciting such information, could presumably work to eliminate run equilibria much in the same way it does in an environment without aggregate uncertainty.

To solicit such information, we expand the message space — in lay terms, we let depositors have a conversation with the bank manager (in doing so, we depart from the constraint of direct mechanisms). Suppose that a patient agent, fearing that a run is on, decides to withdraw early (he does so, in a direct mechanism, by misrepresenting himself as an impatient agent in the sequential service queue). At this stage, our mechanism permits this agent to send an additional message: “I am patient, but I believe a run is on.” In our mechanism, the first patient agent to report such a message is rewarded so that he strictly prefers to say this over misrepresenting his type. Conditional on receiving this message, the bank suspends all early redemptions. Understanding this, all agents (patient agents in particular) have no incentive to run. This is the basic idea — the rest of the paper is devoted to formalizing it.

The paper is organized as follows. The next section describes the economic environment. Section 3 characterizes the best weakly implementable allocation. In Section 4, we provide an illustrative example. In Section 5 we construct an indirect mechanism that uniquely implements the best weakly implementable allocation. In Section 6, we discuss an issue that arises concerning the Peck-Shell environment. Some concluding comments are offered in the final section.

2 Environment

There are three dates: 0, 1 and 2. The economy is endowed with \( Y > 0 \) units of date-1 goods. A constant returns to scale technology transforms \( y \) units of date-1 goods into \( yR > y \) units of date-2 goods.

There are \( N \) ex-ante identical agents. An agent is one of two types \( t \in T = \{1, 2\} \), patient, \( t = 1 \), or impatient, \( t = 2 \). The utility function for an impatient agent is \( u(c^1) \) and the utility function for a patient agent is \( \rho u(c^1 + c^2) \), where \( c^1 \) is date-1 consumption, \( c^2 \) is date-2 consumption and
\( R^{-1} < \rho \leq 1 \). \( u \) is are increasing, strictly concave, and twice continuously differentiable.\(^5\) Agents maximize expected utility.

The number of patient agents in economy is drawn from the distribution \( \pi = (\pi_0, \ldots, \pi_N) \), where \( \pi_n > 0, n \in \{0,1,\ldots,N\} \equiv \mathbb{N}, \) is the probability that there are \( n \) patient agents. A queue is the vector \( t^N = (t_1, \ldots, t_N) \in T^N, \) where \( t_k \in T \) is the type of agent that occupies the \( k^{\text{th}} \) position/coordinate in the queue. Let \( P_n = \{t^N \in T^N | \#2 \in t^N = n\} \) and \( Q_n = \{j | t_j = 2 \text{ for } t^N \in P_n\}, \) where “\#2” is the number of patient agents. \( P_n \) is the set of queues with \( n \) patient agents and \( Q_n \) is the set of queue positions of the \( n \) patient agents in queue \( t^N \in P_n. \) The probability that \( t^N \in P_n \) is \( \pi_n/\#P_n = \pi_n / (N_n), \) where \( \#P_n \) is the number of queues \( t^N \in P_n. \) This implies that given there are \( n \) patient agents, all queues with \( n \) patient agents are equally likely. Agents are randomly assigned a position in the queue, where the (unconditional) probability that an agent is assigned to position \( k \) is \( 1/N. \) Call an agent assigned to position \( k \) agent \( k. \) The queue realization, \( t^N, \) is observed by no one; not by any of the agents nor the planner. Each agent, however, privately observes his type \( t \in T. \)

The timing of events and actions is as follows. At date 0, the planner constructs a mechanism that determines how date-1 and date-2 consumption are allocated among the \( N \) agents, and queue \( t^N \) is realized. A mechanism is a set of announcements, \( M, \) and an allocation rule, \( c = (c^1, c^2) \) where \( c^1 = (c^1_1, \ldots, c^1_N) \) and \( c^2 = (c^2_1, \ldots, c^2_N). \) At date 1, agents sequentially meet the planner, starting with agent 1. Each agent \( k \) makes an announcement \( m_k \in M.\(^6\) Only agent \( k \) and the planner can directly observe \( m_k. \) There is a sequential service constraint at date 1, which means the planner allocates date-1 consumption to agent \( k \in \mathbb{N} \) based on the announcements of agents \( j \leq k, \) i.e.,

\[
 c^1_k \left( m^{k-1}, m_k \right), \quad \text{where } m^{k-1} = (m_1, \ldots, m_{k-1}), \text{ and each agent consumes the date-1 at his meeting with the planner. After all agents meet the planner, the planner simultaneously allocates the date-2 consumption good to each agent based on all of the date-1 announcements made by the agents, i.e., agent } k \text{ receives } c^2_k (m^N), \text{ where } m^N = (m_1, \ldots, m_N) \in M^N. \]

Figure 1 depicts the sequence of actions.

---

\(^5\)These are precisely the preferences used in Diamond and Dybvig (1983).

\(^6\)One could imagine that the planner makes announcement \( a_k \) to agent \( k \) before \( k \) makes his announcement. For example, the planner could tell agent \( k \) his queue position or the set of all of the messages sent in the previous \( k - 1 \) planner-agent meetings or ‘nothing,’ \( a_k = \emptyset. \) The optimal mechanism, however, will have the planner announcing \( a_k = \emptyset \) for all \( k. \) See footnote 4 for a discussion. To reduce notation, and without loss of generality, we simply assume that the planner cannot make announcements to agents.
3 Best Weakly Implementable Outcome

An allocation is weakly implementable if it is an outcome to some equilibrium of the mechanism; it is strongly (or uniquely) implementable if it is an outcome to every equilibrium of the mechanism. Among the set of weakly implementable allocations, the best weakly implementable allocation provides agents with the highest expected utility. To characterize the best weakly implementable allocation, it is without loss of generality to restrict the planner to use a direct revelation mechanism; it is strongly (or uniquely) implementable if it is an outcome to every equilibrium of the mechanism, where agents make truthful announcement, \( m_k = t_k \in T = \{1, 2\} \). The economy-wide welfare—which is the expected utility of an agent before he learns his type—associated with allocation rule \( c \) when agents use strategies \( m_k \in T \) is

\[
\sum_{n=0}^{N} \frac{\pi_n}{\binom{N}{n}} \sum_{t_n \in P_n} \sum_{n=1}^{N} U \left[ c_k^1 \left( m_k^{k-1}, m_k \right), c_k^2 \left( m_1^N \right), t_k \right],
\]

where

\[
U \left[ c_k^1 \left( m_k^{k-1}, m_k \right), c_k^2 \left( m_1^N \right), t_k \right] = u \left[ c_k^1 \left( m_k^{k-1}, m_k \right) \right] \text{ if } t_k = 1
\]

and

\[
U \left[ c_k^1 \left( m_k^{k-1}, m_k \right), c_k^2 \left( m_1^N \right), t_k \right] = \rho u \left[ c_k^1 \left( m_k^{k-1}, m_k \right) + c_k^2 \left( m^N \right) \right] \text{ if } t_k = 2
\]

The allocation rule \( c = (c_1, c_2) \) is feasible if

\[
\forall m^N \in T^N : \sum_{k=1}^{N} \left[ R c_k^1 \left( m_k^{k-1}, m_k \right) + c_k^2 \left( m^N \right) \right] \leq R_Y.
\]

Allocation rule \( c \) must be incentive compatible in the sense that agent \( k \) has no reason to announce \( m_k \neq t_k \). Since impatient agent \( k \) only values date-1 consumption, he always announces \( m_k = 1 \).\(^7\) Patient agent \( k \) has no incentive to depart from the strategy \( m_k = 2 \), assuming that all other agents \( j \) announce \( m_j = t_j \), if

\[
\sum_{n=1}^{N} \frac{\hat{\pi}_n}{\binom{N}{n}} \sum_{t_n \in P_n} \frac{1}{n} \sum_{k \in Q_n} u \left[ c_k^1 \left( t_k^{k-1}, 2 \right) + c_k^2 \left( t_k^{k-1}, 2, t_{k+1}^N \right) \right] \geq \sum_{n=1}^{N} \frac{\hat{\pi}_n}{\binom{N}{n}} \sum_{t_n \in P_n} \frac{1}{n} \sum_{k \in Q_n} u \left[ c_k^1 \left( t_k^{k-1}, 1 \right) + c_k^2 \left( t_k^{k-1}, 1, t_{k+1}^N \right) \right] + \delta,
\]

where \( x_i^j \equiv (x_i, \ldots, x_j) \), \( \delta \geq 0 \) is a parameter, and

\[
\hat{\pi}_n = \frac{\pi_n / \binom{N}{n}}{\sum_{n=1}^{N} \pi_n / \binom{N}{n}}
\]

\(^7\)This anticipates the result that the best weakly implementable allocation provides zero date-1 consumption to agents who announce that they are patient, which implies that the incentive compatibility constraint for impatient agents is always slack.
is the conditional probability that agent $k$ is in a specific queue that has $n$ patient agents. The $1/n$ terms that appear in (3) reflect that a patient agent has a $1/n$ chance of occupying each of the patient queue positions in $Q_n$.

Denote the solution to the problem

$$\max_c \text{ (1) subject to (2) and (3),}$$

where $m_k = t_k$ for all $k \in \mathbb{N}$ in (1) and (2),

as $c^*(\delta) = (c^{1*}(\delta), c^{2*}(\delta))$. Allocation rule $c^*(\delta)$ has the following features: (i) agents $k$ who announce $m_k = 1$ consume only at date 1,

$$c^2_k(m_1, \ldots, m_{k-1}, 1, m_k, \ldots, m_{N-1}) = 0;$$

(ii) agents $k$ who announce $m_k = 2$ consume only at date 2,

$$c^1_k(m_1, \ldots, m_{k-1}, 2) = 0;$$

and (iii) agents $j$ and $k$ who announce $m_j = m_k = 2$ consume identical amounts at date 2,

$$c^2_j(m_N) = c^2_k(m_N) \text{ for all } m_j = m_k = 2.$$

The best-weakly implementable allocation is $c^*(0)$. (Note that allocation rule $c^*(0)$ corresponds to allocation rule derived in PS’s Appendix B.)

Both PS and Ennis and Keister (2009b) demonstrate, by example, that the direct mechanism $(T, c^*(0))$ can have two equilibria: one where agents play truth-telling strategies, $m_k = t_k$ for all $k \in \mathbb{N}/0$, and another where patient agents play bank run strategies, $m_k = 1$ for all $k \in \mathbb{N}/0$. Bank run equilibria arise in these examples because the direct revelation mechanism they use, $(T, c^*(0))$, is not an optimal mechanism. Before we demonstrate that an allocation that is arbitrarily close to the best weakly implementable allocation can be implemented by an indirect mechanism, we provide a simple example that illustrates the basic intuition that underlies the optimal indirect mechanism.

### 4 An Illustrative Example

Consider a stripped-down version of a Diamond-Dybvig model where there are only 2 agents—column and row—and both agents are patient. Agents simultaneously announce that they are either patient, $m = 2$, or impatient, $m = 1$. The payoffs to agents for this game are

---

8To characterize the best weakly implementable allocation, one wants to choose from the largest possible set of incentive compatible allocations. This implies the planner should not make any announcements, as noted in footnote 3. In particular, if the planner does not make any announcements, then there is only one incentive compatibility constraint for all patient agents, (3). If, however, the planner did make announcement $a_k$ to agents $k$, there will be additional incentive constraints for agents $k$ who received information $a_k$. For example, suppose that $a_k = k$ for all $k$, i.e., the planner announces to each agent his place in the queue. Then there would be $N$ incentive compatibility constraints for patient agents, one for each queue position. Since an appropriately weighted average of these distinct incentive constraints reduces to the single incentive constraint (3), the set of incentive compatible allocations when the planner makes announcements is a subset of the set of incentive compatible allocations when he does not.

9The Ennis and Keister (2009b) bank run example is in section 4.2 of their paper. There, agents do not know their position in the queue, as in PS, and the utility functions of patient and impatient agents are the same, $\rho = 1$, as in GL.
This simple normal form game captures two important insights of the Diamond-Dybvig model: (i) there are multiple equilibria, one where both agents announce the truth, \( m = 2 \), and another where both agents announce they are impatient, \( m = 1 \); and (ii) the truth-telling equilibrium generates the highest payoff for agents.

Consider now a related normal form game that augments the announcement space of the original game from \( \{1,2\} \) to \( \{1,2,g\} \), with payoffs

<table>
<thead>
<tr>
<th></th>
<th>( m = 1 )</th>
<th>( m = 2 )</th>
<th>( m = g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 1 )</td>
<td>1 , 1</td>
<td>2 , 0</td>
<td>0 , 1 + ( \epsilon )</td>
</tr>
<tr>
<td>( m = 2 )</td>
<td>0 , 2</td>
<td>3 , 3</td>
<td>3 , 2 + ( \epsilon )</td>
</tr>
<tr>
<td>( m = g )</td>
<td>1 + ( \epsilon ) , 0</td>
<td>2 + ( \epsilon ) , 3</td>
<td>( \epsilon ) , ( \epsilon )</td>
</tr>
</tbody>
</table>

There are three features of the augmented game that we would like to highlight. First, when agents restrict their announcements to \( \{1,2\} \), the payoffs they receive are identical to the original game. Second, announcement \( m = g \) strictly dominates announcement \( m = 1 \). And finally, the payoffs to an agent who announces \( m = 2 \) receives the same payoff if his opponent announces \( m = 2 \) or \( m = g \).

Since agents never play \( m = 1 \) in the augmented game, the “relevant” augmented game that agents play is

<table>
<thead>
<tr>
<th></th>
<th>( m = 2 )</th>
<th>( m = g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 2 )</td>
<td>3 , 3</td>
<td>2 , 0</td>
</tr>
<tr>
<td>( m = g )</td>
<td>2 + ( \epsilon ) , 3</td>
<td>( \epsilon ) , ( \epsilon )</td>
</tr>
</tbody>
</table>

But in this relevant augmented game, announcement \( m = g \) is strictly dominated by announcement \( m = 2 \). Therefore, the unique (iterated strict dominance) equilibrium to the augmented game is truth-telling, \( m = 2 \). Hence, by slightly modifying the game that agents play we are able to get rid of the “bad” (bank run) equilibrium in the original game.

The best weakly implementable allocation described in Section 3, \( c^*(\delta) \), is somewhat more complicated than the payoff structure in the stripped-down example. Nevertheless, it is easy to construct an indirect mechanism \( (M, \hat{c}) \) where \( M = \{1,2,g\} \), announcement \( m_k = 1 \) is strictly dominated by announcement \( m_k = g \) for patient agents, and upon elimination of announcement \( m_k = 1 \), announcement \( m_k = 2 \) strictly dominates announcement \( m_k = g \) for patient agents. What may not be so easy to construct is an allocation rule \( \hat{c} \) that is feasible both on and off the equilibrium path. Note that the definition of feasibility, (2), is independent of any notion of equilibrium; that is, (2) must be valid for \textit{any} announcement vector, \( m^N \in M^N \). In the subsequent section we
construct an indirect mechanism \((M, \hat{\epsilon})\) and demonstrate that: (i) the allocation rule, \(\hat{\epsilon}\), is feasible for any arbitrary announcement vector; and (ii) the indirect mechanism unique implements an allocation can that be made arbitrarily close to the best weakly implementable allocation \(c^* (0)\).

5 An Indirect Mechanism

We consider an indirect mechanism \((\hat{M}, \hat{\epsilon})\), where \(\hat{M} = \{1, 2, g\}\) and \(\hat{\epsilon}\) is described below. Let \(\hat{m}_j \in \{1, 2, g\}\) represents the announcement of agent \(j\) and define \(\hat{\epsilon}^{k-1} \in T^{k-1}\) as a vector of length \(k - 1\) where for each \(j \leq k - 1\), \(\hat{\epsilon}_j = 1\) if \(\hat{m}_j = 1\) and \(\hat{\epsilon}_j = 2\) if either \(\hat{m}_j = 2\) or \(\hat{m}_j = g\). In words, vector \(\hat{\epsilon}^{k-1}\) is constructed from the message vector \(\hat{m}^{k-1}\) by simply replacing all of the \(g\)'s with 2's. The basic construction of the allocation rule \(\hat{\epsilon}\) uses \(c^* (\hat{\epsilon})\). For convenience denote \(\hat{c}^1_k | \hat{m}_k\) as the date-\(s\) payoff that agent \(k\) receives from allocation rule \(\hat{\epsilon}\) when he announces \(\hat{m}_k\).

We now describe \(\hat{c}\). If agent \(j\) announces \(\hat{m}_k = 1\), then

\[
\hat{c}^1_k |_{1} = \begin{cases} 
 c^*_k (\hat{\epsilon}) (\hat{\epsilon}^{k-1}, 1) & \text{if } \hat{m}_j \in \{1, 2\} \text{ for all } j < k \\
 0 & \text{if } \hat{m}_j = g \text{ for some } j < k
\end{cases}
\]

(5)

If agent \(k\) announces \(\hat{m}_k = 1\), then he receives the date-1 consumption payoff that they would get under the direct revelation mechanism \((T, c^* (\hat{\epsilon}))\) only if all earlier agents \(j < k\) announce either \(\hat{m}_j = 1\) or \(\hat{m}_j = 2\); otherwise his date-1 consumption is zero. The date-2 consumption payoff associated with the announcement \(\hat{m}_k = 1\) is zero, as in the direct revelation mechanism \((T, c^* (\hat{\epsilon}))\).

If agent \(k\) announces \(\hat{m}_k = 2\), then

\[
\hat{c}^1_k |_{2} = 0,
\]

\[
\hat{c}^2_k |_{2} = c^* (\hat{\epsilon}) (\hat{\epsilon}^{N}).
\]

(6)

If agent \(k\) announces \(\hat{m}_k = 2\), then he receives the same consumption payoff that they would obtain under the direct revelation mechanism \((T, c^* (\hat{\epsilon}))\) assuming that any announcement \(\hat{m}_j = g\) is treated as if \(\hat{m}_j = 2\).

Finally, if agent \(k\) announces \(\hat{m}_k = g\), then

\[
\hat{c}^1_k |_{g} = 0
\]

\[
\hat{c}^2_k |_{g} = \begin{cases} 
 c^*_k (\hat{\epsilon}^{k-1}, 1) (1 + \epsilon) & \text{if } \hat{m}_j \in \{1, 2\} \text{ for all } j < k \\
 \epsilon & \text{otherwise}
\end{cases}
\]

(7)

where \(\epsilon > 0\) is arbitrarily small. If agent \(k\) announces \(\hat{m}_k = g\), then he receives a zero payoff at date 1. At date 2, he receives a payoff that is slightly bigger than the date-1 payoff he would receive by announcing \(\hat{m}_k = 1\); see (5) which implies that \(\hat{c}^2_k |_{g} = \hat{c}^1_k |_{1} + \epsilon\). Hence, for any patient agent \(k\), announcing \(\hat{m}_k = 1\) is strictly dominated by announcing \(\hat{m}_k = g\).

Since the allocation rule \(\hat{\epsilon}\) given by (5)-(7) depends on \(\hat{\epsilon}\) and on \(\epsilon\), we will denote it as \(\hat{\epsilon} (\hat{\epsilon}, \epsilon)\). Our first result is that allocation rule \(\hat{\epsilon} (\hat{\epsilon}, \epsilon)\) is feasible in the sense that

\[
\forall \hat{m}^N \in \hat{M}^N, \sum_{k=1}^{N} \left[ R^1_k (\hat{\epsilon}, \epsilon) (\hat{\epsilon}^{k-1}, \hat{\epsilon}_k) + c^2_k (\hat{\epsilon}, \epsilon) (\hat{\epsilon}^{N}) \right] \leq RY.
\]

In particular,
Proposition 1. For all $\delta > 0$ that admit a solution to problem (4), there exists an $\varepsilon > 0$ such that if $\varepsilon \in (0, \varepsilon]$ the allocation rule $\hat{c}(\delta, \varepsilon)$ is feasible.

The proof to this proposition, which is tedious, is relegated to the Appendix. We would like to point out, however, that it is not at all obvious that $\hat{c}(\delta, \varepsilon)$ is feasible. This is because, for allocation rule $\hat{c}(\delta, \varepsilon)$, the payoff associated with announcing $\hat{m}_k = g$ is “essentially the same” as the payoff associated with announcement $\hat{m}_k = 1$, but the payoff to announcing $\hat{m}_k = 2$ is calculated by replacing all announcements $\hat{m}_k = g$ with $\hat{m}_k = 2$—not $\hat{m}_k = 1$. And, the payoffs associated with announcing $\hat{m}_k = 1$ and $\hat{m}_k = 2$ are not related to one another in a simple way.

Just as in the stripped-down example from Section 4, it is rather straightforward to demonstrate that the allocation rule $\hat{c}(\delta, \varepsilon)$ admits only one equilibrium, which is characterized by truth-telling for all agents. This means that mechanism $(\hat{M}, \hat{c}(\delta, \varepsilon))$ does not have a bank run equilibrium. Furthermore, allocation $\hat{c}(\delta, \varepsilon)$ can be made arbitrarily close to the best weakly implementable allocation $c^*(0)$ by choosing $\delta$ arbitrarily close to zero.

Proposition 2. For all $\delta > 0$ that admit a solution to problem (4), there exists an $\varepsilon > 0$ such that for all $\varepsilon \in (0, \varepsilon]$ the indirect mechanism $(\hat{M}, \hat{c}(\delta, \varepsilon))$ has a unique equilibrium characterized by truth-telling. Furthermore, the indirect mechanism $(\hat{M}, \hat{c}(\delta, \varepsilon))$ achieves the maximum of problem (4) and the allocation rule $\hat{c}(\delta, \varepsilon)$ can be made arbitrarily close to the best weakly implementable allocation rule $c^*(0)$ by choosing $\delta$ arbitrarily close to zero.

Proof. A mechanism $(\hat{M}, \hat{c}(\delta, \varepsilon))$ implies a symmetric Bayesian game $\Gamma = \{T, S\}$ where: $T = \{1, 2\}$ is the set of types; $s_t \in M$ is the player’s message contingent on his type $t \in T$; and $S = \{(s_1, s_2) \in M^2\}$ is the set of pure strategies. We solve the game by iterated elimination of strictly dominated strategies.

Round 1 - Any strategy $(s_1, s_2) \in S$, with $s_1 \neq 1$, is strictly dominated by $(1, s_2)$ since, contingent on being impatient, an agent only derives utility of period 1 consumption. Additionally, any strategy $(s_1, 1)$ is strictly dominated by $(s_1, g)$ since, contingent on being patient, agents are indifferent between period 1 or period 2 consumption and announcing $g$ always gives total payment $\varepsilon$ higher than announce 1. Let $S^1 = \{(1, 2), (1, g)\}$ denotes the set of strategies which survived the first round of elimination of strictly dominated strategies.

Round 2 - The payment of an agent whose announcement is 2 is independent of other players strategies in $S^1$ — it is always the payment as if everyone else is playing the truth-telling strategy $s = (1, 2)$. Hence, the payoff of a patient player who announces 2 is given by

$$\sum_{n=1}^{N} \tilde{\pi}_n \sum_{t^N \in \pi_n} \frac{1}{n} \sum_{k \in Q_n} \rho u \left( c^*_{k}^{(\delta)}(t^k(2, t^N_{k+1})) \right).$$

Where $t^{k-1} = (t_1, \ldots, t_{k-1})$ and $t^N_{k+1} = (t_{k+1}, \ldots, t_N)$ are the other agents types. We should note that, when agents use strategies in $S^1$ they always reveal their types, so expectations can be taken correctly.

There are two possibilities of payment for a player who announces $g$: either he is the first agent in the queue to announce $g$, in which case his payment is $c_{k}^*(t^{k-1}, 1) + \varepsilon$; or he is not the first agent, in which case his payment is $\varepsilon$. Therefore, the payoff of a patient player who announces $g$ is bounded above by

$$\sum_{n=1}^{N} \tilde{\pi}_n \sum_{t^N \in \pi_n} \frac{1}{n} \sum_{k \in Q_n} \rho u \left( c^*_{k}^{(\delta)}(t^{k-1}, 1) + \varepsilon \right).$$
Since \( u \) is continuous, there exists \( \epsilon > 0 \) such that
\[
\sum_{n=1}^{N} \hat{\alpha}_n \sum_{T \in \mathcal{P}_n} \frac{1}{n} \sum_{k \in Q_n} \left\{ \rho u \left( c^*_{k}(\delta)(t^{k-1},1) + \epsilon \right) - \rho u \left( c^*_{k}(\delta)(t^{k-1},1) \right) \right\} < \delta.
\]

The incentive compatibility given by equation (3) implies
\[
\sum_{n=1}^{N} \hat{\alpha}_n \sum_{T \in \mathcal{P}_n} \frac{1}{n} \sum_{k \in Q_n} \rho u \left( c^*_{k}(\delta)(t^{k-1},2,t_{k+1}^N) \right) > \sum_{n=1}^{N} \hat{\alpha}_n \sum_{T \in \mathcal{P}_n} \frac{1}{n} \sum_{k \in Q_n} \rho u \left( c^*_{k}(\delta)(t^{k-1},1) + \epsilon \right).
\]

The above two inequalities combined implies that
\[
\sum_{n=1}^{N} \hat{\alpha}_n \sum_{T \in \mathcal{P}_n} \frac{1}{n} \sum_{k \in Q_n} \rho u \left( c^*_{k}(\delta)(t^{k-1},2,t_{k+1}^N) \right) > \sum_{n=1}^{N} \hat{\alpha}_n \sum_{T \in \mathcal{P}_n} \frac{1}{n} \sum_{k \in Q_n} \rho u \left( c^*_{k}(\delta)(t^{k-1},1) + \epsilon \right).
\]

Therefore, the strategy \((1, g)\) is strictly dominated by the strategy \((1, \tilde{g})\) in \(S^1\). Let \(S^2 = \{(1, 2)\}\) be the set of strategies which survived the second round of elimination of strictly dominated strategies. Since \(S^2\) is a singleton, the game is iterated strict dominance solvable. The unique survival strategy is the truth-telling \(s = (1, 2)\), i.e., the implied outcome is the same of the truth-telling equilibrium in the optimal direct mechanism.

For the last, Berge’s maximum theorem implies that the allocation rule \(c^*(0)\) is approximated by \(\hat{c}(\delta, \epsilon)\) when \(\delta\) is close to zero. Which concludes the proof. \(\square\)

6 Peck and Shell

PS generate a bank run equilibrium. But they assume that \(\rho R > 1\). Allocation rule (5)-(7) cannot be applied in this case since Proposition 1 may not be valid. We demonstrate below, however, when agents are restricted to using symmetric pure strategies, the best weakly implementable allocation rule, \(c^*(0)\), can be uniquely implemented by an alternative indirect mechanism, \((\hat{M}, \hat{c})\).

The indirect mechanism \((\hat{M}, \hat{c})\) has \(\hat{M} = \{1, 2, g\}\) and \(\hat{c}\) is described below. Let \(\hat{m}_j \in \{1, 2, g\}\) represent the announcement of agent \(j\) and define \(\tilde{k}^{k-1} \in T^{k-1}\) as a vector of length \(k - 1\) where for each \(j \leq k - 1, \tilde{t}_j = 1\) if either \(\hat{m}_j = 1 \) or \(\hat{m}_j = g\) and \(\tilde{t}_j = 2\) if \(\hat{m}_j = 2\). It is important to emphasize that the relationship between \(\tilde{m}_j\) and \(\tilde{t}_j\) is different from that of relationship between \(\hat{m}_j\) and \(\hat{t}_j\) from the previous section. In particular, vector \(\tilde{k}^{k-1}\) is constructed from the message vector \(\hat{m}^{k-1}\) by replacing all of the \(g\)’s with \(1\)’s. (In vector \(\tilde{k}^{k-1}\), all of the \(g\)’s are replaced by \(2\)’s.) The basic construction of the allocation rule \(\hat{c}\) uses the best weakly implementable allocation rule \(c^*(0)\). We now denote the date-\(s\) payoff that agent \(k\) receives from allocation rule \(\hat{c}\) when he announces \(\hat{m}_k\) as \(\hat{c}^s_{k} | m_k\).

We now describe contract \(\hat{c}\). If agent \(k\) announces \(\hat{m}_k = 1\), then
\[
\begin{align*}
\hat{c}^1_{k1} & = c^1_{j}(0) \left( \tilde{t}_j^{-1}, 1 \right) \\
\hat{c}^1_{k2} & = 0.
\end{align*}
\]

If agent \(k\) announces \(\hat{m}_k = 1\), then he receives the date-1 consumption payoff that they would get under the direct revelation mechanism \((T, c^*(\delta))\) assuming that any announcement \(\hat{m}_j = g\) is treated as \(\hat{m}_j = 1\). The date-2 consumption payoff associated with the announcement \(\hat{m}_k = 1\) is zero, as in the direct revelation mechanism \((T, c^*(\delta))\).
If agent $j$ announces $m_j = 2$, then
\begin{align*}
\hat{c}_j^1|_2 &= 0, \\
\hat{c}_j^2|_2 &= \hat{c}_j^2(0)\left(I^N\right).
\end{align*}

If agent $k$ announces $\bar{m}_k = 2$, then he receives the same consumption payoff that they would obtain under the direct revelation mechanism $(T, c^\ast(\delta))$ assuming that any announcement $\bar{m}_j = g$ is treated as if $\bar{m}_j = 2$.

Finally, if agent $j$ announces $m_j = g$, then
\begin{align*}
\hat{c}_j^1|_g &= 0 \\
\hat{c}_j^2|_g &= \begin{cases} c_j^1(0)\left(\bar{t}^{i-1},1\right) + \epsilon & \text{if } \bar{m}_i = 1 \text{ for all } i \neq k \in N/0 \\ 0 & \text{otherwise} \end{cases},
\end{align*}

where $\epsilon > 0$ is arbitrarily small. If agent $k$ announces $\bar{m}_k = g$, he receives a zero date 1 payoff. His date 2 payoff is slightly bigger than what he would receive by announcing $\bar{m}_k = 1$ but only if all other agents $j$ announce $m_j = 1$; otherwise, he receives a payoff of zero.

Note that allocation rule $\hat{e}$ given by (8)-(10) is feasible as long as $\epsilon$ is not “too big,” which is what we assume. In particular, since the payoff associated with announcement $\bar{m}_j = g$ is essentially equal to the payoff associated with the announcement $\bar{m}_i = 1$ and the payoff for the announcement $\bar{m}_k = 2$ is calculated replacing any announcements $\bar{m}_j = g$ with $\bar{m}_j = 1$, the planner can afford to pay $c_j^1(\bar{t}^{i-1}) + \epsilon$ to agent $j$ who announces $\bar{m}_j = g$ in the second period and $c_j^2(0)$ to any any agent $k$ who announces $\bar{m}_k = 2$ if $\epsilon \leq (R-1)c_j^1(\bar{t}^{i-1})$. The indirect mechanism $(\bar{M}, \hat{e})$ has a very nice feature. In particular,

**Proposition 3.** The indirect mechanism $(\bar{M}, \hat{e})$ uniquely implements the best weakly implementable allocation in $c^\ast(0)$ in symmetric pure Nash equilibrium strategies.

**Proof.** All impatient agents $k$ announce truthfully since announcing $\bar{m}_k = 1$ results in a strictly positive date-1 payoff and announcing $\bar{m}_k \neq 1$ results in a date-1 payoff equal to zero.

First, there cannot exist an equilibrium where all patient agents $k$ announce $\bar{m}_k = 1$. Suppose such an equilibrium exists, and some patient agent $j$ defects from proposed equilibrium play by announcing $\bar{m}_j = g$. Agent $j$’s payoff will be strictly greater than the payoff associated with announcing $\bar{m}_j = 1$ by the amount $\epsilon > 0$; a contradiction.

Second, there cannot be an equilibrium where all patient players $k$ announce $\bar{m}_k = g$. To see this, note that if agent $k$ announces $\bar{m}_k = g$, then his payoff will zero if he is not the only patient agent in the economy. The (proposed) equilibrium payoff, therefore, is

\begin{equation}
\frac{\bar{\pi}_1}{N} \sum_{k=1}^{N} \rho u \left[c_k^1\left(1^{k-1},1\right)(1+\epsilon)\right] + (1-\bar{\pi}_1)\rho u(0), \tag{11}
\end{equation}

where $\bar{\pi}_1$ is the probably that patient agent $k$ is the only patient agent in the economy. If instead, agent $k$ announces $\bar{m}_k = 1$, his payoff will be

\begin{equation}
\frac{1}{N} \sum_{k=1}^{N} \rho u \left[c_k^1\left(1^{k-1},1\right)\right]. \tag{12}
\end{equation}

For $\epsilon > 0$ sufficiently small, payoff (12) will exceed (11); a contradiction.
Third, there is an equilibrium where all patient agents $k$ announce $\tilde{m}_k = 2$. By construction, patient agent $j$ has no incentive to announce $\tilde{m}_j = 1$ when all other agents announce truthfully, i.e., allocation rule $c^∗(0)$ is incentive compatible for patient agents when $\tilde{m}_j$ is restricted to the set $\{1, 2\}$. Suppose, instead, patient agent $j$ defects from equilibrium play and announces $\tilde{m}_j = g$. In this case, his payoff will be only slightly greater than the payoff associated with announcing $\tilde{m}_j = 1$ if and only if he is the only patient agent in the economy—an event that occurs with probability $\tilde{\pi}_1$. With probably $1 - \tilde{\pi}_1$, there are other patient agents $k$ who announce $\tilde{m}_k = 2$, which implies that patient agent $j$ receives a zero payoff. For any $\tilde{\pi}_1 < 1$, there exists an $\varepsilon > 0$ sufficiently small so that the expected payoff associated with announcing $\tilde{m}_j = g$ is strictly less than that associated with announcing $\tilde{m}_j = 1$ when all other agents announce truthfully.

Therefore, the unique pure strategy equilibrium for mechanism $(\tilde{M}, \tilde{c})$ is characterized by $\tilde{m}_k = t_k$ for all $j \in \mathbb{N}/0$. These strategies implement the best weakly implementable allocation in $c^∗(0)$.

7 Final Comments

In a way, the message that underlies this paper is a rather negative one: A well designed deposit contract can prevent bank run equilibria in the classic Diamond-Dybvig environment. The message is negative because the Diamond-Dybvig model is supposed to be a model of banking instability. Green and Lin (2003) conjectured that the overlapping generations nature of depositors in the real world and/or moral hazard associated with the people who operate banks may prevent agents from using efficient mechanisms, which has implications for banking instability. These conjectures, unfortunately, do not appear to be supported by subsequent research. An important assumption in the Diamond-Dybvig environment is that the planner can ex ante commit to implement contract allocations. Relaxing this assumption may result in bank run equilibria; see, for example, Ennis and Keister (2009a). Perhaps assuming that agents cannot not fully commit is a fruitful avenue for future work.

References


---

10 The Peck and Shell (2003) can be interpreted as a way of modelling overlapping generations in the sense that there is no “last” depositor in an overlapping generations model, and in Peck and Shell (2003) depositors do not know if they are the last depositor. Andolfatto and Nosal (2008) assume that one of the agents operates the bank in a Diamond-Dybvig environment, and find that there do no exist bank run equilibria.
We can solve the above two equations for \( \lambda \) impatient when every agent announces to be of type \( \lambda \) the Lagrange multiplier of the incentive compatibility (3). By simplicity, Lemma 1.

For all \( k \)

In order to prove proposition (1) we use the following lemma.

Appendix

In order to prove proposition (1) we use the following lemma.

**Lemma 1.** For all \( k \in \mathbb{N} \) and \( \bar{t}^{k-1} \in T^{k-1} \) we have \( c_k^{1*}(\delta)(\bar{t}^{k-1}, 1) < c^{2*}(\delta)(\bar{t}^{k-1}, 2^{N-k+1}) \). Where \( 2^n \) denotes the \( n \)-dimensional vector of twos.

**Proof.** Since \( c^*(\delta) \) solves problem (4), it satisfies the implied first-order conditions.\(^{11}\) Let \( \lambda_iN \) denotes the Lagrange multiplier of the feasibility constraint (2) for each \( t^N \in T^N \) and \( \mu \) denotes the Lagrange multiplier of the incentive compatibility (3). By simplicity, \( \lambda_iN \) is normalized by \( \pi_{n_iN} / \binom{N}{n_iN} \), where \( n_iN \) denotes the number of type 2 players in queue \( t^N \). And \( \mu \) is normalized by \( \bar{\pi} = \sum_{n=1}^{N} \pi_n / (\binom{N}{n}) \). Since \( u'(0) = \infty \) the constraint \( c^1 \geq 0 \) and \( c^2 \geq 0 \) are not binding and the respective Lagrange multipliers can be ignored. The first order conditions of the problem are given below.

\[
\begin{align*}
\left[c_1^1(\bar{t}^{k}), \bar{t}_k = 1\right] : \sum_{n=0}^{N} \frac{\pi_n}{\binom{N}{n}} \sum_{t^N \in P_n} \left\{ u'[c_1^1(\bar{t}^{k})] - \lambda_iN R \right\} - \sum_{n=1}^{N} \frac{\pi_n}{\binom{N}{n}} \frac{\mu u'}{n_iN} \left[c_1^2(\bar{t}^{k})\right] = 0 \tag{13}
\end{align*}
\]

for all \( k \in \mathbb{N} \) and \( \bar{t}^{k-1} \in T^{k-1} \), and

\[
\begin{align*}
\left[c^2(t^N), n_iN > 0\right] : \frac{\pi_n}{\binom{N}{n_iN}} \left\{ \rho u'[c^2(t^N)] - \lambda_iN + \frac{\mu}{n_iN} u'[c^2(t^N)] \right\} = 0 \tag{14}
\end{align*}
\]

We can solve the above two equations for \( \lambda_iN \) and obtain

\[
\lambda_iN = \begin{cases} 
\rho \left(1 + \frac{\mu}{n_iN}\right) u'[c^2(t^N)]; & \text{if } n_iN > 0 \\
\frac{1}{R} u'[c_1^1(t^N)]; & \text{if } n_iN = 0
\end{cases}
\]

Note that \( c^2(t^N) \) is not defined if \( t^N = 1^N = (1, 1, \ldots, 1) \) — there is no second period payments when every agent announces to be of type impatient in the first period. In order to keep the notation short, let us define \( u'[c_1^N(1^N)] = R\rho u'[c^2(1^N)] \) and \( 1/n_iN = 0 \). Then, \( \lambda_iN \) is given by

\[
\lambda_iN = \rho \left(1 + \frac{\mu}{n_iN}\right) u'[c^2(t^N)]. \tag{15}
\]

\(^{11}\)From now on we will denote \( c^*(\delta) \) just by \( c \) in order to keep the notation short.
After replacing equation (15) in equation (13) we obtain that for all \( k \in \mathbb{N} \) and \( \tilde{t}^k = (\tilde{t}^{k-1}, 1) \in T^{k-1}:

\[
\sum_{n=0}^{N} \frac{\pi_n}{(N)} \sum_{i^{N} \in P_{n}^{N}} u' \left[ c_{k}^{i}(\tilde{t}^{k}) \right] - \sum_{n=1}^{N} \frac{\pi_n}{(N)} \sum_{i^{N} \in P_{n}^{N}, i = (1,2)} \frac{\mu P}{n_t N} u' \left[ c_{k}^{i}(\tilde{t}^{k}) \right] = \sum_{n=0}^{N} \frac{\pi_n}{(N)} \sum_{i^{N} \in P_{n}^{N}, i = (1,2)} R \rho \left( 1 + \frac{\mu}{n_t N} \right) u' \left[ c^{2}(t^{N}) \right]
\]

\[
\Rightarrow \quad \mathbb{P} \left[ t^{k} = (\tilde{t}^{k-1}, 1) \right] - \sum_{n=1}^{N} \frac{\pi_n}{(N)} \sum_{i^{N} \in P_{n}^{N}, i = (1,2)} \frac{\mu P}{n_t N} u' \left[ c_{k}^{i}(\tilde{t}^{k}) \right] = \sum_{n=0}^{N} \frac{\pi_n}{(N)} \sum_{i^{N} \in P_{n}^{N}, i = (1,2)} R \rho \left( 1 + \frac{\mu}{n_t N} \right) u' \left[ c^{2}(t^{N}) \right]
\]

We can also write the equation in expectations, which yields to the formula

\[
\left[ 1 - \gamma(\tilde{t}^{k-1}) \right] u' \left[ c_{k}^{i}(\tilde{t}^{k}) \right] = \mathbb{E}_{t^{N}|t^{k} = (\tilde{t}^{k-1}, 1)} \left\{ R \rho \left( 1 + \frac{\mu}{n_t N} \right) u' \left[ c^{2}(t^{N}) \right] \right\}
\]

(16)

where \( \gamma(\tilde{t}^{k-1}) = \mathbb{P} \left[ t^{k} = (\tilde{t}^{k-1}, 2) \right] \mathbb{E}_{t^{N}|t^{k} = (\tilde{t}^{k-1}, 2)} \left[ \mu P / n_t N \right] / \mathbb{P} \left[ t^{k} = (\tilde{t}^{k-1}, 1) \right].

The result will be derived from equation (16). Let us use induction on \( k \in \mathbb{N} \) starting from \( k = N \) and going down until \( k = 1 \).

**Proof for \( k = N \):** Fix any \( t^{N} = (\tilde{t}^{N-1}, 1) \). From equation (16) we have that

\[
\left[ 1 - \frac{\mathbb{P} \left[ t^{N} = (\tilde{t}^{N-1}, 2) \right]}{\mathbb{P} \left[ t^{N} = (\tilde{t}^{N-1}, 1) \right]} \right] \times \frac{\mu P}{n_t (N-1)} u' \left[ c_{N}^{i}(\tilde{t}^{N-1}, 1) \right] = R \rho \left( 1 + \frac{\mu}{n_t(N-1)} \right) u' \left[ c^{2}(\tilde{t}^{N-1}, 1) \right]
\]

which implies that \( u' \left[ c_{N}^{i}(\tilde{t}^{N-1}, 1) \right] > u' \left[ c^{2}(\tilde{t}^{N-1}, 1) \right] \). \( u \) is strictly concave implies that \( c_{N}^{i}(\tilde{t}^{N-1}, 1) < c^{2}(\tilde{t}^{N-1}, 1) \). We know that the resources constraints holds at equality because \( u \) is strictly increasing. Therefore,

\[
n_{t^{N-1}N} c^{2}(\tilde{t}^{N-1}, 2) = n_{t^{N}} c^{2}(\tilde{t}^{N-1}, 1) + R c_{N}^{N}(\tilde{t}^{N-1}, 1)
\]

And after reorganizing the equation above we have that

\[
c^{2}(\tilde{t}^{N-1}, 2) = \frac{n(\tilde{t}^{N-1}, 1)}{n(\tilde{t}^{N-1}, 1) + 1} c^{2}(\tilde{t}^{N-1}, 1) + \frac{1}{n(\tilde{t}^{N-1}, 1) + 1} R c_{N}^{N}(\tilde{t}^{N-1}, 1) > c_{N}^{i}(\tilde{t}^{N-1}, 1).
\]

Hence, for the case \( k = N \), we can conclude that \( c_{k}^{i}(\tilde{t}^{k-1}, 1) < c^{2}(\tilde{t}^{k-1}, 2^{N-k+1}) \).

**Proof for \( k < N \):** Assume the result holds for all \( j > k \) and \( \tilde{t} = (\tilde{t}^{j-1}, 1) \in T^{j} \). That is, for all \( j > k \) we have

\[
c_{j}^{i}(\tilde{t}^{j-1}, 1) < c^{2}(\tilde{t}^{j-1}, 2^{N-j}).
\]

Let us show it also holds for \( k \). Fix some \( \tilde{t}^{k} = (\tilde{t}^{k-1}, 1) \in T^{k-1} \), then equation (16) is given by

\[
u' \left[ c_{k}^{i}(\tilde{t}^{k}) \right] = \frac{1}{1 - \gamma(\tilde{t}^{k-1})} \mathbb{E}_{t^{N}|t^{k} = (\tilde{t}^{k-1}, 1)} \left\{ R \rho \left( 1 + \frac{\mu}{n_t N} \right) u' \left[ c^{2}(t^{N}) \right] \right\}
\]

Note that, for any function \( X : T^{N} \rightarrow \mathbb{R} \), the conditional expectation can be decomposed as

\[
\mathbb{E}_{t^{N}|t^{k} = \tilde{t}^{k}} \left\{ X(t^{N}) \right\} = \sum_{j=k+1}^{N} \mathbb{P} \left[ t^{j} = (\tilde{t}^{j}, 2^{j-k+1}) \bigg| t^{k} = \tilde{t}^{k} \right] \mathbb{E}_{t^{N}|t^{j} = (\tilde{t}^{j}, 2^{j-k+1})} \left\{ X(t^{N}) \right\} + \mathbb{P} \left[ t^{N} = (\tilde{t}^{k}, 2^{N-k}) \bigg| t^{k} = \tilde{t}^{k} \right] X(\tilde{t}^{k}, 2^{N-k}).
\]
Applying this decomposition to equation (16) we obtain

\[ u' \left[ c_k^1 (\hat{p}^k) \right] = \frac{1}{1 - \gamma (\hat{p}^k - 1)} \left\{ \sum_{j=k+1}^{N} \mathbb{P} \left[ t^j = (\hat{p}^k, 2^{i-k-1}, 1) \bigg| t^k = \hat{p}^k \right] \mathbb{E}_{t^N | \hat{t} = (\hat{p}, 2^{i-k-1}, 1)} \left\{ R \rho \left( 1 + \frac{\mu}{n_{1N}} \right) u' \left[ c^2 (t^N) \right] \right\} \right. \\
+ \left. \mathbb{P} \left[ t^N = (\hat{p}^k, 2^{N-k}) \bigg| t^k = \hat{p}^k \right] R \rho \left( 1 + \frac{\mu}{n_{2N}} \right) u' \left[ c^2 (\hat{p}^k, 2^{N-k}) \right] \right\}. \]

By equation (16) we know that

\[ 1 - \gamma (\hat{p}^k, 2^{i-k-1}) u' \left[ c_k^1 (\hat{p}^k, 2^{i-k-1}, 1) \right] = \mathbb{E}_{t^N | \hat{t} = (\hat{p}, 2^{i-k-1}, 1)} \left\{ R \rho \left( 1 + \frac{\mu}{n_{1N}} \right) u' \left[ c^2 (t^N) \right] \right\} \]

for \( j = k + 1, \ldots, N \). Hence,

\[ u' \left[ c_k^1 (\hat{p}^k) \right] = \frac{1}{1 - \gamma (\hat{p}^k - 1)} \left\{ \sum_{j=k+1}^{N} \mathbb{P} \left[ t^j = (\hat{p}^k, 2^{i-k-1}, 1) \bigg| t^k = \hat{p}^k \right] \left[ 1 - \gamma (\hat{p}^k, 2^{i-k-1}) \right] u' \left[ c_k^1 (\hat{p}^k, 2^{i-k-1}, 1) \right] + \right. \\
+ \left. \mathbb{P} \left[ t^N = (\hat{p}^k, 2^{N-k}) \bigg| t^k = \hat{p}^k \right] R \rho \left( 1 + \frac{\mu}{n_{2N}} \right) u' \left[ c^2 (\hat{p}^k, 2^{N-k}) \right] \right\}. \]

By the inductive hypothesis we know that \( c_k^1 (\hat{p}^k, 2^{i-k-1}, 1) < c^2 (\hat{p}^k, 2^{N-k}) \), which implies that

\[ u' \left[ c_k^1 (\hat{p}^k) \right] > \frac{1}{1 - \gamma (\hat{p}^k - 1)} \left\{ \sum_{j=k+1}^{N} \mathbb{P} \left[ t^j = (\hat{p}^k, 2^{i-k-1}, 1) \bigg| t^k = \hat{p}^k \right] \left[ 1 - \gamma (\hat{p}^k, 2^{i-k-1}) \right] u' \left[ c_k^1 (\hat{p}^k, 2^{i-k-1}, 1) \right] + \right. \\
+ \left. \mathbb{P} \left[ t^N = (\hat{p}^k, 2^{N-k}) \bigg| t^k = \hat{p}^k \right] R \rho \left( 1 + \frac{\mu}{n_{2N}} \right) u' \left[ c^2 (\hat{p}^k, 2^{N-k}) \right] \right\} \\
= \frac{1}{1 - \gamma (\hat{p}^k - 1)} \left\{ \sum_{j=k+1}^{N} \mathbb{P} \left[ t^j = (\hat{p}^k, 2^{i-k-1}, 1) \bigg| t^k = \hat{p}^k \right] \left[ 1 - \gamma (\hat{p}^k, 2^{i-k-1}) \right] \right. \\
+ \left. \mathbb{P} \left[ t^N = (\hat{p}^k, 2^{N-k}) \bigg| t^k = \hat{p}^k \right] R \rho \left( 1 + \frac{\mu}{n_{2N}} \right) \right\} u' \left[ c^2 (\hat{p}^k, 2^{N-k}) \right] \\
= \frac{1}{1 - \gamma (\hat{p}^k - 1)} \left\{ \sum_{j=k+1}^{N} \mathbb{P} \left[ t^j = (\hat{p}^k, 2^{i-k-1}, 1) \bigg| t^k = \hat{p}^k \right] \right. \\
- \left. \sum_{j=k+1}^{N} \mathbb{P} \left[ t^j = (\hat{p}^k, 2^{i-k-1}, 1) \bigg| t^k = \hat{p}^k \right] \frac{\mathbb{P} \left[ t^j = (\hat{p}^k, 2^{i-k}) \right]}{\mathbb{P} \left[ t^j = (\hat{p}, 2^{i-k-1}, 1) \right]} \mathbb{E}_{t^N | \hat{t} = (\hat{p}, 2^{i-k-1})} \left[ \frac{\mu \rho}{n_{1N}} \right] \\
+ \left. \mathbb{P} \left[ t^N = (\hat{p}^k, 2^{N-k}) \bigg| t^k = \hat{p}^k \right] R \rho \left( 1 + \frac{\mu}{n_{2N}} \right) \right\} u' \left[ c^2 (\hat{p}^k, 2^{N-k}) \right]. \]
After simplify the above equation we obtain

\[
\mu' \left[ c_k^1(P) \right] > \frac{1}{1 - \gamma(P_{1k})} \left\{ 1 - \mathbb{P} \left[ t_e^{(k)} = (P_{1k}, 2^{N-k}) | t^k = \tilde{P} \right] \right.
\]

\[
- \sum_{j=k+1}^{N} \frac{\mathbb{P} [ t_j = (P_{1k}, 2^{j-k}) ]}{\mathbb{P} [ t^k = \tilde{P} ]} \mathbb{E}_{t^{(k)} = (P_{1k}, 2^{j-k})} \left[ \frac{\mu}{n_{t}} \right] \]

\[
+ \mathbb{P} \left[ t_e^{(k)} = (P_{1k}, 2^{N-k}) | t^k = \tilde{P} \right] R \rho \left( 1 + \frac{\mu}{n_{(P_{1k}, 2^{N-k})}} \right) \mu' \left[ c^2(P_{1k}, 2^{N-k}) \right]. \quad (17)
\]

The fact that the queue position is withdrawn uniformly implies that

\[
\mathbb{P} [ t_j = (P_{1k}, 2^{j-k}) ] = \mathbb{P} [ t_j = (P_{1k-1}, 1, 2^{j-k}) ] = \mathbb{P} [ t_j = (P_{1k-1}, 1, 2^{j-k}) ]
\]

and

\[
\mathbb{E}_{t^{(k)} = (P_{1k}, 2^{j-k})} \left[ \frac{\mu}{n_{t}} \right] = \mathbb{E}_{t^{(k)} = (P_{1k-1}, 2^{j-k})} \left[ \frac{\mu}{n_{t}} \right] = \mathbb{E}_{t^{(k)} = (P_{1k-1}, 2^{j-k})} \left[ \frac{\mu}{n_{t}} \right] .
\]

This implies that

\[
\sum_{j=k+1}^{N} \frac{\mathbb{P} [ t_j = (P_{1k}, 2^{j-k}) ]}{\mathbb{P} [ t^k = \tilde{P} ]} \mathbb{E}_{t^{(k)} = (P_{1k}, 2^{j-k})} \left[ \frac{\mu}{n_{t}} \right] = \sum_{j=k+1}^{N} \frac{\mathbb{P} [ t_j = (P_{1k-1}, 2^{j-k}) ]}{\mathbb{P} [ t^k = (P_{1k-1}, 1) ]} \mathbb{E}_{t^{(k)} = (P_{1k-1}, 2^{j-k})} \left[ \frac{\mu}{n_{t}} \right] \]

\[
= \sum_{j=k+1}^{N} \frac{\mathbb{P} [ t_j = (P_{1k-1}, 2^{j-k}) ]}{\mathbb{P} [ t^k = (P_{1k-1}, 1) ]} \mathbb{E}_{t^{(k)} = (P_{1k-1}, 2^{j-k})} \left[ \frac{\mu}{n_{t}} \right] \]

\[
+ \frac{\mathbb{P} [ t^k = (P_{1k-1}, 2^{N-k}) ]}{\mathbb{P} [ t^k = (P_{1k-1}, 1) ]} \frac{\mu}{n_{(P_{1k-1}, 2^{N-k})}} \cdot \frac{\mu}{n_{(P_{1k-1}, 2^{N-k})}} \quad (18)
\]

Replacing equation (18) in inequality (17) and reorganising the terms in the inequality, we obtain

\[
u' \left[ c_k^1(P_{1k}) \right] > \frac{1}{1 - \gamma(P_{1k})} \left\{ 1 - \gamma(P_{1k-1}) + \frac{\mathbb{P} [ t^k = (P_{1k-1}, 2^{N-k}) ]}{\mathbb{P} [ t^k = (P_{1k-1}, 1) ]} \frac{\mu}{n_{(P_{1k-1}, 2^{N-k})}} \right\}
\]

\[
+ \mathbb{P} \left[ t_e^{(k)} = (P_{1k}, 2^{N-k}) | t^k = \tilde{P} \right] R \rho \left( 1 + \frac{\mu}{n_{(P_{1k}, 2^{N-k})}} - \frac{1}{R \rho} \right) \mu' \left[ c^2(P_{1k}, 2^{N-k}) \right]. \quad (19)
\]
Because \( Rho > 1 \), the inequality (19) implies that

\[
u'[c_k^1(\hat{\pi}^{k-1},1)] = u'[c_k^1(\hat{\pi}^k)] > u'[c^2(\hat{\pi}^k,2^{N-k})] = u'[c^2(\hat{\pi}^{k-1},1,2^{N-k})].
\]

And since \( u \) is concave, it implies that \( c_k^1(\hat{\pi}^{k-1},1) < c^2(\hat{\pi}^{k-1},1,2^{N-k}) \). The resources constraint implies that

\[
\left[ n(\hat{\pi}^{k-1},1,2^{N-k}) + 1 \right] c^2(\hat{\pi}^{k-1},2^{N-k+1}) = n(\hat{\pi}^{k-1},1,2^{N-k}) c^2(\hat{\pi}^{k-1},1,2^{N-k}) + Rc_k^1(\hat{\pi}^{k-1},1)
\]

Thus,

\[
c^2(\hat{\pi}^{k-1},2^{N-k+1}) = \frac{n(\hat{\pi}^{k-1},1,2^{N-k})}{n(\hat{\pi}^{k-1},1,2^{N-k}) + 1} c^2(\hat{\pi}^{k-1},1,2^{N-k}) + \frac{1}{n(\hat{\pi}^{k-1},1,2^{N-k}) + 1} Rc_k^1(\hat{\pi}^{k-1},1) > c_k^1(\hat{\pi}^{k-1},1)
\]

We have shown that the result holds for \( k = N \) and that if it holds for all \( j \in \{k+1, \ldots, N\} \) it holds for \( k \). Therefore, by induction, we can conclude that the result holds for all \( k \in \mathbb{N} \).

**Proof of proposition (1).** We know that, for any vector of announcements \( \hat{m}^N \in \hat{M}^N \), if \( \hat{m}^N \in T^N \) the payments are feasible — they are the same payments of the solution of problem (4). Consider any realized vector of announcements \( \hat{m}^N \in \hat{M}^N \), with \( \hat{m}^N \notin T^N \), and assume w.l.o.g. that the first agent to announce \( g \) has queue position \( k \). As before, let \( \hat{\pi}^N \in T^N \) denotes the vector \( \hat{m}^N \) where \( t_i = \hat{m}_i \) if \( \hat{m}_i \in T \), and \( t_i = 2 \) if \( \hat{m}_i = g \). When agent \( k \) announced \( g \) the first period payments were suspended, hence, the total resources in the beginning of period 2 is

\[
TR = n(\hat{\pi}^{k-1},2^{N-k+1}) c^2(\hat{\pi}^{k-1},2^{N-k+1}).
\]

Let \( l \) denote the total number of agents, besides agent \( k \), who have also announced \( g \) and \( t \) denote the number of agents who announced \( 2 \). The total payments in the second period are given by

\[
TP = c_k^1(\hat{\pi}^{k-1},1) + \epsilon + le + tc^2(\hat{\pi}^N) = c_k^1(\hat{\pi}^{k-1},1) + \epsilon + le + \frac{TR - \sum_{j=k+1}^{N} c_j^1(\hat{\pi}^j)}{t + l + 1} \leq c_k^1(\hat{\pi}^{k-1},1) + \epsilon + le + \frac{TR}{t + l + 1}.
\]

By lemma (1) we know that \( c_k^1(\hat{\pi}^{k-1},1) < c^2(\hat{\pi}^{k-1},1,2^{N-k+1}) \). Thus, we can take \( \epsilon > 0 \) small enough such that \( c_k^1(\hat{\pi}^{k-1},1) + \epsilon \leq c^2(\hat{\pi}^{k-1},2^{N-k+1}) \). Which implies that,

\[
TP \leq c^2(\hat{\pi}^{k-1},2^{N-k+1}) + le + \frac{TR}{t + l + 1} + le = \frac{TR}{n(\hat{\pi}^{k-1},2^{N-k+1})} + le + \frac{TR}{t + l + 1} \leq \frac{TR}{t + l + 1} + le + \frac{TR}{t + l + 1} = \frac{(t + 1)TR}{t + l + 1} + le.
\]

Now we can take \( \epsilon > 0 \) small enough such that \( \epsilon \leq TR/(t + l + 1) \) for any announcement \( \hat{m}^N \in M^N \). Therefore,

\[
TP \leq \frac{(t + 1)TR}{t + l + 1} + l \frac{TR}{t + l + 1} n = TR
\]

which concludes the proof. \( \Box \)