A Model of Technology Assimilation

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Abstract

What makes countries productive and rich? This paper endogenizes technology and total factor productivity (TFP) based on a model of technology assimilation. We consider an economy with a large stock of production ideas, where the factor requirements of ideas are different from its factor endowment. Firms can undergo an assimilation process which modifies ideas with respect to their factor endowment. The equilibrium level of TFP and the shape of the production function depend on the deep parameters that govern the assimilation power and the distribution of ideas. We apply the model to study cross-country income differences. Once foreign productive ideas are free to assimilate, there is symmetry breaking of the autarky equilibrium. Depending on the assimilation power, a laggard country can either catch up with the frontier countries (and their productive ideas) or fall into an assimilation trap with stagnant income. An advance in the world frontier technology polarizes the world economy. Finally, the model is used to study a number of challenging issues in growth and development, namely, the Lucas (1993) miracle, the "Twin Peaks" phenomenon of club convergence, the Flying-Geese Pattern of development, and the leapfrogging in technology.

Keywords: endogenous TFP, technology assimilation, international income disparity, growth miracle, club convergence, Flying-Geese Pattern, leapfrogging

JEL Classification Codes: E23, O11, O14, O33, O41, O47

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"...whether countries at early stages of development, ..., should take advantage of the modern technology developed by advanced countries, ..., or whether they should ... use production methods which are obsolete in countries abroad." (Kindleberger 1956, p. 249)

"...all industrializing countries have managed to grow more rapidly by borrowing foreign technologies. This was, as widely acknowledged, true of Japan in the twentieth century, but it was also a central element of America’s rapid industrialization in the nineteenth century, an industrialization that built upon the already existing technologies of metallurgy, power generation, railroads, and textiles in Great Britain." (Rosenberg 1994, p. 101)

1 Introduction

What makes countries productive and rich? Why there is huge, if not widening, and persistent gap in international income levels?1 In standard growth models, productivity is usually indexed by a single coefficient of total factor productivity (TFP), which is independent to the factor endowment of the economy.2 Unless the improvements are country-specific, technological improvement in one country should then also improve productivity of the others through technology adoption. Given the vast examples of technological transfer across countries, standard growth models will tend to predict similar, at least narrowing, levels of productivity and income in the long run instead. How can we avoid this counterfactual prediction?

Existing literature offers alternative explanations and here we highlight two strands.3 One strand emphasizes the barrier to technology adoption due to different costs and frictions, which results in a low level of TFP. For instance, Parente and Prescott (1994) and Acemoglu (2007) consider models of costly technology adoption. Grossman and Helpman (1991) and Barro and Sala-i-Martin (1997) consider the costly-imitation models of technology diffusion. Lagos (2006) proposes an aggregate model to illustrate how the TFP depends on the search frictions in the labor markets. Comin et al (2012) considers the role of spatial frictions on technology diffusion.4

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1To be precise, in the convergence literature there is not much disagreement on the divergence amongst countries in two dimensions: firstly, growth rates of poor countries have been lower than the growth rates of their rich counterparts and secondly, the dispersion of income per capita across countries has tended to increase over time.

2See Prescott (1998) for a discussion. Empirically, it is found that cross-country differences in total factor productivity are significant (e.g., Hall and Jones 1999).

3Here we compare our paper with the literatures on technology adoption and TFP. There is also a growing literature which focuses on the effects of distortion and misallocation of resources across firms and sectors on aggregate productivity, see Hsieh and Klenow (2009). Another related literature, for example Duarte and Restuccia (2010), emphasizes the consequences of structural change on aggregate productivity.

4Other interesting work in this area includes the quality-ladder model of Segerstrom et al (1990), the
On the other hand, by allowing for free and instant technology adoption across countries, there is another strand of the literature focusing on the "appropriateness" of technology. Here, at least in our paper, the definition of "technology" goes beyond the standard production functions which can be ranked by a single measure of TFP. Instead, "technologies" represent a set of input-output relationships with different factor requirements. The main idea is that "countries with different factor endowments should choose different technologies." [Caselli and Coleman (2006), p.501] The origin of the appropriate-technology literature goes back to the work of Atkinson and Stiglitz (1969) on the "localised technology." In their model productivity improvement from adopting productive techniques is based on learning by doing on the firm’s current production technique, which is specified by a given combination of capital and labor. Basu and Weil (1998) extends the Atkinson-Stiglitz model by considering that such technological improvement is not only for a specific combination of capital and labor but also of similar techniques.\(^5\) The North-South model of Acemoglu and Zilibotti (2001) differs from Atkinson and Stiglitz (1969) and Basu and Weil (1998) by focusing on the ratio of skilled to unskilled labor instead of capital intensities. As a result, productivity differences come from the technology-skill mismatch between the North and the South.\(^6\)

Our paper belongs to the latter strand of literature on appropriate technology. Having acknowledged the tension between the factor requirement by technologies and the factor endowment of the economy, this paper goes one step further by allowing technology assimilation to ease the tension: an endogenous modification of the technology making the best use of the factor endowment.\(^7\) We consider an economy with a large stock of production ideas. Similar to Jones (2005), the factor requirements of the ideas are different from the factor endowment of the economy. Unlike Jones (2005), firms in this economy can undergo an assimilation process which modifies the idea with respect to their factor endowment, and it results in a global production function. Technology assimilation is documented in, for example, Rosenberg (1994), which points out that many developed countries nowadays, such as Japan and the U.S., are successful examples of borrowing and modifying foreign technology.\(^8\) In addition, Los and

\(^5\) According to Basu and Weil (1998), it is in the neighborhood of "\(\gamma\)" (an arbitrarily small magnitude) of the specific capital-labor ratio.

\(^6\) In the words of Acemoglu and Zilibotti (2001), it is "because unskilled workers in the South perform some of the tasks performed by skilled worker in the North." (p.566)

\(^7\) Our concept of technology assimilation can be regarded as a formalization of the idea of the intermediate technology proposed by Schumacher (1973). According to Schumacher, capital-intensive processes are unproductive in developing countries because they are "lack of marketing and financial infrastructure, inappropriate inputs, and untrained workers." (Basu and Weil, 1998, p.1026) In our understanding, the lack of these factors is the cause for the failure of technology assimilation.

\(^8\) See also Nelson and Pack (1999) for a discussion of the literature of technology assimilation for accounting Asian growth miracle.
Timmer (2005) point out that the assumption of the Basu-Weil model that "new knowledge generated in one country is immediately available to all other countries" (p.519) is not empirically relevant and needs to be relaxed. They then suggest that "a country’s ability to assimilate appropriate knowledge should probably be considered as a separate determinant of growth" (p.520). This theoretical paper can be conceived as an attempt to model the important process of technology assimilation.

A quantitative support for the importance of technology assimilation on cross-country growth experience can be found in a recent paper by Wang et al. (2013). Building on the theoretical framework developed in this paper, they apply the assimilation model to growth accounting. They show that the ability to assimilate can be instrumental for differentiating between trapped and miracle economies. Table 1 is borrowed from Wang et al. (2013), which shows the percentage of relative (to U.S.) income growth of countries contributed by technology assimilation, compared to other components like factor accumulation and TFP. For most of the miracle countries, 40-70% of its relative growth performance can be attributed to assimilation, while it is above 50% for trap countries due to backward or negative assimilation.

<table>
<thead>
<tr>
<th>OECD</th>
<th>Asian Miracles</th>
<th>High Growth</th>
<th>Trapped Countries</th>
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<tbody>
<tr>
<td>France 31%</td>
<td>Hong Kong 60%</td>
<td>Botswana 69%</td>
<td>Comoros 50%</td>
</tr>
<tr>
<td>Germany 0%</td>
<td>Japan 45%</td>
<td>Brazil 70%</td>
<td>Cote d’Ivoire 54%</td>
</tr>
<tr>
<td>Greece 46%</td>
<td>South Korea 45%</td>
<td>China 52%</td>
<td>Kenya 57%</td>
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<td>Portugal 57%</td>
<td>Singapore 43%</td>
<td>India 53%</td>
<td>P. New Guinea 91%</td>
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<td>Spain 67%</td>
<td>Taiwan 50%</td>
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<td>U.K. 0%</td>
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Table 1: Contribution to relative income growth from technology assimilation, 1960-2007

One of our key features is that now the TFP and the production function are endogenized as the result of the interplay between technology assimilation and factor accumulation. In the autarky steady state, cross-country differences in income and productivity come from the deep parameters that govern the ability to assimilate and the distribution of ideas. We apply the model to study the situation when foreign productive ideas are free to assimilate as if the domestic one. We find that assimilating foreign technology can have long-lasting dramatrical consequences. Depending on the assimilation power, a laggard country can either catch up with the frontier countries (and their productive ideas) or fall into an assimilation trap with stagnant income. In this regard our model also provides some guidances on the optimal industrial policy. Furthermore, given its assimilation power, an advance in foreign technique leads to polarization of countries, depending on their autarkic income levels. It lowers the income threshold of attaining the foreign frontier so that richer countries can reach
the full-assimilation steady state and become richer. But it also lowers the income level of the assimilation trap so that poorer countries get worse and become poorer.

The potential of our model is demonstrated by a number of interesting applications which have been challenging to standard theory of growth and development. One application is the "miracle" story of South Korea given by Lucas (1993). It is pointed out that both South Korea and the Philippines are of similar situations back in the 1960s, but their growing experiences in the end are dramatically different. Standard convergence theory will tend to predict that they should end up the same instead. Another application is to extend the model to generate the "Twin-Peak" phenomenon observed by Quah (1997). In particular, there are two groups of countries: in one group the income levels of its members converge amongst themselves, while in another group there is divergence of income levels. Standard convergence theory will tend to predict a single convergence club instead. The third application is to study the “Flying-Geese” pattern of technology adoption documented by Lin (2009). In particular, it is observed that developing countries borrow technologies from those developed countries "whose stage of development is higher than but not far away from theirs" (p.37). For instance, consider the post-World War II development of the four East Asian NIEs (Hong Kong, Singapore, South Korea and Taiwan, known as the four Asian Tigers) in the 1960s-1970s. They borrowed technology from Japan instead of USA and Western Europe and were successful. In turn, the technologies of the four Tigers are assimilated by other late-comers such as Malaysia, Thailand and Vietnam. Standard technology adoption theory will tend to predict immediate adoption of the leader technology instead. Finally, we apply our model to explain how economic leapfrogging can happen in the last three centuries, as documented by Maddison (1982). In particular, there is recurring overtaking of economic leadership by the laggard countries which mimic the technology of the leading country. The early dominance of the Dutch was ended by the rise of England; England dominance was ended by the rise of America and Germany. It cannot be easily explained by standard endogenous growth theory; as it suggests, on the contrary, technological change should reinforce the position of the leading nation.

The organization of the paper is as follows. The autarkic benchmark consists of the technology concepts is developed in the next section so as to highlight the production structure of the economy and the characteristics of the assimilation function. Section 3 presents the full version of the model of technology assimilation. Applications of the model are studied in section 4, and section 5 concludes.

9See also Azariadis (1996) and Galor (1996) for this idea of club convergence.
2 The Autarky Benchmark

To understand how firms make their choice of technology, we first introduce the notion of “production technique”\textsuperscript{10}, which is a mapping from a vector of required inputs to a scalar of output. The idea of technique follows the terminology of Atkinson and Stiglitz (1969). Technology assimilation is the adjustment mechanism of the required input vector in reference to the factor endowment. The technique after assimilation is referred as the local production techniques. Given the factor endowment, firms freely choose the most profitable one from the collection of the local production techniques. The collection of the most profitable local production techniques based on the variation of the factor endowment then constitutes the global production function.\textsuperscript{11}

2.1 The Production Technique

A production technique is a blueprint specifying the quantity of factors input for a given output level. We assume such blueprint is divisible and can be perfectly replicated, so a production technique must be constant return to scale. Each technique \(i\) specifies the quantity of capital \((K)\) and labor \((N)\) for a given output level \(Y\), defined by two parameters of factor-augmented productivity, \(a_i\) and \(b_i\):

\[
Y = \min(a_i K, b_i N). \tag{1}
\]

Implementing technique \(i\) requires \(1/a_i\) units of capital and \(1/b_i\) units of labor for each unit of output. Define \(\kappa_i \equiv b_i/a_i\) and \(k \equiv K/N\). Then we can index technique \(i\) by the duple \((b_i, \kappa_i) \in \mathcal{P}\), where \(\mathcal{P}\) is a compact set of all given techniques in a country. Because of constant return to scale, it is the capital-labor ratio \(k\) that matters for the output per capita. We note that \(\kappa_i\) is the factor input required by technique \(i\), and \(k/\kappa_i\) is the factor endowment in term of \(\kappa_i\). When the factor input requirement \(\kappa_i\) under technique \(i\) exactly matches the factor endowment \(k\) (i.e., \(k = \kappa_i\)), then \(b_i\) is the per-capita output produced by the technique.

2.2 The Assimilated Technique

We now introduce the concept of technology assimilation: the possibility of modifying the factor requirements specified by a given technique. The output after assimilating technique \(i\) is given by \(Y = \tilde{F}(a_i K, b_i N)\), where \(\tilde{F}(a_i K, b_i N)\) is given by

\[
\tilde{F}(a_i K, b_i N) \equiv \left[ \lambda (a_i K)^{\frac{\sigma - 1}{\sigma}} + (1 - \lambda) (b_i N)^{\frac{\sigma - 1}{\sigma}} \right]^{\frac{1}{\sigma - 1}}, \tag{2}
\]

\textsuperscript{10}Also called the production process, activity or blueprint in the literature.

\textsuperscript{11}As all the premises of the Welfare theorems are satisfied in our frictionless environment, we are able to use the "efficient" technology to characterize the profitable technology at the competitive equilibrium.
where $\sigma \in [0, 1)$. We refer $\tilde{F}(a_iK, b_iN)$ as the \textit{local production technique}. To see how the assimilation is captured by the functional form of $\tilde{F}$, notice that without assimilation output per capita is given by $y = \min a_i(k, \kappa_i)$. Suppose labor endowment is abundant with respect to technique $\kappa_i$ such that $\kappa_i > k$. Without assimilation, labor is not fully used and the scarcity of capital limits output per capita to be $y = a_ik < a_i\kappa_i = b_i$. It will be preferable, whenever possible, to modify $\kappa_i$ such as to raise the labor requirement (decrease $b_i$) and/or lower the capital requirement (increase $a_i$) to fully utilizes all factor endowment. With assimilation, the factor requirement $\kappa_i$ is modified to $\tilde{\kappa}_i = \left[\lambda k^\psi + (1 - \lambda) \kappa_i^\psi\right]^{\frac{1}{\psi}} \in (k, \kappa_i)$, where $\psi \equiv (\sigma - 1) / \sigma$. Then after assimilating technique $i$, output per capita becomes $y = \min a_i(k, \tilde{\kappa}_i) = a_i\left[\lambda k^\psi + (1 - \lambda) \kappa_i^\psi\right]^{\frac{1}{\psi}} > a_ik$. A similar logic applies to the opposite case of scarce labor endowment $\kappa_i < k$. In other words, technology assimilation improves a country’s output by adjusting the factor requirement of a technique toward its relative abundant factor. The share parameter $\lambda$ captures the relative contribution of capital to output after assimilation.

The assimilation parameter $\sigma \in [0, 1)$ captures how well firms can modify the factor requirement with respect to the factor endowment. Notice that $\tilde{F}$ is increasing in $\sigma$. The production technique under perfect assimilation becomes Cobb-Douglas:

$$\tilde{F}(a_iK, b_iN) = (a_iK)^{\lambda} (b_iN)^{1-\lambda}$$

when $\sigma \to 1$.

Therefore, the share parameter $\lambda$ in the CES specification of the assimilated technique can be interpreted as the capital share of the technique (when assimilation is perfect).

### 2.3 The Production Function

Firms maximize their profit by choosing the optimal technique given the factor endowment. The outcome is the (global) production function, which is the envelope of all optimal local production techniques for all feasible combinations of factor endowments.

**Definition 1** The global production function $F(K, N)$ is given as

$$F(K, N) \equiv \max_{i=1,...,n} \tilde{F}(a_iK, b_iN),$$

where the $n$ draws are independent.

$$\frac{\partial}{\partial \sigma} \log \tilde{F} = \frac{-1}{(\sigma - 1)^2} \left[ \pi \log \left( \frac{\lambda}{\pi} \right) + (1 - \pi) \log \left( \frac{1 - \lambda}{1 - \pi} \right) \right],$$

$$\geq \frac{-1}{(\sigma - 1)^2} \left[ \pi \left( \frac{\lambda}{\pi} - 1 \right) + (1 - \pi) \left( \frac{1 - \lambda}{1 - \pi} - 1 \right) \right] = 0,$$

where $\pi \equiv \frac{\lambda(a_i, K)}{\lambda(a_i, K) + (1 - \lambda)(b_i, N)}$ and the inequality follows the fact that $\log x \leq x - 1$. 

6
To derive the production function at the limit, we follow Jones (2005) and assume:

**Assumption P.** The factor requirement parameters \((a_i, b_i)\) of a technique \(i\) are drawn from independent Pareto distributions:

\[
\Pr(a_i \geq a) = H_a(a) = \left(\frac{\gamma_a}{a}\right)^{\theta_a},
\]

\[
\Pr(b_i \geq b) = H_b(b) = \left(\frac{\gamma_b}{b}\right)^{\theta_b},
\]

where \(\theta_j > 0\) and \(\gamma_j > 0\) for \(j = a, b\).

Recall that the per-capita output after assimilating blueprint \((a_i, b_i)\) is

\[
y_i = \left[\lambda (a_i k)^{\frac{\sigma-1}{\sigma}} + (1 - \lambda) b_i^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}}, \sigma \in [0, 1),
\]

then we have the following lemma:

**Lemma 1** Given \(\sigma \in [0, 1)\) and \(y_i\) given by (3), the event \(\{y_i \geq y\}\) is equivalent to

\[
\left\{ a_i \geq \frac{y}{k} \left[ \lambda^{-1} + (1 - \lambda^{-1}) \left( \frac{b_i}{y} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}} \right\} \cap \left\{ b_i \geq y (1 - \lambda)^{\frac{\sigma}{\sigma-1}} \right\}.
\]

**Proof.** See Appendix. ■

Suppose \(y \geq \gamma_b (1 - \alpha)^{\frac{\sigma}{\sigma-1}}\), then Lemma 1 and the law of total probability together yield

\[
\Pr\{y_i \geq y\} = -\int_{y/(1-\alpha)^{-\frac{\sigma}{\sigma-1}}}^{\infty} H_a \left( \frac{y}{k} \left[ \lambda^{-1} + (1 - \lambda^{-1}) \left( \frac{b_i}{y} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{1}{\sigma-1}} \right) dH_b(b).
\]

Substituting \(x = (1 - \alpha)^{\sigma\theta_b/(1-\sigma)} \left( \frac{b}{y} \right)^{-\theta_b}\) we have

\[
\Pr\{y_i \geq y\} = -k^{\theta_a \gamma_a \theta_b} \gamma_b y^{-\theta_a} \int_{y/(1-\alpha)^{-1/\phi}}^{\infty} \left[ \lambda^{-1} + (1 - \lambda^{-1}) \left( \frac{b_i}{y} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\sigma\theta_a/(\alpha) - 1} \sigma^{\theta_a/(\alpha)} - \theta_a \int_{0}^{1} \left[ 1 - x^{1-\theta_a \gamma_b \alpha} \right]^{\sigma\theta_b/(\sigma-1)} \frac{\sigma\theta_b}{\sigma-1} dx.
\]

where \(\theta \equiv \theta_a + \theta_b\), \(\alpha = \theta_a/\theta \in (0, 1)\), \(n\) is the number of techniques and \(z\) is the constant given by

\[
z = \left[ n^{\theta_a \gamma_b \lambda \sigma^{\theta_a/(\sigma-1)} (1 - \lambda) \sigma^{\theta_b/(\sigma-1)}} \int_{0}^{1} \left[ 1 - x^{1-\sigma\theta_a} \right]^{\sigma\theta_b/(\sigma-1)} dx \right]^{1/\sigma}.
\]
Since $\lim_{n \to \infty} (1 - x/n)^n = \exp(-x)$ for a given value of $x$, we have

$$\lim_{n \to \infty} \Pr\left\{ \max_{i \in n} y_i \leq \varepsilon z \kappa^\alpha \right\} = \lim_{n \to \infty} \{1 - \Pr\{y_i > \varepsilon z \kappa^\alpha\}\}^n = \lim_{n \to \infty} \left[1 - \frac{\varepsilon^{-\theta}}{n}\right]^n = \exp\left(\varepsilon^{-\theta}\right),$$

which is known as Frechet distribution. This means that as $n$ gets larger and larger, the global production function converges to:

$$y = f(k) = \varepsilon z \kappa^\alpha,$$

where $\varepsilon$ is a random variable drawn from a Frechet distribution with parameter $\theta$. Thus, summarizing the above we have

**Proposition 1** In the limit where the number of techniques goes to infinity, the global production function is Cobb-Douglas regardless the assimilation parameter $\sigma$.

From (6) the global production function consists of the endogenous TFP $\varepsilon$, the TFP shock following a Frechet distribution with parameter $\theta$, and the global capital share $\alpha$ which depends on the Pareto parameter $\beta_a$ and $\beta_b$. The global production function represents the technology frontier of all local production techniques. Each point of the technology frontier represents the best local production technique given the factor endowment. For example, given capital $k$ and unit labor, the best technique $i$ on the global production function is the one such that $\kappa_i = k$, $a_i = \varepsilon z \kappa_i^{\alpha-1}$ and $b_i = \varepsilon z \kappa_i^\alpha$.

Proposition 1 concludes that when the number of techniques becomes larger and larger, the limiting global production function admits the form of Cobb-Douglas function and the underlying assimilation power is exactly unity. The local production technique admits the form of CES with the assimilation power $\sigma < 1$. Higher assimilation power can be understood as the rising flexibility from the substitution between different techniques and different factor endowment.

**Remark.** Jones (2005) assumes that there is no assimilation $\sigma = 0$ and the underlying techniques are all Leontief. So it is a special case of our result because

$$\lim_{\sigma \to 0} \lambda^{\sigma \theta_a/(\sigma-1)} (1 - \lambda)^{\sigma \theta_b/(\sigma-1)} \int_0^1 \left[1 - x \frac{1 - \varepsilon^2}{\pi x}\right]^{\sigma \theta_b} dx = 1,$$

which is a straightforward application of Lebesgue’s monotone convergence theorem. Illustrated with simulation Jones (2005) "conjectures" the limiting global production function should be also Cobb-Douglas for the general case of $\sigma > 0$. Here we formally prove the conjecture for the case $\sigma \in [0,1)$. Our proof fails when $\sigma \geq 1$ since the event $\{y_i \geq y\}$ is no longer equivalent to the joint events (4).
2.4 Endogenous TFP under Technology Assimilation

To echo the literature of technology and growth, we frame our assimilation analysis into an endogenous TFP idea. Recall (5):

\[ z = \left( n^{\gamma_a} \cdot \gamma_b \cdot \lambda^{\gamma_a/(\sigma-1)} (1 - \lambda)^{\gamma_b/(\sigma-1)} \int_0^1 \left[ 1 - x^{1-\sigma} \right] x^{\sigma_a} d\lambda \right)^{1/\gamma} . \]

So the TFP \( z \) depends on parameters including: \( n, \gamma_a, \gamma_b, \) and \( \sigma \). The next proposition establishes the connection between the endogenous TFP and these parameters:

**Proposition 2** \( \partial z / \partial \gamma_a > 0, \partial z / \partial \gamma_b > 0, \partial z / \partial \sigma > 0 \) and \( \partial z / \partial \lambda \geq 0 \) iff \( \lambda \geq \alpha \).

**Proof.** See Appendix.

The intuition of Proposition 2 is straightforward to understand. Higher \( \gamma_j \) (\( j = a, b \)) imply that it is easier to search ideas from the Pareto distributions, resulting in a higher TFP. Also, the more techniques available (larger \( n \)) for use, the higher the TFP. A result that is analogous to the main finding of the product variety models of endogenous technological change [see Romer (1990)]. An improvement in technology assimilation (higher \( \sigma \)) increases productivity (higher TFP \( z \)). This is intuitive as we can show that, given the quantities of factor inputs, a rise in \( \sigma \) leads to higher output. This is captured by an increase in the TFP \( z \) for any given \( k \) in (6). Finally, the effect of \( \lambda \) on \( z \) is ambiguous, depending on the relative magnitude between \( \lambda \) and \( \alpha \). We recall that \( \lambda \) is the capital share of the assimilated technique under perfect assimilation, whereas \( \alpha \) is the capital share of the global production function. The comparative statics effect of \( \lambda \) highlights the fact that the further \( \lambda \) away from \( \alpha \), the higher the TFP. Intuitively, the further we are from the global production function, the larger the productivity gain (higher resulting TFP) from perfect assimilation.

2.5 Household and Savings

To complete the model, we need to introduce savings behavior from the preference side. We consider a discrete-time economy populated by an infinite sequence of two-period lived, overlapping generations as in Reichlin (1986). At each date a set of young agents is born, and care only about old age consumption. Hence all the wage in the young period is saved and invested in capital. Agents other than the initial old have no endowment of capital or the final good at any date. We assume that the initial old of a country is each endowed with \( k_0 \) units of capital. There is no population growth and capital depreciates at the rate \( \delta \) in each period. Each agent is endowed with one unit of labor when young, which is supplied inelastically and
earns a wage income $w$. Agents of cohort $t$ are retired in period $t+1$ and consume all his investment return $(r_{t+1} + 1 - \delta) w_t$.

Having taken assimilation into account, given factor prices, the firm’s optimization problem is to solve

$$\max_{N,k,y} \left[ y - rk - w \right] \text{ s.t. } y = f(k).$$

(7)

Now we are ready to define the equilibrium in the autarky economy:

**Definition 2** An autarky equilibrium consists of factor prices $\{w_t, r_t\}_{t=0}^{\infty}$ and capital $\{k_t\}_{t=0}^{\infty}$ such that

a. (household optimization) $k_{t+1} = w_t$, and

b. (firm optimization and markets clearing) Given $w_t$ and $r_t$, $N_t = 1$ and $k_t$ solve (7)

Standard marginal product conditions yield\footnote{We incorporate the stochastic shock $\varepsilon$ into $z$ from now on.}

$$r = \alpha z k^{\alpha-1},$$

(8)

$$w = (1 - \alpha) z k^{\alpha}.$$  

(9)

Equilibrium requires that savings equal investment in each period:

$$k_{t+1} = w_t = (1 - \alpha) z k_t^{\alpha}.$$  

(10)

In autarky, the unique fixed point of (10) [i.e., $k^* = w(k^*)$] is

$$k^* = [(1 - \alpha) z]^{\frac{1}{1-\alpha}}.$$  

(11)

Notice that the right hand side of (9) is increasing and concave so that the fixed point given by (11) is globally stable. Thus the law-of-motion equation (10) implies that the convergence toward the fixed point $k^*$ must be monotone.

### 3 A Model of Technology Assimilation

So far all the influence of the assimilation power $\sigma$ on the autarky steady state goes through the productivity $z$. Output is given by the domestic global production function in autarky as follows:

$$f(k) \equiv \max_{(a,b) \in \mathcal{P}} \left[ \lambda (ak)^{\frac{\sigma-1}{\sigma}} + (1 - \lambda) b \right]^{\frac{\sigma}{\sigma-1}} = z k^\alpha.$$  

(12)

It is no longer the case when firms can also assimilate foreign technique. Consider an advanced foreign country with higher productivity $\bar{z} > z$, maybe due to favorable fundamental
parameters studied in Proposition 2. The advanced foreign country is in its autarky steady state with technique \( k = \tilde{k} = [(1 - \alpha)z]^{\frac{1}{1-\alpha}} \) and \( b = \tilde{b} = \tilde{z} \tilde{k}^{\alpha} \). It implies \( a = \tilde{a} \equiv \tilde{z} \tilde{k}^{\alpha - 1} \). Let \( f(k; \tilde{k}, \tilde{z}, \sigma) \) denote the output level resulted from assimilating the foreign technique:

\[
\tilde{f}(k; \tilde{k}, \tilde{z}, \sigma) \equiv \left[ \lambda (\tilde{a}k)^{\frac{\alpha-1}{\sigma}} + (1 - \lambda) \tilde{b} \right]^{\frac{\sigma}{\alpha-1}} = \tilde{z}^{\frac{\alpha}{\sigma-1}} \left[ \lambda \left( \frac{k}{\tilde{k}} \right)^{\frac{\alpha-1}{\sigma}} + 1 - \lambda \right]^{\frac{\sigma-1}{\alpha-1}}. 
\]  

(13)

Firms can choose either domestic technology (12) or assimilating foreign technique (13) for production. Given the factor prices \( r_t \) and \( w_t \), we solve

\[
\max_{N_t, k_t, y_t} N_t (y_t - r_t k_t - w_t) \text{ s.t. } y_t = \tilde{f}(k_t; \tilde{k}, \tilde{z}, \sigma) \text{ or } y_t = f(k_t).
\]  

(14)

We are ready to define the equilibrium under the possibility of assimilating foreign technique:

**Definition 3** An assimilation equilibrium consists of a factor price system \( \{w_t, r_t\}_{t=0}^{\infty} \) and capital \( \{k_t\}_{t=0}^{\infty} \) such that

a. (household optimization) \( k_{t+1} = w_t \).

b. (firm optimization) Given \( w_t \) and \( r_t \), \( N_{i,t} = 1 \) and \( k_{i,t} \) solve (14).

c. (markets clearing) \( k_t = \int_{i \in [0,1]} k_{i,t} di \) and \( \int_{i \in [0,1]} N_{i,t} di = 1 \).

The definition of an assimilation equilibrium has several crucial differences from the autarky equilibrium. Firstly, although firms now can assimilate the foreign technique which reflects in the additional technology choice of \( y_t = \tilde{f}(k_t; \tilde{k}, \tilde{z}, \sigma) \) in the firm’s optimization problem, they can only observe and assimilate the technique which is currently in use in the foreign country rather than the entire set of foreign techniques. Secondly, in general the homogenous firms can have different technology choices, which happens when they are indifferent between assimilating foreign technique and using domestic production technology. The definition of equilibrium does not exclude this possibility by allowing firm-specific factor demand \( k_{i,t} \) and \( N_{i,t} \).

It is useful to exploit the duality of the firm’s optimization problem to analyze the technology choice, which we proceed as follows. Given the factor prices \( r \) and \( w \), define the unit cost function of assimilating foreign technique as \( \tilde{c}(r, w) \equiv \min L(rk + w) \text{ s.t. } L f(k; \tilde{k}, \tilde{z}, \sigma) = 1 \).

The first-order conditions of \( k \) and \( L \) imply

\[
\frac{w}{r} = \left( \frac{\tilde{k}}{k} \right)^{\frac{1}{\lambda}} \left( \frac{k}{\tilde{k}} \right)^{\frac{\alpha - 1}{\alpha - 1}}. 
\]  

(15)

\[14\] Recall that the optimal choice in the autarky equilibrium is that \( k = k \), so we unify the notation and have \( k \) represented both the capital-labor endowment and the associated technique ratio.
Then the unit cost function $\bar{c}(r, w)$ of assimilating foreign technique follows a CES form:

$$
\bar{c}(r, w) \equiv \frac{1}{\alpha} \left[ \lambda^\sigma (rk)^{1-\sigma} + (1 - \lambda)^\sigma w^{1-\sigma} \right]^{\frac{1}{1-\sigma}}. 
$$

Similarly, define the unit cost function of domestic production as $c(r, w) \equiv \min L(rk + w)$ s.t. $Lf(k) = 1$. The first order conditions of $k$ and $L$ imply

$$
\frac{w}{r} = \frac{1 - \alpha}{\alpha} k. 
$$

Then the unit cost function $c$ of the domestic production is Cobb-Douglas:

$$
c(r, w) \equiv \frac{r^\alpha w^{1-\alpha}}{C}, 
$$

where $C \equiv z\alpha^\alpha (1 - \alpha)^{1-\alpha}$. Given $r$ and $w$, the firm optimization problem is equivalent to solve $\max \{Ly - \min [c(r, w), \bar{c}(r, w)] Ly\}$. At the competitive equilibrium, the zero profit condition implies factor prices $r$ and $w$ are chosen such that $\min \{c(r, w), \bar{c}(r, w)\} = 1$.

Assimilating foreign technique is weakly preferred if and only if it leads to lower unit cost than the domestic technique, i.e.,

$$
c(r, w) \geq \bar{c}(r, w) \iff 0 \geq \Lambda(\chi) \equiv (1 - \lambda) Z\chi - \chi^{1-\alpha} + \lambda Z, 
$$

where $\chi \equiv \left[ \frac{\lambda}{(1-\lambda)k} \frac{w}{r} \right]^{1-\sigma}$, $\eta \equiv \frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{\lambda^\alpha (1-\lambda)^{1-\sigma}}$ and $Z \equiv (\eta^2)^{1-\sigma}$. The comparison of the unit costs between assimilating foreign technique and using the technique from the domestic technology frontier depends on the sign of $\Lambda(\chi)$. We can interpret $\chi \equiv \left[ \frac{\lambda}{(1-\lambda)k} \frac{w}{r} \right]^{1-\sigma}$ as the normalized factor-cost ratio of assimilating foreign technique.

In general, it is not clear whether there will be solution to $\Lambda(\chi) = 0$. If the solution does not exist, then it means $\Lambda(\chi) > 0$ for all $\chi$, in other words domestic technology is always less costly and foreign technique is never assimilated. The following lemma characterizes the sign of $\Lambda(\chi)$ in different regions of $\chi$:

**Lemma 2** Given the maintained assumptions, $\Lambda(\chi) = 0$ has two distinct roots, $\chi_1$ and $\chi_2$ such that

$$
\chi_1 < \frac{\lambda (1 - \alpha)}{\alpha (1 - \lambda)} < \chi_2.
$$

Furthermore, $\Lambda(\chi) < 0$ if and only if $\chi \in (\chi_1, \chi_2)$.

**Proof.** See Appendix. ■

Now we are ready to characterize the equilibrium technology choice. The following proposition summarizes our findings about firms’ choice of techniques based on unit cost comparison.
Proposition 3 Define $\omega_j \equiv [(1 - \lambda) / \lambda] \tilde{k} \chi_j^{1/(1-\sigma)}$, for $j = 1, 2$. Then we have

- a. $c(r, w) = \bar{c}(r, w)$ if $w/r = \omega_1$ or $w/r = \omega_2$;
- b. $c(r, w) > \bar{c}(r, w)$ if $\omega_1 < w/r < \omega_2$;
- c. $c(r, w) < \bar{c}(r, w)$ if $w/r < \omega_1$ or $w/r > \omega_2$.

Proposition 3 concludes that foreign production technique is assimilated if the wage-rental ratio falls within an appropriate range. The intuition is that since the advanced foreign technique is fixed at the capital-labor ratio $\tilde{k}$, factor prices have to be "appropriate" (i.e., cannot be too high or too low) in order to have $\tilde{k}$ to be cost efficient for domestic firms. Otherwise, it does not pay domestic firms to assimilate the technique. For instance, when $k < \tilde{k}$ the foreign technique requires more capital than the economy’s endowment. In order to be profitable to assimilate the capital-intensive foreign technique $\tilde{k}$, the rental price has to be low enough relative to the wage, i.e., $w/r > \omega_1$. On the other hand, when $k > \tilde{k}$ the foreign technique requires more labor than the economy’s endowment. In order to be profitable to assimilate the labor-intensive foreign technique $\tilde{k}$, the wage has to be low enough relative to the rent, i.e., $w/r < \omega_2$. In summary, firms find assimilating foreign technique profitable if the wage-rent ratio falls within an appropriate range, i.e., $\omega_1 < w/r < \omega_2$.

3.1 Equilibrium Analysis

So far we have characterized the technology choice based on the relative factor prices, $w/r$, rather than the fundamental of the economy. So it is helpful to introduce different regions of capital intensity that feature different technology choices of firms. First of all, suppose all firms assimilate foreign technique, then from the definition $\chi \equiv \left[\frac{\lambda}{(1 - \lambda) k w} \frac{w}{r}\right]^{1-\sigma}$ and (15), define $\tilde{k}_j$ as follows:

$$\tilde{k}_j \equiv \tilde{k} \chi_j^{\sigma/(1-\sigma)}, \quad j = 1, 2. \tag{20}$$

Since the right hand side of (15) is increasing in $k$, $\tilde{k}_j$ are the threshold levels of capital where firms with $\tilde{k}_1 < k < \tilde{k}_2$ will assimilate foreign technique, corresponding to Proposition 3b.

Next, suppose all firms use domestic technology, then from the definition $\chi \equiv \left[\frac{\lambda}{(1 - \lambda) k w} \frac{w}{r}\right]^{1-\sigma}$ and (17), define $k_j$ as follows:

$$k_j \equiv \frac{\alpha (1 - \lambda)}{\lambda (1 - \alpha)} \tilde{k} \chi_j^{1/(1-\sigma)}, \quad j = 1, 2. \tag{21}$$

Since the right hand side of (17) is increasing in $k$, $k_j$ are the threshold levels of capital where only firms with $k < k_1$ or $k > k_2$ will use domestic technology, corresponding to Proposition 3c. The following lemma ranks the thresholds $k_j$ and $\tilde{k}_j$:
Lemma 3 Given the maintained assumptions, we have the following ordering:

\[ k_1 < \bar{k}_1 < \bar{k}_2 < k_2. \]

Proof. See Appendix. ■

Then we can restate Proposition 3 in terms of the fundamental instead of factor prices:

Corollary 1 Given the maintained assumptions,

1. when \( k_1 < k < \bar{k}_1 \) or \( \bar{k}_2 < k < k_2 \), both the domestic production technology and the foreign production technique are used by some firms;
2. when \( \bar{k}_1 \leq k \leq \bar{k}_2 \), all firms assimilate the foreign technique;
3. when \( k \leq k_1 \) or \( k \geq k_2 \), all firms use the domestic technology.

In the first case of Corollary 1, firms are indifferent between domestic technology and foreign techniques. Let \( h \) be the share of firms using the foreign technique \( \bar{f} \), which is given by the market clearing condition of \( k \):

\[
h = \begin{cases} 
0, & \text{if } k \leq k_1, \\
(k - k_1) / (k_1 - k), & \text{if } k_1 < k < \bar{k}_1, \\
1, & \text{if } \bar{k}_1 \leq k \leq \bar{k}_2, \\
(k - k_2) / (k_2 - k_2), & \text{if } \bar{k}_2 < k < k_2, \\
0, & \text{if } k \geq k_2,
\end{cases}
\]

This then implies that whenever \( k \) is in between \( k_j \) and \( \bar{k}_j \), the wage function is flat. It is because any increase in \( k \) in that region will be directed to support the increase in the number of domestic firms assimilating the capital-intensive foreign technique. As a result, wage would not increase as capital accumulates. Thus, by using (15) and (17) as well as the fact that \( \min(c, \bar{c}) = 1 \) in equilibrium, the equilibrium wage \( w(k) \) under technology assimilation is given by

\[
w(k) = \begin{cases} 
W(k), & \text{if } k \leq k_1, \\
W(k_1) = \bar{W}(\bar{k}_1), & \text{if } k_1 < k < \bar{k}_1, \\
\bar{W}(k), & \text{if } \bar{k}_1 \leq k \leq \bar{k}_2, \\
W(k_2) = \bar{W}(\bar{k}_2), & \text{if } \bar{k}_2 < k < k_2, \\
W(k), & \text{if } k \geq k_2,
\end{cases}
\]

where \( W(k) = (1 - \alpha) zk^\alpha \) and \( \bar{W}(k) = (1 - \lambda) \bar{z}\bar{k}^\alpha \left[ \lambda (k/\bar{k})^{\frac{\alpha}{\sigma}} + (1 - \lambda) \right]^{-\frac{1}{\sigma - 1}} \). The graphical representation of \( w(k) \) is given in Figure 1.
3.2 The Wage Function under Assimilation

In this subsection, we study the wage function under assimilation $w(k)$, given by (23), in more details. We investigate the effects of technology assimilation and productivity, which are captured by the parameters $\sigma$, $z$ and $\bar{z}$, on the wage function. We first note that productivity always increases wage so that an increase in $z$ ($\bar{z}$) shifts up the technique-based wage function $W(k)$ [$\bar{W}(k)$]. In addition, as an increase in assimilation ability raises efficiency in production, so an increase in $\sigma$ also shifts up the technique-based wage function $\bar{W}(k)$.

Before studying the effects of assimilation on the range of capital resulting in the adoption of foreign technique, that is the comparative statics of $\sigma$ on $k_j$ and $\bar{k}_j$ ($j = 1, 2$), we first compute the effects of productivity changes, $z$ and $\bar{z}$. By delegating the algebra to the Appendix, we provide the comparative findings as follows:

$$\frac{\partial k_1}{\partial z} > 0 \quad \text{and} \quad \frac{\partial \bar{k}_1}{\partial z} > 0,$$

$$\frac{\partial k_1}{\partial \bar{z}} < 0 \quad \text{and} \quad \frac{\partial \bar{k}_1}{\partial \bar{z}} < 0.$$  

Other things equal, an improvement in the productivity of the domestic economy makes domestic technology relatively attractive to the foreign technique. Recall that $k_1$ represents the minimum level of $k$ such that domestic firms start to adopt foreign technique, whereas $\bar{k}_1$ is the critical level where all firms adopt the foreign technique. As a result, these critical levels of

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15Strictly speaking, this requires the restriction that $k < \bar{k}$ which is the relevant case that we focus on in the analysis.

16We recall that $\bar{k}_2$ is the critical level of capital intensity where domestic firms start switching back to use domestic technology after all firms have adopted the foreign technique in production. This is not the focus of the analysis since we are studying the case of adopting an advanced foreign technique where $k_t < \bar{k}$. In addition, the derivation is symmetric (see the Appendix) so our discussion below mainly concentrate on the effects of $k_1$ and $\bar{k}_1$. 

---
capital must rise to reflect the increased competitiveness of the domestic technology. On the contrary, given \( \bar{k} \), a rise in the productivity of the foreign country makes foreign technology relatively attractive so that the corresponding critical levels of capital for technology adoption have to fall.\(^{17}\)

The analysis applies symmetrically to the thresholds \( k_2 \) and \( \bar{k}_2 \):

\[
\frac{\partial k_2}{\partial z} < 0 \quad \text{and} \quad \frac{\partial \bar{k}_2}{\partial z} < 0,
\]

\[
\frac{\partial k_2}{\partial z} > 0 \quad \text{and} \quad \frac{\partial \bar{k}_2}{\partial z} > 0.
\]

In terms of the "assimilated" wage function delineated in Figure 1, it can be shown that a rise in \( z \) (\( \bar{z} \)) shifts up \( W(k) \) (\( \bar{W}(k) \)). Based on our comparative statics results above, then lower (upper) flat portion of the assimilated wage function also shifts up (down). This confirms that an increase in domestic (foreign) productivity leads to a contraction (expansion) of the range of \( (\bar{k}_1, \bar{k}_2) \) of the wage function and hence discourages (encourages) technology adoption. For an increase (decrease) in \( z \) (\( \bar{z} \)), then the assimilation part of \( w(k) \) [i.e., \( \bar{W}(k) \)] where \( \bar{k}_1 \leq k \leq \bar{k}_2 \) is shrinking; in the limit where \( z = \bar{z} \), the fraction vanishes.

For the effects of \( \sigma \), we have

\[
\frac{\partial k_1}{\partial \sigma} > 0 \quad \text{and} \quad \frac{\partial \bar{k}_1}{\partial \sigma} > 0.
\]

To understand these effects, we note that an improvement in technology assimilation implies that firms are more capable in making use of the idle labor when they adopt the advanced foreign capital-intensive techniques. As \( k_1 \) can be interpreted as the minimum level of capital required for domestic firms to start adopting foreign techniques (or to start giving up the domestic technology) and \( \bar{k}_1 \) as the critical or threshold level of capital where all domestic firms adopt the foreign technique (or all firms abandon the domestic technology), we expect that better assimilation in technology (an increase in \( \sigma \)) would directly result in a decrease of both these levels. This direct assimilation effect is qualitatively equivalent to an increase in \( \bar{z} \) because better assimilation can be interpreted as capable of adopting a higher foreign technology \( \bar{z} \), given \( z \) and \( \sigma \). On the other hand, we also note that there is an indirect effect of \( \sigma \) on \( k_1 \) and \( \bar{k}_1 \) via the endogenous TFP \( z \). For instance, when better assimilation improves domestic TFP, the domestic technology becomes more productive. Other things being equal, the capital requirement for switching from domestic to foreign technique (i.e., \( k_1 \)) then rises. This is qualitatively the same effect as an increase in \( z \). As we have shown above that the effects of an increase in \( z \) and \( \bar{z} \) are working in opposite directions, the net effects on the

\(^{17}\)Formally speaking, a change in \( \bar{z} \) also alter \( \bar{k} \). However, the latter involves two and conflicting effects on \( \bar{W} \). For illustrative purposes, we simply consider the case where the direct effect of \( \bar{z} \) dominates.
thresholds $k_j$ and $\bar{k}_j$ become ambiguous. In summary, we have

**Proposition 4** Given a level of per capita capital, an increase in domestic (foreign) productivity raises the domestic wage $W(k)$ [the assimilation wage $\bar{W}(k)$]. An increase in assimilation power $\sigma$ consists of two opposing effects: an endogenous domestic TFP improvement effect and an assimilation effect similar to a foreign productivity increase.

Finally, from (22), we derive the effects on the firm share of technology adoption, $h$:

$$\frac{\partial h}{\partial z} < 0, \quad \frac{\partial h}{\partial \bar{z}} > 0, \quad \frac{\partial h}{\partial \sigma} > 0.$$ 

The intuition is clear. When domestic productivity increases, then foreign technique is relatively less competitive so that the share of firms that adopt it declines. The opposite situation occurs for an increase in foreign productivity. The ambiguity of $\sigma$ on $h$ is due to the conflicting forces that a higher $\sigma$ makes adoption easier but it also leads to a higher domestic TFP $z$.

### 3.3 Steady-State Analysis

Recall the law-of-motion equation of capital accumulation $k_{t+1} = w(k_t)$, a steady state level of capital $k$ then solves

$$k = w(k).$$

We characterize the non-trivial steady states associated with the wage function (23) by the following propositions and lemmas: (i) an *autarky* steady state where all firms are still using domestic technology, (ii) a *full-assimilation* steady state where all firms assimilate foreign technique, (iii) a *partial-assimilation* steady state where both foreign technique and domestic technology are used.

#### 3.3.1 Symmetry Breaking

In the autarky steady state, all firms continue using techniques from the domestic technology frontier, even though a productive foreign technique is available to assimilate. The autarky steady-state level of capital $k = k^* = [(1 - \alpha)z]^{\frac{1}{1-\alpha}}$, if it exists, is always unique and stable. Recall from the equilibrium wage function (23), the autarky steady state exists if and only if using domestic technology is less costly in the steady state, which is the case when $k^* \leq k_1$ or $k^* \geq k_2$ according to Collorary 1. The following proposition summarizes our characterization of the autarky steady state:

**Proposition 5** Given the maintained assumptions, there does not exist any autarky steady state.
Proof. See Appendix. ■

The elimination of the autarky steady state once the economy is opened echoes the symmetry breaking mechanism of Matsuyama (2004). There Matsuyama (2004) studies the incomplete market due to financial frictions. Here, markets are frictionless and the fundamental forces that break the autarky symmetry are different. After the introduction of foreign technique, the country will eventually end up with either a higher or a lower level of output, but never remains status quo. The autarky steady state no longer survives because of the comparative advantage of its relatively cheap wage, which makes foreign technique desirable to assimilate. Suppose both the domestic country and the technology source country were in the autarky steady state, then the rental price of capital are given by $k^*/y^* = \bar{k}/\bar{y} = 1-\alpha$ due to constant labor income share. Given this background, now consider the case even without any assimilation ($\sigma = 0$). To produce one unit of output with the foreign technique, firms need to have $k = y/(1-\alpha)$ units of capital and $1/(1-\alpha)$ units of labor which only costs $\alpha/(1-\alpha) + k^*/y = \alpha + (1-\alpha) k^*/\bar{k} < 1$. This is less than the the unit cost of the domestic technology (which is one). So firms will always assimilate the foreign technique in the autarky steady state, as a result autarky can not be sustained. The proof of Proposition 5 shows that it is true in general for all the remaining cases $\sigma \in [0,1)$.

3.3.2 Assimilation Patterns and Multiple Steady States

An equilibrium with capital $k$ is a full-assimilation steady state only if $k = \bar{k}(k)$, which is equivalent to (recall, in equilibrium, we have $\chi = (k/\bar{k})^{1-\sigma}/\sigma$ in this case):\footnote{This can be derived as follows:

\[
k = \bar{k}(k) \iff k = (1-\alpha) \bar{k}^{\alpha} \left[ \lambda \left( \frac{k}{\bar{k}} \right)^{\frac{\alpha-1}{\sigma}} + (1-\lambda) \right]^\frac{1}{1-\alpha}, \text{ where } \bar{k} = (1-\alpha) \bar{k}^{\alpha} \\
\iff \frac{k}{\bar{k}} = \frac{1-\lambda}{1-\alpha} \left[ \lambda \left( \frac{k}{\bar{k}} \right)^{\frac{\alpha-1}{\sigma}} + (1-\lambda) \right]^\frac{1}{1-\alpha} \\
\iff \chi^{1-\sigma} = \left( \frac{1-\lambda}{1-\alpha} \right)^\sigma [\lambda + (1-\lambda) \chi] \\
\iff 0 = S(\chi) \equiv \left( \frac{1-\alpha}{1-\lambda} \right)^{1-\sigma} [\lambda + (1-\lambda) \chi] - \left( \frac{1-\lambda}{1-\alpha} \right)^{1-\sigma}.
\]}

$$k = \bar{k}(k) \iff 0 = S(\chi) \equiv (1-\lambda) \chi - \left( \frac{1-\lambda}{1-\alpha} \right)^{1-\sigma} + \lambda.$$ It is straightforward to show that $S''(\chi) > 0$, $S(0) = \lambda$ and $S(\infty) = \infty$, so $S(\chi)$ is U-shaped and has a minimum at $\chi_{\text{min}} \equiv \frac{1-\alpha}{1-\lambda} \left( \frac{1-\alpha}{1-\lambda} \right)^{1-\sigma}$. Thus $S(\chi) = 0$ at most has two roots, denoted...
as $\chi_L$ and $\chi_H \geq \chi_L$ respectively. To characterize the solutions, we first compute

$$S(\chi_{\min}) = \lambda - \sigma \left( \frac{1 - \sigma}{1 - \alpha} \right)^{\frac{1 - \sigma}{\alpha}}.$$  

So the existence of the non-trivial steady states where all firms assimilate foreign technology requires $S(\chi_{\min}) \leq 0$, which requires the following necessary condition:

**Condition E.** $\frac{\sigma}{\lambda} \left( \frac{1 - \sigma}{1 - \alpha} \right)^{\frac{1 - \sigma}{\alpha}} > 1$.

The following lemma characterizes the parameterization where Condition E is satisfied:

**Lemma 4** $E(\sigma) \equiv \frac{\sigma}{\lambda} \left( \frac{1 - \sigma}{1 - \alpha} \right)^{\frac{1 - \sigma}{\alpha}}$ is U-shape in $\sigma$ with the minimum at $\sigma = \alpha$ where $E(\alpha) = \alpha/\lambda$, $\lim_{\sigma \to 0} E(\sigma) = \infty$ and $\lim_{\sigma \to 1} E(\sigma) = \lambda^{-1}$.

**Proof.** See Appendix.

Lemma 4 implies that Condition E is always satisfied if $\alpha \geq \lambda$. Otherwise, Condition E is satisfied for either sufficiently low value of $\sigma$, or values sufficiently close to one.

Recall from the equilibrium wage function (23), the steady state with capital level $k$ features full assimilation of foreign technique if and only if it is less costly under wage $w(k)$, ie, $\bar{k}_1 < k < \bar{k}_2$. Define $\bar{k}_L = \bar{k}_{\chi_L}^{\sigma/(1-\sigma)}$ and $\bar{k}_H = \bar{k}_{\chi_H}^{\sigma/(1-\sigma)}$ as the levels of capital associated with $\chi_L$ and $\chi_H$, respectively. Then we need to check $\bar{k}_1 < k_i < \bar{k}_2$ for $i = H, L$. From the equilibrium wage function (23), we have $\bar{k}_1 < k_i < \bar{k}_2$ if and only if $\chi_i \in (\chi_1, \chi_2)$. The following lemma summarizes our characterization of the full-assimilation steady state:

**Lemma 5** Given maintained assumptions,

a. if Condition E does not hold, then there is no full-assimilation steady state, ie $\chi_i$, where $i = L, H$, does not exists.

b. Suppose Condition E holds with strict inequality (equality), there are at most two (one) full-assimilation steady states, namely $k = \bar{k}_L$ and $k = \bar{k}_H$ ($k = \bar{k}_L = \bar{k}_H$). In particular, $k = \bar{k}_i$ is a full-assimilation steady states if $\chi_i \in [\chi_1, \chi_2]$, for $i = H, L$. Moreover, we always have $w'(\bar{k}_L) \geq 1 \geq w'(\bar{k}_H)$.

**Proof.** See Appendix.

Finally we turn to characterize of partial-assimilation steady states. In this case we have $k = W(k_j)$, for $j = 1, 2$. Recall from the equilibrium wage function (23), the steady state $k = W(k_j)$ features partial assimilation of foreign technique if and only if $k_1 < W(k_1) < \bar{k}_1$ or $\bar{k}_2 < W(k_2) < k_2$. By dividing both sides by $k_j$, it is equivalent to

$$1 < (1 - \alpha) z \chi_i^{\alpha-1} < \frac{\lambda(1 - \alpha)}{\alpha(1 - \lambda)} \chi_i^{-1} \quad \text{or} \quad \frac{\lambda(1 - \alpha)}{\alpha(1 - \lambda)} \chi_i^{-1} < (1 - \alpha) z k_2^{\alpha-1} < 1.$$


Substituting $k_j = \frac{\alpha(1-\lambda)}{\lambda(1-\alpha)} k \chi_j^{1/(1-\sigma)}$ and $1 = (1-\alpha) z_k^{\alpha-1}$ into the above inequalities, we have:

**Lemma 6** Given the maintained assumptions, there are at most two partial-assimilation steady states, which are given by $k = W(k_j)$.

a. For $j = 1, 2$, $k = W(k_j)$ is a partial-assimilation steady-state level of capital if

$$Z\left(\frac{\lambda}{\alpha}\right)^{1-\sigma} \in \left(\chi_1^{1-\sigma}, \left[\frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}\right]^{1-\sigma} \chi_1^{\sigma-\alpha}\right) \text{ if } j = 1,$$

$$Z\left(\frac{\lambda}{\alpha}\right)^{1-\sigma} \in \left(\left[\frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}\right]^{1-\sigma} \chi_2^{\sigma-\alpha}, \chi_2^{1-\sigma}\right) \text{ if } j = 2.$$  

b. We never have both $\chi_H \in [\chi_1, \chi_2]$ and $Z\left(\frac{\lambda}{\alpha}\right)^{1-\sigma} \in \left(\left[\frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}\right]^{1-\sigma} \chi_2^{\sigma-\alpha}, \chi_2^{1-\sigma}\right)$ satisfied.

c. If Condition E does not hold, then $Z\left(\frac{\lambda}{\alpha}\right)^{1-\sigma} \in \left(\chi_1^{1-\sigma}, \left[\frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}\right]^{1-\sigma} \chi_1^{\sigma-\alpha}\right)$ is never satisfied.

d. If $\chi_L > \chi_2$ or $\chi_1 \geq \chi_H$, then $Z\left(\frac{\lambda}{\alpha}\right)^{1-\sigma} \in \left(\left[\frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}\right]^{1-\sigma} \chi_2^{\sigma-\alpha}, \chi_2^{1-\sigma}\right)$ is never satisfied.

e. If $\chi_L < \chi_1 < \chi_H$, then $Z\left(\frac{\lambda}{\alpha}\right)^{1-\sigma} \in \left(\chi_1^{1-\sigma}, \left[\frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}\right]^{1-\sigma} \chi_1^{\sigma-\alpha}\right)$ is never satisfied.

**Proof.** See Appendix. ■

Part b of Lemma 6 implies that the full-assimilation steady state with $k = \tilde{k}_2$ and the partial-assimilation steady state with $k = W(k_2)$ cannot co-exist. Since $\chi_L > \chi_2$ also implies $\chi_H > \chi_2$ as $\chi_L > \chi_H$, part c and d of Lemma 6 imply that if there is no full-assimilation steady state then there is also no partial-assimilation steady state with $k = W(k_2)$. Then we are able to characterize the steady states lexicographically by $\chi_L$ and $\chi_H$, which is given by the following proposition:

**Proposition 6** Given the maintained assumptions,

a. there are three steady states if and only if $\chi_L \in (\chi_1, \chi_2)$, where one of them is an unstable full-assimilation steady state with capital $k = \tilde{k}_L$, and another one is a stable partial-assimilation steady state with capital $k = W(k_1)$. The remaining one is either the full-assimilation steady state with capital $k = \tilde{k}_H$, or the partial-assimilation steady state with capital $k = W(k_2)$. Furthermore, the former is the case if and only if $\chi_H \in (\chi_1, \chi_2)$;

b. if either Condition E is violated or $\chi_L > \chi_2$ or $\chi_H \leq \chi_1$, then there is a unique steady state, which also happens to be stable and is the partial-assimilation steady state with capital $k = W(k_1)$;

c. if $\chi_L < \chi_1 < \chi_H$, then there is a unique steady state, which also happens to be stable and is either the full-assimilation steady state with capital $k = \tilde{k}_H$, or the partial-assimilation
steady state with capital $k = W(k_2)$. Furthermore, the former is the case if and only if $\chi_H \in (\chi_1, \chi_2)$;

d. if $\chi_L = \chi_1$, then there are two steady states, where one of them is a full-assimilation steady state with capital $k = \overline{k}_L = W(k_1)$, which is unstable from above and stable from below, and the other one is either a full-assimilation steady state $k = \overline{k}_H$ or a partial-assimilation steady state with $k = W(k_2)$. In either case it is stable;

e. if $\chi_L = \chi_2$, then there are two steady states, where one of them is a stable partial-assimilation steady state with capital $k = W(k_1)$, and the other one is an unstable full-assimilation steady state with $k = \overline{k}_L = W(k_2)$.

Proof. See Appendix.

Proposition 6 completely characterizes the qualitative properties of the steady states by according to the different ordering of $\chi_1, \chi_2, \chi_L$ and $\chi_H$. The big picture of Proposition 6 is as follows. Generically, there is a unique steady state if $\chi_L \notin [\chi_1, \chi_2]$; three steady states if $\chi_L \in (\chi_1, \chi_2)$; two steady states (the bifurcation case) if $\chi_L = \chi_1$ or $\chi_L = \chi_2$. By Proposition 5 the autarky steady state no longer survives due to the reason outlined above. By Lemma 5 the low full-assimilation steady state with $k = \overline{k}_L$ exists if and only if $\chi_L \in (\chi_1, \chi_2)$, which is always unstable. The high full-assimilation steady state with $k = \overline{k}_H$ exists if and only if $\chi_H \in (\chi_1, \chi_2)$, which is always stable. By Lemma 6 the high full-assimilation steady state with $k = \overline{k}_H$ and the high partial-assimilation steady state with $k = W(k_2)$ are mutually exclusive, so if we have $\chi_L \in [\chi_1, \chi_2]$ or $\chi_L < \chi_1 < \chi_2 < \chi_H$ then the steady state with $k = W(k_2)$ exists. The steady state with $k = W(k_2)$ is always stable unless in the bifurcation case $\chi_L = \chi_2$. On the other hand, the steady state with $k = W(k_1)$ exists if either $\chi_L \geq \chi_1$ or $\chi_H \leq \chi_1$ or Condition E not satisfied. The steady state with $k = W(k_1)$ is always stable (at least from below).

3.4 Further Results

So far we have established the general results based on $\chi_1, \chi_2, \chi_L$ and $\chi_H$ rather than the fundamentals. In order to provide interesting and tractable findings based on fundamentals, we need to impose a stronger structure on the distributive parameters $\alpha$ and $\lambda$. To this end, we start with a heuristic interpretation on these parameters by applying a normalization analysis on the local production technique at the point $k = \bar{k}$. Since technology assimilation provides flexibility in production, the relation between the assimilated output and the foreign output can be illustrated in Figure 2 (see Klump and de la Grandville, 2000). Based on the normalized CES property, we know that the marginal products of capital or the slope of the
production functions are equal at the normalization point \( k = \bar{k} \). The marginal products of capital are given by:

\[
f'(k; \bar{k}, \bar{z}, \sigma) = \frac{\bar{z}\bar{k}^\alpha}{k} \left[ \lambda \left( \frac{k}{\bar{k}} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \lambda \right]^{\frac{\sigma}{\sigma-1}} \left[ \frac{\lambda \left( \frac{k}{\bar{k}} \right)^{\frac{\sigma-1}{\sigma}}}{\lambda \left( \frac{k}{\bar{k}} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \lambda} \right]
\]

and

\[
f'(\bar{k}) = \alpha \bar{z}\bar{k}^{\alpha-1},
\]

hence we have

\[
f'(k; \bar{k}, \bar{z}, \sigma) = f'(\bar{k}) \Leftrightarrow \lambda \bar{z}\bar{k}^{\alpha-1} = \alpha \bar{z}\bar{k}^{\alpha-1} \Leftrightarrow \alpha = \lambda.
\]

We take this as our benchmark case of the analysis.

**Assumption BM.** \( \alpha = \lambda \).

![Figure 2: Output of an assimilated technique under Assumption BM](image)

### 3.4.1 Convergence

Assumption BM has several implications. First, under Assumption BM, Lemma 2 implies

\[
\chi_1 < 1 < \chi_2.
\]  

(24)

---

19 For a good discussion on the normalized CES approach, see the recent survey by Klump et al. (2012).

20 We relax this assumption in section 4.4 where we study the phenomena of leapfrogging and bounded development.
So (20) and (21) imply \( k_j = \bar{k} \chi_j^{1/(1-\sigma)} \) and \( \bar{k}_j = \bar{k} \chi_j^{\sigma/(1-\sigma)} \), hence we can rank capital thresholds with the foreign steady-state level of capital \( \bar{k} \)

\[
    k_1 < \bar{k}_1 < \bar{k} < \bar{k}_2 < k_2. \tag{25}
\]

Second, Lemma 4 implies that Assumption BM is stronger than Condition E, so the necessary condition of the existence of the full-assimilation steady state is satisfied. Third, under Assumption BM, \( \chi_i = 1 \) is a root to the following

\[
    0 = S(\chi) \equiv (1 - \alpha) \chi - \chi^{1-\sigma} + \alpha, \tag{26}
\]

so that either \( \chi_L \) or \( \chi_H \) equals to one, corresponding to \( k_i = \bar{k} \). Since \( \chi_i = 1 \in (\chi_1, \chi_2) \), Lemma 5 implies now the foreign steady-state level of capital \( \bar{k} \) is a full-assimilation steady state in this economy. Let \( \tilde{k} = \bar{k} \chi_i^{\sigma/(1-\sigma)} \) denote another full-assimilation steady state if exists. Lemma 5 can be strengthened to establish the stability of the full-assimilation steady states, if they exist, which is given by the following Corollary:

**Corollary 2** Given Assumption BM, we always have a full-assimilation steady state with capital \( k = \bar{k} \). Suppose there are two full-assimilation steady states, if \( \sigma > \alpha \), then \( \tilde{k} > \bar{k} \) and \( w'(\tilde{k}) < 1 < w'(\bar{k}) \); if \( \sigma < \alpha \), then \( \tilde{k} < \bar{k} \) and \( w'(\tilde{k}) > 1 > w'(\bar{k}) \).

We are ready to complete our steady state characterization with a stability analysis. Whenever a unique steady state exists, it must be \( k = \bar{k} \) and is stable. In this case all firms assimilate foreign technique. With multiple steady states, the stability of the steady states depends on the parameterization of the model. Define

\[
    B(\sigma) \equiv \chi_L(\sigma)^{\frac{\sigma-1}{1-\sigma}},
\]

where \( \chi_L(\sigma) \) is the smallest root solving (26) given \( \sigma \). The boundary function \( B(\sigma) \) only depends on \( \sigma \) but not \( z \) or \( \bar{z} \). We strengthen our steady-state characterization with Assumption BM in the following proposition:

**Proposition 7** Given Assumption BM, there does not exists autarky steady state. If \( \sigma > \alpha \), then \( B(\sigma) < 1 \) is strictly decreasing in \( \sigma \) with \( \lim_{\sigma \to 1} B(\sigma) = 0 \) such that

a. if \( z < \bar{z} B(\sigma) \) then there are three steady states where one is stable partial-assimilation with capital \( k = W(k_1) < k^* \), and two are full-assimilation with capital \( k = \bar{k}_L \) and \( k = \bar{k} > k^* \), which are unstable and stable respectively;

b. if \( z = \bar{z} B(\sigma) \) then there are two steady states which are full-assimilation \( k = k_L = W(k_1) \) and \( k = \bar{k} \), where only the former is unstable;
c. if $z > \varepsilon B(\sigma)$ then there is a unique stable steady state which is full-assimilation $k = \bar{k}$.
d. If $\sigma < \alpha$ then there are three steady states as in case (a) above, but the one with capital $k = \bar{k}$ is unstable and the one with $k = \bar{k}_H > \bar{k}$ is stable.
e. If $\sigma = \alpha$ then there are two steady states, where one is stable partial-assimilation with capital $k = W(k_1)$ and another one is unstable full-assimilation steady state with $k = \bar{k}_L = \bar{k}_H = \bar{k}$.

Proof. See Appendix. ■

Figure 3 illustrates different cases of Proposition 7. Now by assimilating foreign technique, it is possible to attain the foreign steady state $k = \bar{k}$, as illustrated by all but the bottom-right plot in Figure 3. The intuition is as follows. Since foreign technique requires high level of capital per labor $\bar{k}$, the assimilation of foreign technique is profitable only when the capital rent is low, which is the case when the economy is also endowed with high level of capital per labor. It is possible when the labor wage is high so that there is also high level of investment. However, due to capital-labor complementary in production wage is high only when capital per labor is high. There is a virtuous cycle: when capital is high enough, then wage is high resulting in a high level of capital accumulated and low rent. Domestic firms find it profitable to assimilate the capital-intensive foreign technique, which is also more productive and a even higher wage is then paid. Eventually the country will converge the steady state of the technology source country.

3.4.2 Assimilation Trap

There is also a vicious cycle where when the level of capital is not high enough to surpass the threshold, ie, the unstable steady-state level of capital in all but the bottom-left plot in Figure 3, such that the economy will gravitate to an "assimilation trap". In the assimilation trap wage is stagnant even though the economy is endowed with rising level of capital from $k = k_1$ to $k = \bar{k}_1$. To understand the intuition we need to distinguish the intensive margin and the extensive margin of capital demand. The intensive margin of capital demand is the capital demanded by each firm. Due to capital-labor complementarity, wage grows only when there is growing intensive margin of capital demand. On the other hand, notice that when the economy is open there are now two types of firm: one is using technique along the domestic technology frontier, and another is assimilating the foreign technique. The later has higher capital demand than the former since foreign technique is capital intensive. In the assimilation trap, any increase in capital endowment is absorbed by the increase in the extensive margin of
firms assimilating the foreign technique. So the capital rent is stagnant rather than decreases. As a result each firm does not increase their intensive margin of capital demand and wage becomes stagnant, although there is an increase in capital endowment. Eventually there is no further capital accumulation and the economy remains in the assimilation trap.

The condition $z \leq \bar{z} B(\sigma)$ in Proposition 7 is only necessary for an autarky economy to gravitate to the assimilation trap. Define

$$C(\sigma) \equiv \chi_L(\sigma)^{\frac{1}{1-\sigma}},$$

where $\chi_L(\sigma)$ is the smallest root solving (26) given $\sigma$. The boundary function $C(\sigma)$ only depends on $\sigma$ but not $z$ or $\bar{z}$. The necessary and sufficient condition for an autarky economy to gravitate to the assimilation trap is summarized by the following proposition:

**Proposition 8** Given the maintained assumptions, starting from an autarky steady state
the economy will converge to the low partial-assimilation steady state with \( k = W(k_1) \) if and only if \( z < \bar{z}C(\sigma) \), where \( C(\sigma) \leq B(\sigma) \) for all \( \sigma \in [0, 1) \) and \( C(\sigma) \leq 1 \) is strictly decreasing in \( \sigma \) with \( \lim_{\sigma \to 1} C(\sigma) = 0 \).

**Proof.** See Appendix. □

The intuition of Proposition 8 is as follows. There are two cases where the condition \( z < \bar{z}C(\sigma) \) is satisfied. First, for any \( \sigma \leq \alpha \) we have \( \chi_L = C(\sigma) = 1 \) and the condition \( z < \bar{z}C(\sigma) \) is satisfied automatically. What this means is that when assimilation power \( \sigma \) is low (\( \sigma \leq \alpha \)), then by the result of comparative statics the marginal product of labor of assimilated foreign technique is also low.\(^{21}\) Assimilating foreign technique will lead to low wage, then low level of capital is accumulated, leading to a further decrease of wage until the assimilation trap is reached. Thus, to avoid the assimilation trap it is necessary to have sufficient assimilation power given by \( \sigma > \alpha \).

Second, given any \( \sigma > \alpha \), if \( z \) is sufficiently low such that \( z < \bar{z}C(\sigma) \) then an autarky economy will still converge to the assimilation trap. In this case, it is possible for the economy to converge to the steady state of the technology source country. It is reflected by the result of Proposition 8 that \( C(\sigma) < B(\sigma) \). But the domestic productivity is so low that the autarky capital level is less than the unstable steady-state \( \bar{k} \), which serves as a capital threshold beyond which an economy can take off. In this case, the initial capital level fails short of the required capital of foreign technique to support enough marginal product of labor for further growth of wage. Then again the low-wage-low-capital vicious cycle emerges until the assimilation trap is reached.

Figure 4 provides a graphical representation of Proposition 7 and Proposition 8 in the parameter space of \((\sigma, z/\bar{z})\). The region in the \((\sigma, z/\bar{z})\) space where an economy gravitates to the assimilation trap is given by the boundary \( z/\bar{z} \leq C(\sigma) \). On the other hand, the region in the \((\sigma, z/\bar{z})\) space where an economy converges to the foreign steady state regardless of the initial condition is given by the boundary \( z/\bar{z} \geq B(\sigma) \). Figure 4 delivers our main message: assimilating foreign techniques can be beneficial and catching up the foreign steady state only if the assimilation power \( \sigma \) is high enough. There are two effects from a higher assimilation power \( \sigma \). The first one is the direct effect from assimilation, whereas the second one is the indirect of improving the domestic TFP \( z \) through the endogenous technology choice mechanism. An improvement in the assimilation power \( \sigma \) thus is captured by a north-east movement in the parameter space of \((\sigma, z/\bar{z})\) in Figure 4.

\(^{21}\)The Leontief case is an extreme example.
3.4.3 Comparative Statics

In this subsection, we investigate the effects of technology assimilation and productivity, which are captured by changes in the parameters $\sigma$, $z$ and $\bar{z}$, on the steady states. Recall the comparative statics of these parameters on the wage function derived in Section 3.2: a rise in $z$ ($\bar{z}$) shifts up the technique-based wage function $W(k)$ [$\bar{W}(k)$]. From Proposition 7, we know that when $\sigma \geq \alpha$, the equilibrium characterization depends on the comparison between $z/\bar{z}$ and $B(\sigma) \in [0, 1)$, where the latter is decreasing in $\sigma$. So the global effect of a change in $z$ (or $\bar{z}$) is straightforward to characterize. Specifically, a rise (fall) in $z$ ($\bar{z}$) then is more likely leading to the case where $z/\bar{z} > B(\sigma)$ so that a unique equilibrium $\bar{k}$ will be achieved. The intuition is easy to understand. Given a high enough assimilation ability where $\sigma \geq \alpha$, when the productivity gap between the two countries closes up (a rise in $z/\bar{z}$ toward unity), then assimilation becomes more likely to be successful. As a result, the high full-assimilation steady state $\bar{k}$ becomes the dominating equilibrium outcome. On the other hand, if the assimilation ability is low so that $\sigma < \alpha$, then an assimilation trap is a must. Regardless of the relative productivity ratio $z/\bar{z}$, the high full-assimilation steady state $\bar{k}$ is a dominated equilibrium outcome.

We then move to study the local effects of an increase in $z$. According to Proposition 7, the full-assimilation steady states are not affected because $\bar{W}(k)$ is not affected. However, the partial-assimilation steady state $k = W(k_1)$ increases. For the case where the partial-assimilation steady state is below the autarky steady state, then the output gap between assimilation and autarky diminishes. This is because, other things equal, a higher domestic productivity raises firm’s intensive margin of capital demand so that the assimilation trap in the partial-assimilation steady state is improved. Similarly, the effects of an increase in $\bar{z}$ shifts up $\bar{W}(k)$ with $W(k)$ remained unaffected. In this case, the high full-assimilation steady state increases whereas the low full-assimilation steady state decreases. For the case of $\sigma > \alpha$, then the fall in the threshold (low) full-assimilation steady state makes the convergence toward $\bar{k}$
easier. This is because if the level of $k$ is relatively high before (in the neighborhood of the initial $\hat{k}$), then the increase in $\bar{z}$ raises wage so that $\hat{k}$ can be now achieved. However, for the partial-assimilation steady state that is below the autarky level, it moves further away from the autarky steady state so that the assimilation trap deteriorates. This is due to the fact that, at low level of $k$, the improvement in foreign technique only makes the assimilation process more difficult without much improvement in the wage so that polarization occurs.

For the comparative statics of assimilation ability $\sigma$, it is in general ambiguous due to the presence of different conflicting forces, namely, the direct effect of $\sigma$ and the indirect TFP effect of $z$. Hence its effects on the steady-state equilibria are ambiguous too. However, the global comparative statics is clear because a rise in $\sigma$ shifts up $\bar{W}(k)$ for $k < \hat{k}$. For tractability, let us focus on the case of $\sigma > \alpha$. Then as shown in Proposition 7, we have $B'(\sigma) < 0$. Similar to the analysis of an increase in $z/\bar{z}$, a rise in the assimilation power $\sigma$ also more likely leads to the case where $z/\bar{z} > B(\sigma)$ so that a unique equilibrium $\bar{k}$ will be achieved. The intuition again is straightforward. Given a productivity gap between the two countries $z/\bar{z}$, then assimilation is more likely to be successful when a country has a better assimilation ability. As a result, the high full-assimilation steady state $\bar{k}$ becomes the dominating equilibrium outcome when $\sigma$ increases.

For the local comparative statics, the results are in general ambiguous. As shown in Proposition 4, an increase in $\sigma$ consists of the direct effect of $\sigma$ and the indirect TFP effect of $z$. The former mimics the assimilation effect of an increase in foreign technology $\bar{z}$, whereas the latter represents the efficiency effect an increase in domestic TFP $z$. Since the two have opposite effects on $k_1$ and $\bar{k}_1$, so the net effect on the partial-assimilation steady state cannot be determined. For the full-assimilation steady states, an improvement in assimilation ability lower the threshold (which is the low full-assimilation steady state) so that the high full-assimilation steady state $\bar{k}$ is more likely to be achieved. So we conclude:

**Proposition 9** For $\sigma > \alpha$, an increase in the relative productivity ratio $z/\bar{z}$ always leads to a higher partial-assimilation steady state $k = W(k_1)$. For the full-assimilation steady states, a decrease in $\bar{z}$ raises the low full-assimilation steady state so that the assimilation trap is more likely to occur; but a change in $z$ has no effect. For an increase in assimilation ability $\sigma$, it is a combined effect of an increase in both $z$ and $\bar{z}$. Its effect on the partial-assimilation steady state is ambiguous, whereas it reduces the low full-assimilation steady state so that the high full-assimilation steady state $\bar{k}$ is a more likely outcome.

From Proposition 9, we can see that an improvement in foreign technology has complicated effects on the equilibrium outcomes of the domestic economy. For a reasonably capable country
in assimilation where $\sigma > \alpha$, it is polarized. On the one hand, if it is relatively capital abundant (rich) in autarky, then a rise in $\tilde{z}$ makes the high full-assimilation steady state $\tilde{k}$ to be a more likely outcome. Successful assimilation improves the autarky situation through its higher reward. On the other hand, when it is relatively capital scarce (poor) in autarky, then the partial-assimilation steady state is the equilibrium outcome whose level is reduced by the increase in $\tilde{z}$ and the assimilation trap becomes worse.

### 3.5 Optimal Open Policy

The model also sheds light on the optimal timing to open to foreign technique. Specifically, the timing to open depends on both the productivity gap $z/\bar{z}$ and the assimilation power $\sigma$, which are summarized by the threshold $\tilde{k} = \chi_1(\sigma, z/\bar{z})^{\sigma/(1-\sigma)}$. It is clear from the wage function that it is optimal to open only after the first stopping time $t$ when $W(k_t) < W(k_\ell)$ and $k_t > \tilde{k}$. In this case the accumulation of capital is the fastest and it is guaranteed to converge to the foreign steady state. It is optimal to remain in the autarky forever if such a stopping time does not exist. Opening too early would lead to an assimilation trap. It echoes the argument for first protecting domestic industry to grow from the foreign technique until it is deepened enough for the structural change to foreign technique.

### 4 Applications

In this section, we apply our model of technology assimilation to study four interesting phenomena of growth and development, namely:

1. The growth miracle of South Korea relative to the Philippines where their initial conditions in the 1960s are almost identical as discussed by Lucas (1993);
2. The "Twin Peaks" phenomenon of Quah (1997) or the club convergence hypothesis of Azariadis (1996) and Galor (1996);
3. The Flying-Geese pattern of Asian growth experience emphasized by Akamatsu (1962) and Lin (2011);
4. The recurring pattern of "leapfrogging" during the last three centuries documented by Maddison (1982).

#### 4.1 The Lucas (1993) Miracle

In this subsection, we apply the assimilation model to study the miracle of South Korea discussed by Lucas (1993). According to Lucas, back in the 1960s, both South Korea and the Philippines were very similar in many respects, such as the standard of living, wealth and
endowment levels. However, in the next twenty years, the two countries grow in a drastically different manner:

From 1960 to 1988, GDP per capita in the Philippines grew at about 1.8 percent per year, about the average for per capita incomes in the world as a whole. In Korea, over the same period, per capita income grew at 6.2 percent per year, a rate consistent with the doubling of living standards every 11 years. Korean incomes are now similar to Mexican, Portuguese, or Yugoslavian, about three times incomes in the Philippines, and about one third of incomes in the United States. (p,251)

Our model of technology assimilation can explain the Lucas (1993) miracle in terms of different assimilation power $\sigma$ of the countries. Notice that if Korea and the Philippines have different assimilation power $\sigma$ but the same domestic productivity $z$ then they still attain the same autarky steady state, consistent with what Lucas observes. Specifically, let us denote the assimilation power of South Korea (the Philippines) as $\sigma_{\text{Kor}} (\sigma_{\text{Phi}})$. If $\sigma_{\text{Kor}}$ and $\sigma_{\text{Phi}}$ estimated by the data is consistent with $\sigma_{\text{Kor}} > \alpha > \sigma_{\text{Phi}}$, then the model predicts that, once both countries are open to a more advanced foreign technique, it is possible for South Korea converging to the high full-assimilation steady state $k = \bar{k}$, while the Philippines converging to the assimilation trap $k = W (k_1)$.

To proceed we estimate $\sigma$ by using the time series from Penn World Table 7.1. Since there is only data of investment available, we need to construct a time series of capital. To recover the initial level of capital, we assume both countries are already at their autarky steady states in 1960 so that

$$k_{1960} = \frac{I^*}{\delta + n^*}, \quad (27)$$

where $I^*$ is the level of average real investment per capita before 1960 and $n^*$ the average population growth rate before 1960. We use the standard value $\delta = 0.04$ for the depreciation rate and $\alpha = \lambda = 1/3$ for the capital share. Equation (27) implies at the autarky steady state investment is used to compensate the depreciated capital and population growth. Using (27) we can recover the level of capital per capita in year 1960 for both countries. To construct the subsequent level of capital per capita, we use the accounting identity:

$$k_{t+1} = \frac{I_t + (1 - \delta) k_t}{1 + n_{t+1}}, \quad \text{for } t > 1960, \quad (28)$$

where $n_t \equiv N_t/N_{t-1} - 1$ is the population growth rate. Assuming one generation correspond to twenty five years. Table 2 reports the level of capital per capita relative to the U.S.
Then we estimate the assimilation parameters $\sigma_{\text{Kor}}$ and $\sigma_{\text{Phi}}$ with the constructed capital time series. Since in data there can be growth in productivities, we need to stationarize the time series of capital with respect to the advanced country. We assume both Korea and the Philippines assimilate the U.S. technology. For the U.S. its dynamics of capital accumulation according to our model is

$$k_{\text{US},t+25} = \frac{N_{\text{US},t}}{N_{\text{US},t+25}} (1 - \alpha) z_{\text{US},t} (k_{\text{US},t})^\alpha. \quad (29)$$

For countries that adopt the U.S technology, says Korea, our model characterizes the dynamics of capital accumulation as

$$k_{\text{Kor},t+25} = \exp(\varepsilon_t) \frac{N_{\text{Kor},t}}{N_{\text{Kor},t+25}} (1 - \alpha) z_{\text{US},t} (k_{\text{US},t})^\alpha \left[ \alpha \left( \frac{k_{\text{Kor},t}}{k_{\text{US},t}} \right)^{\frac{\sigma_{\text{Kor}}}{\sigma_{\text{Kor}}+1}} + 1 - \alpha \right]^{\frac{\sigma_{\text{Kor}}}{\sigma_{\text{Kor}}+1}}, \quad (30)$$

where $\varepsilon_t$ is the error term, which can be interpreted as the distortion (subsidy if positive) to assimilate foreign technique. Dividing (30) by (29), we have a stationary capital level of Korea:

$$\log \left( \frac{k_{\text{Kor},t+25}}{k_{\text{US},t+25}} \right) = \log \left( \frac{N_{\text{US},t+25}}{N_{\text{US},t}} \right) - \frac{\sigma_{\text{Kor}}}{\sigma_{\text{Kor}}+1} \log \left( \alpha \left( \frac{k_{\text{Kor},t}}{k_{\text{US},t}} \right)^{\frac{\sigma_{\text{Kor}}}{\sigma_{\text{Kor}}+1}} + 1 - \alpha \right) + \varepsilon_t. \quad (31)$$

Thus, the stationarized capital level is not affected by the growth of US productivity. We adopt the same construction for the Philippines. Applying nonlinear least square on (31), the estimated assimilation parameters of both countries are given by

$$\sigma_{\text{Kor}} = 0.7675, \sigma_{\text{Phi}} = 0.1632.$$  

It is consistent with our hypothesis that $\sigma_{\text{Kor}} > \alpha > \sigma_{\text{Phi}}$. So by Proposition 7 the Philippines case is in the situation of converging to the assimilation trap of the low partial-assimilation steady state. For Korea, since the condition

$$\frac{z}{\bar{z}} = (\chi_L)^{(\sigma_{\text{Kor}}-\alpha)/(1-\sigma_{\text{Kor}})} = (k_{\text{Kor,1960}}/k_{\text{US,1960}})^{1-\alpha} - (\chi_L)^{(\sigma_{\text{Kor}}-\alpha)/(1-\sigma_{\text{Kor}})} = 0.1606 > 0$$

Table 2: Capital per capita relative to the U.S., Korea vs. the Philippines

<table>
<thead>
<tr>
<th></th>
<th>$t = 1960$</th>
<th>$t = 1985$</th>
<th>$t = 2010$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{\text{Kor},t}/k_{\text{US},t}$</td>
<td>6.45%</td>
<td>22.95%</td>
<td>79.20%</td>
</tr>
<tr>
<td>$k_{\text{Phi},t}/k_{\text{US},t}$</td>
<td>7.83%</td>
<td>9.86%</td>
<td>5.92%</td>
</tr>
</tbody>
</table>

Source: calculation based on Penn World Table 7.1
is satisfied, Proposition 7 implies that there is no assimilation trap and Korea converges to the unique high full-assimilation steady state.

4.2 The Twin-Peak Phenomenon

The challenge to the conditional convergence hypothesis of the neoclassical growth model has been one of the main contribution of the endogenous growth models in the 1990s. An alternative is the concept of club convergence, which is described by Galor (1996) as follows:

The club convergence hypothesis (polarisation, persistent poverty, and clustering) - per capita incomes of countries that are identical in their structural characteristics converge to one another in the long-run provided that their initial conditions are similar as well. (p.1056)

By examining the relative income patterns for the period of 1961-1988 of 105 countries from the standard Summer-Heston (1991) data set, Quah (1997) concludes that the club convergence hypothesis seems to be relevant:

There is a wide spectrum of intradistribution dynamics - overtaking and catching-up occur simultaneously with persistence and languishing - while overall the twin-peaks shape in the cross-sectional distribution emerges. (p.38)

To generate the twin-peak phenomenon through the channel of technology assimilation, we extend our two-country setting to a continuum of countries with heterogenous productivity $z \in [0, z]$ and assimilation power $\sigma \in [0, 1)$. Denote $g(z/\bar{z}, \sigma) : [0, 1] \times [0, 1) \rightarrow \mathbb{R}_+$ to be the joint density function of the country with relative productivity $z/\bar{z}$ and assimilation power $\sigma$. To keep the model simple, we assume techniques are freely assessible across countries at $t = 0$, while firms can only assimilate at most one foreign technique and factors of production are kept within the national boundary.

In autarky firms in country $i$ are choosing domestic techniques along the frontier of the global production function, which is given by

$$f_i(k) = z_i k^\alpha,$$  \hspace{1cm} (32)

Once a foreign technique is available for assimilation, firms can also assimilate the technique from country $j \in [0, 1]$, where $j \neq i$ and the local production technique is given by

$$f_j(k) = z_j \left( k^*_j \right)^\alpha \left[ \alpha \left( \frac{k}{k^*_j} \right)^{\frac{\sigma - 1}{\bar{\sigma}}} + (1 - \alpha) \right]^{\frac{\bar{\sigma}}{\bar{\sigma} - 1}}, \text{ for } j \neq i,$$ \hspace{1cm} (33)
where \( k_j^* \) is the factor requirement of the technique from country \( j \). In autarky (before \( t = 0 \)), each country is endowed with capital \( k_i^* \), which is given by

\[
k_i^* = [(1 - \alpha) z_i]^{1/(1-\alpha)}. \tag{34}
\]

Then the corresponding unit cost function for the domestic technique is given by

\[
\hat{c}_i \equiv \frac{x\alpha w^{1-\alpha}}{z_i\alpha(1-\alpha)^{1-\alpha}}. \tag{35}
\]

The unit cost function for assimilating the technique from country \( j \) is given by

\[
c_j \equiv \frac{1}{z_j (k_j^*)^{\alpha}} \left[ \alpha^{\sigma_i} (r k_j^*)^{1-\sigma_i} + (1 - \alpha)^{\sigma_i} w^{1-\sigma_i} \right]^{1/\sigma_i}, \text{ for } j \neq i, \tag{36}
\]

We first compare the domestic techniques with all the foreign techniques. By the symmetry breaking result of Proposition 5, it is straightforward to show that assimilating the technique from country \( j \) is less costly than any domestic technique along the global production function if and only if \( z_j > z_i \).

What remains to show is the comparison amongst the foreign techniques with \( z_j > z_i \). Combining (34) and (36), the unit cost function for assimilating technique from country \( j \) is given by

\[
c_j = \left[ \alpha (r/r_j)^{1-\sigma_i} + (1 - \alpha) (w/w_j)^{1-\sigma_i} \right]^{1/\sigma_i}, \tag{37}
\]

where \( r_j = \alpha/(1-\alpha) \) and \( w_j = k_j \). But (37) implies that given the factor prices \( w \) and \( r \), the foreign technique with the lowest unit cost \( c_j \) is the one with the largest \( k_j \), which is the one from the global leader with productivity \( \bar{z} \). In sum, although firms in each country are facing a continuum of foreign techniques, the only relevant choice is the one from the global leader and we are back to the previous analysis of the two-country case.

As illustrated in Figure 4, Proposition (8) can be used to predict which countries are converging to the global leader and which are converging to the assimilation trap. Define the measure \( G \) as

\[
G \equiv \int_0^1 \int_0^{C(\sigma)} g(x, \sigma) \, dx \, d\sigma. \tag{38}
\]

Then the following corollary from Proposition (8) predicts the twin-peak convergence pattern in the global economy:

**Corollary 3** Given the maintained assumptions, then

a. There is a measure \( G \) of countries converging to the assimilation trap.

b. For the remaining measure \( 1 - G \) of countries, each converges to the steady state of the country with the highest productivity \( \bar{z} \).
The phenomenon of "club convergence" is generated as follows. There exists a threshold $C(\sigma)$ (depends on $\sigma$) to the relative productivity $z/\overline{z}$ which divides the world economy into two groups. For countries with $z/\overline{z} < C(\sigma)$ there is divergence in the income level within the group, where each of them remains in their own assimilation trap. For countries with $z/\overline{z} > C(\sigma)$ each of them converges to the steady state of the global leader. This produces the "Twin-Peak" phenomenon that emphasizes the stratification in the world income levels.

4.3 Flying-Geese Pattern of Development

The phrase “flying geese pattern of development” was coined originally by Kaname Akamatsu in the 1930s (in Japanese) to describe the catching-up process of industrialization in latecomer countries. There is a specific pattern: the developing economy usually goes through the three successive stages of import, production, and export before its takeoff. Akamatsu then applied the “flying geese” concept to industrial development in Asia: from a lead goose (Japan) to follower geese (NIEs, ASEAN 4, etc.).

To formalize the idea, we specify the flying-geese pattern as the phenomenon that a laggard country, says Taiwan or Korea, does not assimilate the frontier technology, says from the U.S. Instead, it waits for an intermediate country, says Japan, to assimilate the frontier technology and kick off. Then it assimilates the technology of the intermediate country rather than of the frontier country. This is what Justin Lin (2011) describes to be the "secret winning formula" for economic development:

The secret winning formula for developing countries is to exploit the latecomer advantage by building up industries that are growing dynamically in more advanced countries that have endowment structures similar to theirs. By following carefully selected lead countries, latecomers can emulate the leader-follower, flying-geese pattern that has served well all successfully catching-up economies since the 18th century. (p.4)

To see how our model of technology assimilation can be applied to study the flying-geese pattern, suppose there are three countries in their autarky steady states, with productivities $z_H$, $z_M$, and $z_L$, respectively, in descending order, i.e., $z_H > z_M > z_L$.\footnote{Since $z$ is endogenous, the fundamental difference between the $z$’s is due to the deep parameters mentioned in section 2. Since in the analysis below we assume $\alpha$ and $\sigma$ are the same across countries, so we can think of the difference in $z$’s comes from the differences in the Pareto distribution parameters such as $\gamma_a$ and $\gamma_b$.} Suppose after having opened to foreign techniques, country $M$ assimilates the technology of country $H$ and will catch up to its steady state, while country $L$ remains in autarky. The fact that foreign
technique is not assimilated in country $L$ implies
\[
\left[ \alpha \left( \frac{k_L}{k_H} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha \right]^{\frac{\sigma}{\sigma-1}} < \frac{z_L}{z_H} \left( \frac{k_L}{k_H} \right)^\alpha .
\] (39)

Now denote $k_{M,t} \in (k_L, k_H)$ as the capital per labor of country $M$ (on the path converging to $k_H$). The output per labor of country $M$ is given by
\[
y_{M,t} = z_H k_H^{\sigma - \alpha} \left[ \alpha \left( \frac{k_{M,t}}{k_H} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha \right]^{\frac{\sigma}{\sigma-1}} .
\]

If firms in country $L$ assimilate the technique of country $M$, which in turn is assimilated from country $H$, then the output per labor is given by
\[
y_{t} = y_{M,t} \left[ \alpha \left( \frac{k_L}{k_{M,t}} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha \right]^{\frac{\sigma}{\sigma-1}} .
\]

The technique of country $M$ is assimilated in country $L$ if and only if
\[
\left[ \alpha \left( \frac{k_M}{k_H} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha \right]^{\frac{\sigma}{\sigma-1}} \geq \frac{z_L}{z_H} \left( \frac{k_L}{k_H} \right)^\alpha .
\] (40)

The necessary condition for (39) and (40) to hold is
\[
\left[ \alpha \left( \frac{k_{M,t}}{k_H} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha \right]^{\frac{\sigma}{\sigma-1}} > \left[ \alpha \left( \frac{k_L}{k_{M,t}} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha \right]^{\frac{\sigma}{\sigma-1}} ,
\]
\[
\Leftrightarrow \left[ \left( \frac{k_{M,t}}{k_H} \right)^{\frac{\sigma-1}{\sigma}} - 1 \right] \left[ \left( \frac{k_L}{k_{M,t}} \right)^{\frac{\sigma-1}{\sigma}} - 1 \right] > 0
\]
which is always satisfied for any $k_{M,t} \in (k_L, k_H)$. Similarly, by induction, we have the $m$-country extension as follows:
\[
\prod_{i=1, \ldots, m-1} \left[ \alpha \left( \frac{k_{i+1}}{k_i} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha \right]^{\frac{\sigma}{\sigma-1}} > \left[ \alpha \left( \frac{k_m}{k_1} \right)^{\frac{\sigma-1}{\sigma}} + 1 - \alpha \right]^{\frac{\sigma}{\sigma-1}} ,
\]
for any $k_1 > k_2 > \ldots > k_m$.

Also, we can provide an interpretation for the flying-geese pattern in terms of technology assimilation. We note that the left hand side of (40) (out of $k_{M,t}$) is at its maximum if and only if $k_{M,t} = (k_H k_L)^{1/2}$. Hence there exists a $k_{M,t}$ such that (40) is satisfied if
\[
\left[ \alpha \left( \frac{k_L}{k_H} \right)^{\frac{\sigma'-1}{\sigma'}} + 1 - \alpha \right]^{\frac{\sigma'}{\sigma'-1}} \geq \frac{z_L}{z_H} \left( \frac{k_L}{k_H} \right)^\alpha
\]
where
\[ \sigma' = \frac{2\sigma}{1 + \sigma} > \sigma. \]

Thus, we can conclude:

**Proposition 10** By following the flying-geese pattern of development, a country is able to raise its assimilation power.

So allowing for the flying-geese pattern is qualitatively equivalent to having a higher assimilation power. The intuition is as follows. By following the flying-geese pattern, a developing country can raise its relative productivity ratio due to a lower frontier target (from \( \bar{z} \) to \( z_M \)). By appropriately choosing the middle goose, the country can catch up to a higher income level of the middle goose. Once \( y_M \) is achieved via this intermediate step, then its relative productivity ratio in terms of the world frontier has risen (from \( z/\bar{z} \) to \( z_M/\bar{z} \)) so that it is possible to converge to \( \bar{k} \) from \( k_M \) step-by-step. However, from the perspective of technology assimilation, this two-step development strategy is qualitatively equivalent to the case where we have an increase in the assimilation ability (say from \( \sigma \) to \( \sigma' \)) given the original level of relative productivity level \( z/\bar{z} \). As the flying-geese pattern is rather common in Asian growth experience [e.g., evidence found in Wang et al. (2013)], our analysis provides an explanation in terms of technology assimilation.

### 4.4 Leapfrogging v.s. Bounded Development

In this section we apply the model to explain how economic leapfrogging can happen, as documented in Maddison (1982) and studied in Brezis et al (1993) and many others. While the process of international growth may be marked by convergence, and countries remain in their economic ranking for long periods, we also observe economic leapfrogging from time to time. In particular, after some long periods of economic leadership, the leading countries are overtaken by laggards that mimic and assimilate the leader's technology. The early dominance of the Dutch was ended by the rise of England; England dominance was ended by the rise of America and Germany; and following the current trend we may be seeing the United States overtaken by the fast growing Asian countries. Table 3 reports the change of economic leadership during the last three centuries. Such overtaking phenomena cannot be easily explained by a standard endogenous growth theory; as it suggests, on the contrary, technological change should reinforce the position of the leading nation.
Here we use our model to generate scenarios of economic leapfrogging and scenarios of development bounded by the leading countries. This section also serves a robustness check to the consequences of relaxing some maintained assumptions. Consider the same two-country model, but now focus on the general case $\alpha \neq \lambda$. If $\alpha > \lambda$, then from Lemma 4, Condition E is satisfied and there are at most two full-assimilation steady states by proposition 5. Notice that $H > 1$ as $S(1) < 0$ under the premise $\alpha > \lambda$, so the definition of $\chi$ implies $k_H > \bar{k} > k_L$. Suppose $\chi_H \in (\chi_1, \chi_2)$, then there exists a full-assimilation steady state with capital $k_H$, which is also stable by Proposition 6. On the other hand, if $\chi_H \geq \chi_2$, then we have the following lemma:

**Lemma 7** Given $\alpha > \lambda$ and suppose $\chi_H \geq \chi_2$, there exists a stable partial-assimilation steady state with capital $k > \bar{W}(\bar{k}_2)$.

**Proof.** See Appendix.

With the help of Lemma 7, the following proposition gives the necessary conditions for leapfrogging to happen:

**Proposition 11** Given maintained assumptions,

a. (Leapfrogging) suppose $\alpha > \lambda$ and $z \leq \eta^{-1}\bar{z}$, then there exists a stable steady state with capital $k > \bar{k}$.

b. (Bounded development) Otherwise and, furthermore, suppose $\sigma \geq \alpha$, the highest level of steady-state capital is never greater than $\bar{k}$.

**Proof.** See Appendix.

Proposition 11 states that if, firstly, the labor share of the assimilated foreign technique $1 - \lambda$ is greater than the labor share of the original foreign technology $1 - \alpha$ and if, secondly, the domestic productivity $z$ is sufficiently lower than the foreign productivity $\bar{z}$, then it is possible for the laggard country to surpass the output level of the technology source country. It is only "possible" because there can be multiple steady states, whether the laggard country "leapfrogs" to a steady state with higher capital level depends on its initial condition. The
leapfrogging steady state can feature either full assimilation or partial assimilation of foreign technique.

Leapfrogging in this model is the result of the interplay between technology assimilation and rising wage. The two conditions for the existence of leapfrogging steady state can be understood as follows: it is less costly to assimilate foreign technique, and assimilating foreign technique will lead to higher wage. In this economy capital is accumulated from the saving of labor wage. By assimilating foreign technique, the equilibrium wage of the laggard country depends on the labor share parameter \(1 - \lambda\) of the assimilation. However, in the technology source country the equilibrium wage depends on the frontier of its production techniques, which is featured by the labor share parameter \(1 - \alpha\), which basically comes from the distribution of production techniques. So if \(\lambda < \alpha\) then the steady-state level of wage is higher in the laggard country than the technology source country, hence more capital is accumulated and higher output is produced eventually. However it may not be desirable to assimilate foreign technique when wage is high. For assimilating foreign technique to dominate any technique from the domestic technology frontier, the domestic productivity \(z\) has to be sufficiently lower than the foreign productivity \(\bar{z}\) such that \(z \leq \eta^{-1} \bar{z}\). Once these two conditions satisfied it is then possible to have leapfrogging to happen.

5 Concluding Remarks

In this paper we have studied the mechanism of technology assimilation and have examined its effects on economic development. The fundamental determinant of aggregate productivity is the power to assimilate production technology with respect to factor endowment. Assimilating foreign technology can have dramatical consequences. Once foreign technology is open to assimilate, there is symmetry breaking of the autarky equilibrium. Depending on the assimilation power, a laggard country can either catch up with the frontier countries (and their productive ideas) through the virtuous cycle of technology assimilation and rising wage, or fall to an assimilation trap with stagnant income. Finally, given a reasonably high ability of assimilation, an advance in the world frontier technology polarizes the world economy so that the richer countries become richer whereas the poorer countries get poorer.

There are a number of policy implications from our paper. If the ability of technology assimilation of a country is not strong, then a big-push type of development policy is called for so as to make sure that the country has enough capital to start off. However, if the ability of technology assimilation is almost absent, then the big push or subsidizing for technology adoption no longer works, which is in contrast to the existing literature of economic development. For instance, Rustichini and Schmitz (1991) argue that imitation plays an important
role to developing countries so that it is optimal to subsidize imitation in their model. In this case, protecting industries from foreign technology can be desirable. Even in the case of strong assimilation power, the timing open to foreign technology can also have significant long-run impact on development. Our paper provides a mechanism to generate a different result.\footnote{The conventional wisdom of subsidizing technology-advancing activities has also been challenged by others. For example, from the perspective of political economy, Boldrin and Levine (2004) show that policies of fostering innovations and their adoption like patents may not be optimal due to rent-seeking behavior of the private and public sectors.} We emphasize that the key to success depends crucially on a country’s ability of technology assimilation.

We demonstrate the potential of the model by a number of interesting applications which have been challenging to standard convergence theory and endogenous growth theory. For the potential of quantitative research, the concept of assimilation provides a new approach to cross-country comparison. By controlling on the same production technique for different countries, we can have a direct comparison of countries’ assimilation power, which are available by standard empirical tools. For instance, a recent paper of Wang et al. (2013) applies the assimilation model to perform growth, development and volatility accounting exercises. It is found that technology assimilation is able to reduce the unexplained component of Lucas (2000) by 40-70\% for miracle countries and by over 50\% for trapped economies. We believe that ongoing researches in this direction will be fruitful in contributing to our understanding in cross-country income differences.
References


6 Appendix

6.1 Proof of Lemma 1

Proof. The "if" part, \( \left\{ a_i \geq \frac{y_i}{\lambda} \left[ \lambda^{-1} + (1 - \lambda^{-1}) \left( \frac{b_i}{\lambda} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right\} \cap \left\{ b_i \geq y (1 - \lambda)^{\frac{\sigma-1}{\sigma}} \right\} \Rightarrow \{ y_i \geq y \} \), is straight-forward to verify and omitted here. To show the "only if" part, consider two cases: first \( \left\{ a_i < \frac{y_i}{\lambda} \left[ \lambda^{-1} + (1 - \lambda^{-1}) \left( \frac{b_i}{\lambda} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right\} \) and second \( \left\{ b_i < y (1 - \lambda)^{\frac{\sigma-1}{\sigma}} \right\} \). In the first case, since \( \frac{\sigma}{\sigma-1} < 0 \), we have \( \left\{ a_i < \frac{y_i}{\lambda} \left[ \lambda^{-1} + (1 - \lambda^{-1}) \left( \frac{b_i}{\lambda} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right\} \) equivalent to

\[
y^{\frac{\sigma-1}{\sigma}} < \lambda (a_i k)^{\frac{\sigma-1}{\sigma}} + (1 - \lambda) b_i^{\frac{\sigma-1}{\sigma}} \Leftrightarrow y > \left[ \lambda (a_i k)^{\frac{\sigma-1}{\sigma}} + (1 - \lambda) b_i^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} = y_i,
\]

thus we have established \( \left\{ a_i < \frac{y_i}{\lambda} \left[ \lambda^{-1} + (1 - \lambda^{-1}) \left( \frac{b_i}{\lambda} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right\} \Leftrightarrow \{ y_i < y \} \). In the second case, since \( \left\{ b_i < y (1 - \lambda)^{\frac{\sigma-1}{\sigma}} \right\} \) is equivalent to \( \left\{ \left( 1 - \lambda \right) b_i^{\frac{\sigma-1}{\sigma}} < y \right\} \), for any \( \sigma \in [0, 1) \) we have

\[
y > \left( 1 - \lambda \right) b_i^{\frac{\sigma-1}{\sigma}} \geq \left[ \lambda (a_i k)^{\frac{\sigma-1}{\sigma}} + (1 - \lambda) b_i^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} = y_i.
\]

Hence we have established that \( \left\{ b_i < y (1 - \lambda)^{\frac{\sigma-1}{\sigma}} \right\} \Rightarrow \{ y_i < y \} \). In sum, we have shown that \( \left\{ a_i < \frac{y_i}{\lambda} \left[ \lambda^{-1} + (1 - \lambda^{-1}) \left( \frac{b_i}{\lambda} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right\} \cup \left\{ b_i < y (1 - \lambda)^{\frac{\sigma-1}{\sigma}} \right\} \Rightarrow \{ y_i < y \} \), hence by negation we establish the "only if" part \( \{ y_i \geq y \} \Rightarrow \left\{ a_i \geq \frac{y_i}{\lambda} \left[ \lambda^{-1} + (1 - \lambda^{-1}) \left( \frac{b_i}{\lambda} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \right\} \cap \left\{ b_i \geq y (1 - \lambda)^{\frac{\sigma-1}{\sigma}} \right\} \). \( \blacksquare \)

6.2 Proof of Proposition 2

Proof. Define \( \psi \equiv (\sigma - 1) / \sigma \). We only need to derive \( \partial z / \partial \psi \) and \( \partial z / \partial \lambda \) since showing the remaining comparative statics is straightforward. Define \( G(\psi, x) \equiv -\frac{\theta_a}{\psi} \log \left[ \lambda^{-1} + (1 - \lambda^{-1}) \frac{\psi}{x} \right] \).

Then we have

\[
\frac{\partial G(\psi, x)}{\partial \psi} = \frac{\theta_a}{\psi^2} \left[ - (1 - \lambda^{-1}) \pi \log \pi - [1 - (1 - \lambda^{-1}) \pi] \left[ \log [1 - (1 - \lambda^{-1}) \pi] + \log \lambda \right] \right]
\]

where

\[
\pi = \frac{x^{\frac{\psi}{\psi}}}{\lambda^{-1} + (1 - \lambda^{-1}) x^{\frac{\psi}{\psi}}}.
\]

Notice that

\[
\frac{\partial^2 G(\psi, x)}{\partial \psi \partial \pi} = \frac{\theta_a}{\psi^2} (1 - \lambda^{-1}) \log \left( \frac{x^{\psi}}{x^{\psi}} \right).
\]
Also notice that $\pi$ is strictly decreasing in $x^\psi$, so we have $\frac{\partial^2 G(\psi, x)}{\partial \psi}$ is U-shape in term of $\pi$, with the minimum at $\pi = 1$. It means $\frac{\partial G(\psi, x)}{\partial \psi} > 0$ almost everywhere. Recall that

$$\log z \propto \log \int_0^{(1-\lambda)^{\theta/\psi}} \exp(G(\psi, x)) \, dx.$$ 

Since both $(1-\lambda)^{\theta/\psi}$ and $G(\psi, x)$ are strictly increasing in $\psi$ almost everywhere, $z$ is also strictly increasing in $\psi$.

Finally, since

$$\frac{1}{z} \frac{\partial z}{\partial \lambda} = \frac{1}{\psi \lambda (1-\lambda)} > 0$$

iff $\lambda > \alpha$ since $\psi < 0$.  

6.3 Proof of Lemma 2

**Proof.** First note that $\eta \equiv \frac{\sigma(1-\alpha)^{1-\alpha}}{\lambda(1-\lambda)^{1-\alpha}} \geq 1$, since

$$\ln \left[ \frac{\lambda^\alpha (1-\lambda)^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \right] = \alpha \ln \left( \frac{\lambda}{\alpha} \right) + (1-\alpha) \ln \left( \frac{1-\lambda}{1-\alpha} \right),$$

$$\leq \alpha \left( \frac{\lambda}{\alpha} - 1 \right) + (1-\alpha) \left( \frac{1-\lambda}{1-\alpha} - 1 \right) = 0,$$

where the inequality comes from the fact that the logarithmic function is concave. The fact that $\eta \geq 1$, $\frac{\lambda}{\alpha} < 1$ and $\sigma \in [0, 1)$ implies $Z \equiv (\eta \frac{\lambda}{\alpha})^{1-\sigma} < \eta$. Second, define $\chi_0 \equiv \frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}$, then $\Lambda(\chi_0)$ is given by

$$(1-\lambda) Z \frac{\lambda(1-\alpha)}{\alpha(1-\lambda)} - \left[ \frac{\lambda(1-\alpha)}{\alpha(1-\lambda)} \right]^{1-\alpha} + \lambda Z = \frac{\lambda}{\alpha} (Z - \eta) < 0.$$ 

Note that $\Lambda(\chi)$ is U-shape with $\Lambda(0) = \lambda Z > 0$, $\Lambda(\infty) = \infty > 0$ and $\Lambda(\chi_0) < 0$ from the above. Thus $\Lambda(\chi) = 0$ must have two distinct roots $\chi_1$ and $\chi_2$. The fact that $\Lambda(\chi)$ is U-shape implies $\Lambda(\chi) < 0$ if and only if $\chi \in (\chi_1, \chi_2)$. ■

6.4 Proof of Lemma 3

**Proof.** First, we have $\bar{k}_1 < \bar{k}_2$ from (20). Second, from (21) and (20), we have $k_1 < \bar{k}_1$ if and only if $\chi_1 < \frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}$, which is always true given Lemma 2. The same logic applies to show $\bar{k}_2 < k_2$. ■
6.5 Comparative Statics of the Wage Function under Assimilation

To study the comparative statics of technology assimilation and productivity on the wage function under assimilation $w(k)$, we first derive the effects of technology assimilation and productivity, which are captured by the parameters $\sigma$, $z$, and $\bar{z}$, on the location of the wage functions $W(k)$ and $\bar{W}(k)$. We first recall that $W(k) = (1 - \alpha)zk^\alpha$ and $\bar{W}(k) = (1 - \lambda) \bar{z}k^\alpha \left[ \lambda \left( \frac{k}{\bar{k}} \right)^{\frac{\alpha - 1}{\sigma}} + (1 - \lambda) \right]^{\frac{1}{\sigma - 1}}$. So an increase in $z$ ($\bar{z}$) then shifts up the wage function $W(k)$ ($\bar{W}(k)$). For the effect of $\sigma$, we can see that it affects only the assimilated wage function $W(k)$. We also note that, from (13), an improvement in assimilation power raises output:

$$df(k; k, \bar{k}, \bar{z}, \sigma) = f(k; \bar{k}, \bar{z}, \sigma) \left( \frac{1}{1 - \sigma} \right)^2 \left[ \pi \log \left( \frac{\pi}{\alpha} \right) + (1 - \pi) \log \left( \frac{1 - \pi}{1 - \alpha} \right) \right] > 0$$

where $\pi(k) = f'(k; \bar{k}, \bar{z}, \sigma)/f(k; \bar{k}, \bar{z}, \sigma)$ is the capital share and $\pi(k) = \alpha$.\textsuperscript{24} It is straightforward to show that

$$\bar{W}(k) = \frac{1 - \lambda}{\lambda \left( \frac{k}{\bar{k}} \right)^{\frac{\alpha - 1}{\sigma}} + (1 - \lambda)} f(k; \bar{k}, \bar{z}, \sigma),$$

so for $k < \bar{k}$, we have $\partial \bar{W}(k)/\partial \sigma > 0$ and an increase in $\sigma$ shifts up the wage function $\bar{W}(k)$.

To get the effects on the overall wage function $w(k)$, we need to study the effects on $k_j$ and $\bar{k}_j$, $j = 1, 2$. To this end, we recall the following definitions: $\chi \equiv \left[ \frac{\lambda \left( \frac{w}{(1 - \lambda)k^\alpha} \right)^{1 - \sigma}}{(1 - \lambda)k^\alpha} \right]$, $Z \equiv (\eta z/\bar{z})^{1 - \sigma} < 1$, $\eta \geq 1$ and

$$\Lambda(\chi) \equiv (1 - \lambda)Z\chi - \chi^{1 - \alpha} + \lambda Z.$$

There are two roots for $\Lambda(\chi) = 0$ under FT. Let $\chi_1 < 1$ be the smaller root of $\Lambda(\chi) = 0$. Then we obtain the following relations between $Z$ and $\chi_1$:

$$\Lambda(\chi_1) = 0 \iff \chi_1 \left[ (1 - \lambda) Z - (1 - \alpha) \chi_1^{-\alpha} \right] = \alpha \chi_1^{1 - \alpha} - \lambda Z,$$

$$\Lambda'(\chi_1) < 0 \iff (1 - \lambda) Z - (1 - \alpha) \chi_1^{-\alpha} = \alpha \chi_1^{1 - \alpha} - \lambda Z < 0.$$ (41)

These together then yield

$$\chi_1^{1 - \alpha} > \lambda Z > \alpha \chi_1^{1 - \alpha} \quad \text{and} \quad \chi_1^{-\alpha} > Z > \chi_1^{1 - \alpha}.$$

From the definition of $Z$ and (41), we have

$$\frac{dZ}{d\sigma} = -Z \ln \left( \frac{\eta z}{\bar{z}} \right) > 0,$$

\textsuperscript{24}The positivity of the square bracket on the RHS of $df(k; \bar{k}, \bar{z}, \sigma)/d\sigma$ comes from the concavity of the logarithmic function. See the algebraic details given in Klump and de la Grandville (2000).
and

\[
\frac{d\chi_1}{dZ} = -\frac{\lambda + (1 - \lambda) \chi_1}{(1 - \lambda) Z - (1 - \alpha) \chi_1^\alpha} > 0,
\]

where \( Z < 1 \) under Assumption FT. Thus, we have

\[
\frac{d\chi_1}{dz} = \frac{d\chi_1}{dZ} \frac{dZ}{dz} > 0.
\]

Recall \( k_1 \equiv \frac{1}{\lambda} k (\chi_1)^{\frac{1}{1 - \alpha}} \) and \( \bar{k}_1 \equiv \bar{k} (\chi_1)^{\frac{\sigma}{1 - \alpha}} \), then we have

\[
\frac{1}{k_1} \frac{dk_1}{d\sigma} = \frac{1}{(1 - \sigma)^2} \left[ \ln (\chi_1) + \frac{1 - \sigma}{\chi_1} \frac{d\chi_1}{d\sigma} \right] > 0
\]

and

\[
\frac{1}{k_1} \frac{d\bar{k}_1}{d\sigma} = \frac{1}{(1 - \sigma)^2} \left[ \ln (\chi_1) + \frac{1 - \sigma}{\chi_1} \sigma \frac{d\chi_1}{d\sigma} \right] > 0.
\]

Next for the comparative statics effects of productivity changes, we first note that

\[
\frac{dZ}{dz} = \frac{(1 - \sigma) Z}{z} > 0, \quad \frac{dZ}{dz} = -\frac{(1 - \sigma) Z}{z} < 0.
\]

Then we simply differentiate the definitions of \( k_1 \) and \( \bar{k}_1 \) to get (given \( \bar{k} \))

\[
\frac{1}{k_1} \frac{\partial k_1}{\partial z} = \frac{\partial \chi_1}{(1 - \sigma) \chi_1} \frac{d\chi_1}{dZ} \frac{dZ}{dz} > 0,
\]

\[
\frac{1}{k_1} \frac{\partial k_1}{\partial z} = \frac{\partial \chi_1}{(1 - \sigma) \chi_1} \frac{d\chi_1}{dZ} \frac{dZ}{dz} < 0,
\]

\[
\frac{1}{k_1} \frac{\partial \bar{k}_1}{\partial z} = \frac{\sigma}{(1 - \sigma) \chi_1} \frac{d\chi_1}{dZ} \frac{dZ}{dz} > 0,
\]

\[
\frac{1}{k_1} \frac{\partial \bar{k}_1}{\partial z} = \frac{\sigma}{(1 - \sigma) \chi_1} \frac{d\chi_1}{dZ} \frac{dZ}{dz} < 0.
\]

Likewise, the effects on \( k_2 \) and \( \bar{k}_2 \) can be derived as follows:

\[
\frac{d\chi_2}{dz} < 0, \quad \frac{d\chi_2}{d\sigma} < 0, \quad \frac{d\bar{k}_2}{d\sigma} < \frac{d\chi_2}{d\sigma} < 0, \quad \frac{\partial \bar{k}_2}{\partial \sigma} > 0, \quad \frac{\partial \bar{k}_2}{\partial \sigma} > 0.
\]

Finally, to study the effects on \( h \), we recall (22):

\[
h = \min \left\{ \max \left\{ \frac{k - k_j}{\frac{\alpha(1 - \lambda)}{\lambda(1 - \alpha)} \bar{k}_j - k_j}, 0 \right\}, 1 \right\}.
\]

Direct differentiation yields

\[
\frac{\partial h}{\partial k_1} = \frac{k - \frac{\alpha(1 - \lambda)}{\lambda(1 - \alpha)} \bar{k}_1}{\left( \frac{\alpha(1 - \lambda)}{\lambda(1 - \alpha)} \bar{k}_1 - k_1 \right)^2} < 0,
\]

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and
\[
\frac{\partial h}{\partial k_1} = - \frac{\alpha(1-\lambda)}{\lambda(1-\alpha)} \left( k - k_1 \right) < 0,
\]
for the case of \( \chi_1 < 1 \). Then we can conclude
\[
\frac{dh}{dz} = \frac{\partial h}{\partial k_1} \frac{dk_1}{dz} + \frac{\partial h}{\partial k_1} \frac{dk_1}{dz} < 0.
\]
Likewise, we have \( \frac{dh}{d\sigma} < 0, \frac{dh}{d\sigma} > 0 \).

6.6 Proof of Proposition 5

Proof. First notice that there is at most one autarky steady state, which is given by \( k = k^* \equiv [(1-\alpha)z]^{1/\alpha} \). The autarky steady state \( k = k^* \) does not exist if and only if \( k^* \in (k_1, k_2) \), which is equivalent to \( Z \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} \in (\chi_1, \chi_2) \), which is also equivalent to

\[
\Lambda \left[ Z \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} \right]^{1/(1-\alpha)} < 0.
\]

Notice that \( \Lambda \left[ Z \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} \right]^{1/(1-\alpha)} \) can be expressed as

\[
\Lambda \left[ Z \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} \right]^{1/(1-\alpha)} = Z \left( 1 - \lambda \right) \left[ Z \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} \right]^{1/(1-\alpha)} - \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} + \lambda,
\]

\[
= Z \left( 1 - \lambda \right) \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} + \lambda,
\]

\[
< Z \left( 1 - \lambda \right) \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} + \lambda,
\]

\[
= Z \Phi(\sigma).
\]

where \( \Phi(\sigma) \equiv (1 - \lambda) \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} - \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} + \lambda \). Notice \( \Phi(0) = \Phi(1) = 0 \). Also, notice that

\[
\Phi'(\sigma) \equiv \frac{d\Phi}{d\sigma} = - \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} \left[ (1 - \lambda) \left( \frac{\lambda}{\alpha} \right)^{1-\sigma} \left[ \log \left( \frac{\lambda}{\alpha} \right) + \log \left( \frac{1 - \alpha}{1 - \lambda} \right) \right] - \log \left( \frac{\lambda}{\alpha} \right) \right],
\]

which implies that there is unique \( \sigma_0 \) solving \( \Phi'(\sigma_0) = 0 \). Together with the result \( \Phi(0) = \Phi(1) = 0 \), it implies that \( \Phi(\sigma) \) is either U-shape or inverted U-shape in \( \sigma \). Finally, notice that \( \Phi'(1) \) is given by

\[
\Phi'(1) = - \left[ (1 - \lambda) \log \left( \frac{1 - \alpha}{1 - \lambda} \right) + \lambda \log \left( \frac{\lambda}{\alpha} \right) \right],
\]

\[
> - \left[ (1 - \lambda) \left( \frac{1 - \alpha}{1 - \lambda} - 1 \right) + \lambda \left( \frac{\lambda}{\alpha} - 1 \right) \right],
\]

\[
= 0.
\]
where the second inequality uses the fact that \( \log(x) < x - 1 \). Similarly we can show \( \Phi'(0) < 0 \), hence \( \Phi(\sigma) \) is U-shape and \( \Phi(\sigma) < 0 \) for all \( \sigma \in [0, 1) \). Since \( Z > 0 \), we have established that 

\[
\Lambda \left[ Z \left( \frac{1}{Z} \right)^{1-\sigma} \right]^{1/(1-\alpha)} < 0 \text{ for all } \sigma \in [0, 1).
\]

### 6.7 Proof of Lemma 4

**Proof.** Differentiate \( \log E(\sigma) \) with respect to \( \sigma \) and we have 

\[
\frac{d}{d\sigma} \log E(\sigma) = \frac{1}{\sigma} \log \left( \frac{1}{1-\alpha} \right),
\]

where \( \frac{d}{d\sigma} \log E(\sigma) \leq 0 \) if \( \sigma \leq \alpha \). So we have shown \( E(\sigma) \) is U-shape in \( \sigma \) with the minimal at \( \sigma = \alpha \) where \( E(\alpha) = \alpha/\lambda \). Finally, notice that \( \lim_{x \to 0} x^\sigma = 1 \) and rewrite \( E(\sigma) = \frac{1}{\lambda} \left[ \sigma^\sigma \left( \frac{1-\sigma}{1-\alpha} \right)^{1-\sigma} \right]^{1/\sigma} \), then we have \( \lim_{\sigma \to -0} E(\sigma) = \frac{1}{\lambda} (1-\alpha)^{1/\sigma} = \infty \) and \( \lim_{\sigma \to 1} E(\sigma) = \frac{1}{\lambda} \).

### 6.8 Proof of Lemma 5

**Proof.** The proof of part a is straight-forward and is omitted here. What remains to be shown is \( W(1/\kappa_L) \geq 1 \geq W(1/\kappa_H) \). Since \( W(k) - k \) cuts the x-axis with slopes of opposite signs at \( k = \kappa_L \) and \( k = \kappa_H \), we only need to show \( W(1/\kappa_H) \leq 1 \), while the remaining case \( W(1/\kappa_L) \geq 1 \) follows immediately. Recall first

\[
W'(\kappa_H) = \frac{\lambda}{\sigma} \left[ \frac{1 - \lambda}{1 - \alpha} \chi(\kappa_H) \right]^{1/\sigma - 1}.
\]

From the fact that \( S'[\chi(\kappa_H)] \geq 0 \), we get

\[
\left[ \frac{1 - \lambda}{1 - \alpha} \chi(\kappa_H) \right]^{1/\sigma - 1} \leq \left[ \frac{1 - \alpha}{1 - \sigma} \right]^{1/\sigma).
\]

Thus we have

\[
W'(\kappa_H) = \frac{\lambda}{\sigma} \left[ \frac{1 - \lambda}{1 - \alpha} \chi(\kappa_H) \right]^{1/\sigma - 1} \leq \frac{\lambda}{\sigma} \left[ \frac{1 - \alpha}{1 - \sigma} \right]^{1/\sigma} \leq 1,
\]

where the last inequality follows from Condition E.

### 6.9 Proof of Lemma 6

**Proof.** Part a is straight-forward and the proof omitted here. To show part b, suppose not and we have \( \chi_H \in [\chi_1, \chi_2] \) and \( Z(\frac{1}{Z})^{1-\sigma} \in \left( \left[ \frac{\lambda(1-\alpha)}{\alpha(1-\lambda)} \right]^{1-\sigma} \chi_2^{\sigma-\alpha}; \chi_2^{1-\alpha} \right) \). The premise that
\[\left[\frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}\right]^{1-\sigma} \chi_2^{\sigma-\alpha} < Z \left(\frac{\lambda}{\alpha}\right)^{1-\sigma}\] implies

\[\chi_2^{1-\alpha} < Z \left(\frac{1-\lambda}{1-\alpha} \chi_2\right)^{1-\sigma},\]

\[\Rightarrow \chi_2^{1-\alpha} < (1-\lambda)Z\chi_2 + \lambda Z,\]

\[\Rightarrow \chi_2^{1-\alpha} < \chi_2^{1-\alpha},\]

which is contradiction. The second line follows the premise that \(\chi_2 \geq \chi_H\) implies \(S(\chi_2) \geq 0\) and hence we have \(\left(\frac{1-\lambda}{1-\alpha} \chi_2\right)^{1-\sigma} \leq (1-\lambda) \chi_2 + \lambda\). The last line follows the definition of \(\chi_2\) from \(\Lambda(\chi_2) = 0\) and hence we have \((1-\lambda)Z\chi_2 + \lambda Z = \chi_2^{1-\alpha}\).

To show part c, notice that if Condition E does not hold, then we have \(S(\chi) > 0\) for any \(\chi\). Suppose \(Z \left(\frac{\lambda}{\alpha}\right)^{1-\sigma} \in \left[\left(\frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}\right)^{1-\sigma}, \chi_2^{1-\alpha}\right]\) is satisfied, then we must have \(\chi_2^{1-\alpha} < Z \left(\frac{1-\lambda}{1-\alpha} \chi_2\right)^{1-\sigma}\) as in the proof of part b. But the fact that \(S(\chi_2) > 0\) implies \(\left(\frac{1-\lambda}{1-\alpha} \chi_2\right)^{1-\sigma} < (1-\lambda) \chi_2 + \lambda\). Hence following the same proof of part b we can establish contradiction.

To show part d, notice that since \(S(\chi)\) is U-shape with \(S(\chi_L) = S(\chi_H) = 0\), either the premise \(\chi_2 < \chi_L\) or the premise \(\chi_2 > \chi_1 \geq \chi_H\) implies \(S(\chi_2) > 0\). Hence following the same proof of part c we can establish part d.

To show part e, notice that since \(S(\chi)\) is U-shape with \(S(\chi_L) = S(\chi_H) = 0\), the premise that \(\chi_L < \chi_1 < \chi_H\) implies \(S(\chi_1) < 0\). Suppose we have \(Z \left(\frac{\lambda}{\alpha}\right)^{1-\sigma} \in \left(\chi_1^{1-\alpha}, \left[\frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}\right]^{1-\sigma} \chi_1^{\sigma-\alpha}\right)\).

The premise that \(\left[\frac{\lambda(1-\alpha)}{\alpha(1-\lambda)}\right]^{1-\sigma} \chi_1^{\sigma-\alpha} > Z \left(\frac{\lambda}{\alpha}\right)^{1-\sigma}\) implies

\[\chi_1^{1-\alpha} > Z \left(\frac{1-\lambda}{1-\alpha} \chi_1\right)^{1-\sigma},\]

\[\Rightarrow \chi_1^{1-\alpha} > (1-\lambda)Z\chi_1 + \lambda Z,\]

\[\Rightarrow \chi_1^{1-\alpha} > \chi_1^{1-\alpha},\]

which is contradiction. The second line follows the definition of \(\chi_L\) from \(S(\chi_1) < 0\) and hence we have \(\left(\frac{1-\lambda}{1-\alpha} \chi_1\right)^{1-\sigma} > (1-\lambda) \chi_1 + \lambda\). The last line follows the fact that \(\Lambda(\chi_1) = 0\) and hence we have \((1-\lambda)Z\chi_1 + \lambda Z = \chi_1^{1-\alpha}\) ■

6.10 Proof of Proposition 6

**Proof.** To show part a, notice that Proposition 5 rules out the existence of the autarky steady state. So steady states can not occur in the first and the last region of the equilibrium wage function (23) such that \(k = w(k)\). Also from Lemma 5 we know there are at most two full-assimilation steady states which occur in the third region of the equilibrium wage function (23). From Lemma 6 we know there are at most two partial-assimilation steady states, one
in the second region and another in the forth region of the equilibrium wage function (23).

But Lemma 6 also implies that the later one cannot coexist with one of the full-assimilation steady state. So at most we have three steady states, \( k = W (k_1), k = k_L \) and either \( k = k_H \) or \( k = W (k_2) \), in the order of capital level. Notice that \( w' (0) > 1 \) and \( w (\infty) < 1 \), so if it is the case of three steady states, then the lowest steady state \( k = W (k_1) \) and the highest steady state, either \( k = k_H \) or \( k = W (k_2) \), must be stable. Since the stability of steady state must be alternating, so it is equivalent to that there are three steady states if and only if there exists a unstable middle steady state \( k = k_L \). By using Lemma 5, it is the case if and only if \( \chi_L \in (\chi_1, \chi_2) \).

To show part b, if Condition E is not satisfied or \( \chi_L > \chi_2 \) or \( \chi_H \leq \chi_1 \), then neither \( \chi_L \) nor \( \chi_H \) can be the steady-level of capital. By Lemma 6 the only steady state is the one with \( k = W (k_1) \). The steady state always exists since \( w' (0) > 1 \) and \( w (k) < k \) for sufficiently large \( k \).

Similarly, to show part c, if \( \chi_L < \chi_1 < \chi_H \), then \( \chi_L \) cannot be the steady-level of capital. So there can not be three steady states. By Lemma 6 the partial-assimilation steady state with \( k = W (k_1) \) is also ruled out. So it could only be either the full-assimilation steady state with capital \( k = \kappa_H \), or the partial-assimilation steady state with capital \( k = W (k_2) \). By Lemma 5 the former is the case if and only if \( \chi_H \in (\chi_1, \chi_2) \).

The remaining bifurcation cases of part d and part e are straight-forward and are omitted here. ■

6.11 Proof of Proposition 7

**Proof.** First we want to show \( B (\sigma) \equiv \chi_L (\sigma) \frac{\sigma - \alpha}{1 - \alpha} \) is decreasing when \( \sigma > \alpha \). Notice that given \( \sigma > \alpha \chi_L (\sigma) < 1 \) is the smallest root solving \( S (\chi_L) = 0 \) with \( S' (\chi_L) < 0 \). Then we have \( B (\sigma) \equiv \chi_L (\sigma) \frac{\sigma - \alpha}{1 - \alpha} < 1 \frac{\sigma - \alpha}{1 - \alpha} = 1 \). Also, total differentiating \( S \) we have

\[
\frac{d\chi_L (\sigma)}{d\sigma} = -\frac{\chi_L^{1-\sigma} \log \chi_L}{S' (\chi_L)} < 0.
\]

Thus we establish:

\[
\frac{dB (\sigma)}{d\sigma} = \frac{\sigma - \alpha}{1 - \alpha} \chi_L^{\frac{\sigma - \alpha}{1 - \alpha}} \frac{d\chi_L (\sigma)}{d\sigma} + \frac{1 - \alpha}{(1 - \sigma)^{\frac{\sigma - \alpha}{1 - \alpha}}} \chi_L^{\frac{\sigma - \alpha}{1 - \alpha}} \log \chi_L < 0
\]

which implies \( B (\sigma) \) is strictly decreasing. Next, notice that \( \lim_{\sigma \to 1} \chi_L (\sigma) = 0 \) and hence \( \lim_{\sigma \to 1} B (\sigma) = 0^\infty = 0 \).

What remains to show is to verify that under \( \sigma > \alpha \) we have \( z < \varepsilon B (\sigma) \) satisfied if and only if \( \chi_L \in (\chi_1, \chi_2) \). But notice that \( \chi_2 > 1 \) and \( \chi_L < 1 \), so the condition \( \chi_L \in (\chi_1, \chi_2) \) is
reduced to $\chi_L < \chi_1$, which is equivalent to $z < \bar{z}B(\sigma)$ under Assumption BM. So part a to c is the corollary to Proposition 6. Combining the result of Corollary 2 we establish Proposition 7.

6.12 Proof of Proposition 8

**Proof.** Notice that if $\sigma \leq \alpha$ then the definition of $S(\chi)$ implies the smallest root to $S(\chi) = 0$ is $\chi_L(\sigma) = 1$, hence we have $C(\sigma) = B(\sigma) = 1$. If $\sigma > \alpha$ then since $\frac{1-\alpha}{1-\sigma} > \frac{\sigma}{1-\sigma}$ and $\chi_L(\sigma) < 1$, then we have $C(\sigma) = \chi_L(\sigma)^{\frac{1-\alpha}{1-\sigma}} < \chi_L(\sigma)^{\frac{\sigma}{1-\sigma}} = B(\sigma)$. Finally, applying the similar proof to Proposition 7 we can show $C(\sigma) \leq 1$ is strictly decreasing in $\sigma$ with $\lim_{\sigma \to 1} C(\sigma) = 0$.

Notice that given the result $C(\sigma) \leq B(\sigma)$ the condition $z < \bar{z}C(\sigma)$ implies $z < \bar{z}B(\sigma)$. So there are three steady state if $z < \bar{z}C(\sigma)$ is satisfied. Then it is straight-forward to show that $k^* < \hat{k}$ is the necessary and sufficient condition for an autarky economy will converge to the low partial-assimilation steady state with $k = W(k_1)$, where the condition $k^* < \hat{k}$ is equivalent to $z < \bar{z}C(\sigma)$ under Assumption BM.

6.13 Proof of Lemma 7

**Proof.** Suppose we have $\bar{W}(\bar{k}_2) \leq \bar{k}_2$. By combining with the fact that $\bar{W}(\bar{k}) > \bar{k}$ given $S(1) < 0$ under the premise $\alpha > \lambda$, there must exists some $k' \in (\bar{k}, \bar{k}_2]$ such that $\bar{W}(k') = k'$. But it means that $k'$ is a full-assimilation steady state with $k' = \bar{k}_H$ (it cannot be the case that $k' = \bar{k}_L$ since $k' > \bar{k} > \bar{k}_L$) and hence contradicts to the premise that $\chi_H \geq \chi_2$. So we must have $\bar{W}(\bar{k}_2) > \bar{k}_2$.

Finally, given the fact that $\bar{W}(\bar{k}_2) \geq \bar{k}_2$ and $w(k) < k$ for sufficient large $k$, then there must exist a steady state with capital $k$ such that $k = w(k) > \bar{W}(\bar{k}_2)$.  ■

6.14 Proof of Proposition 11

**Proof.** Given $\alpha > \lambda$ and $z \leq \eta^{-1}\bar{z}$, suppose $\chi_H \in (\chi_1, \chi_2)$, then $\bar{k}_H > \bar{k}$ is the full-assimilation steady-state level of capital, so the part a of Proposition 11 is shown. Suppose $\chi_H \geq \chi_2$. The premise $z \leq \eta^{-1}\bar{z}$ implies $Z \leq 1$, then we have

$$\Lambda(1) = Z - 1 \leq 0.$$ 

Thus we have $\chi_2 \geq 1$. Notice that $\bar{W}(\bar{k}_2)$ is given by

$$\bar{W}(\bar{k}_2) = \bar{k} [\alpha \chi_2^{-1} + 1 - \alpha]^{-1/\sigma} \geq \bar{k}.$$

Thus by using Lemma 7, we have shown the part a of Proposition 11 under the case $\chi_H \geq \chi_2$. The remaining case $\chi_H \leq \chi_1$ does not exists since we have $\chi_H > 1$ and $\chi_1 < \chi_2 \leq 1$. 

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On the other hand, suppose $\alpha \leq \lambda$ and $\sigma \geq \alpha$. Then we have $S(1) = 1 - \left(\frac{\lambda}{\lambda - \alpha}\right)^{1-\sigma} \geq 0$ and $S'(1) = (\sigma - \alpha) \left(\frac{\lambda}{\lambda - \alpha}\right) \geq 0$. Since $S(\chi)$ is U-shape in $\chi$, it implies $\chi_H \leq 1$. The definition of $\chi$ implies $k_H \leq k$. Proposition (7) implies that the highest level of steady-state capital is the one of full-assimilation, i.e. $k_H$. So we have shown the part b of Proposition 11 under the case $\alpha \leq \lambda$ and $\sigma \geq \alpha$. Suppose $\alpha > \lambda$ and $z > \eta^{-1}z$, then we have $Z > 1$, $\Lambda'(1) = Z - 1 > 0$ and $\Lambda'(1) = (1 - \lambda)(Z - 1) + (\alpha - \lambda) > 0$. Since $\Lambda(\chi)$ is U-shape in $\chi$, it implies $\chi_2 < 1$ and $W(k_2) = k \left[\alpha \chi_2^{-1} + 1 - \alpha\right]^{-1/\sigma} < k$. Proposition (7) implies that the range of $w(k)$ relevant for the steady state is never greater than $W(k_2)$. So we have shown the part b of Proposition 11 under the remaining case. ☐