Long Term Risk

With Lars Peter Hansen

Motivation

- Evaluation of economic models of preferences and technologies using asset prices.
  - Market microstructure, behavioral biases, transaction costs... may make it difficult to evaluate models using short run data.
  - Economics more revealing for modeling long-run phenomena.
- Interest in full term-structure of risk-prices, but here focus on long run (slope).
- How risk averse agents value risks of permanent shocks
  - Long run risk-return frontier
- Complementary to work on long run risk using short run data (Bansal-Yaron...)

- Use Markov formulation and martingale methods to produce factorizations of model implications.
  - Separate effect of altering the standard model into transitory and permanent components.
  - Hansen, Heaton and Li (log-normal environments with linear state dynamics and constant volatility)

Stochastic discount factor

- X_t a Markov process, in a space $(\Omega, \mathcal{F}, P)$, $\mathcal{F}_t$ the associated (completed) filtration.
- A *Stochastic Discount Factor* S is a strictly positive adapted process such that
$$E\left[S_t \Pi_t | \mathcal{F}_0\right] \tag{1}$$
is the price at time 0 of a claim to the payoff Π_t at t.
  - Arrow-Debreu prices
- $$\mathbb{S}_t \psi(x) = E\left[S_t \psi(X_t) | X_0 = x\right],$$
is the time-zero price of payoff $\psi(X_t)$.
- Frictionless trading, law of one price $\Rightarrow$
- $\mathbb{S}_0 = \mathbb{I}$ and $\mathbb{S}_{t+u} = \mathbb{S}_t \mathbb{S}_u$
- **Semigroup property**

- θ_t the *shift operator*:

$$(\theta_t X)_u = X_{t+u}.$$

- Since S_u only depends on the history of the Markov process X between dates 0 and u, $S_u(\theta_t)$ only depends on the history of X between dates t and $t+u$.
- Frictionless trading, law of one price $\Rightarrow$
- $S_0 = 1$ and $S_{t+u} = S_t S_u(\theta_t)$.
- S_t is a **multiplicative functional.**

Example

- $$\begin{aligned} dX_t^f &= \xi_f(\bar{x}_f - X_t^f)dt + \sqrt{X_t^f}\sigma_f dB_t^f, \\ dX_t^o &= \xi_o(\bar{x}_o - X_t^o)dt + \sigma_o dB_t^o \end{aligned}$$

  with $\xi_i > 0$, $\bar{x}_i > 0$ for $i = f, o$ and $2\xi_f \bar{x}_f \geq \sigma_f^2$ where $B = (B^f, B^o)$ is a bivariate standard Brownian motion.

- Per-capita consumption

  $$dc_t = X_t^o dt + \sqrt{X_t^f}\vartheta_f dB_t^f + \vartheta_o dB_t^o$$

  where $c_t = \log(C_t)$

- Interesting case:
  - $\sigma_o > 0$, $\vartheta_0 \geq 0$ (positive B^o's are unambiguously good)
  - $\sigma_f < 0$, $\vartheta_f \geq 0$ (positive B^f's are unambiguously good)

Breeden-Lucas model

- Representative investor preferences are given by:

$$E \int_0^\infty \exp(-\mathsf{b}t) \frac{{C_t}^{1-\mathsf{a}} - 1}{1 - \mathsf{a}}$$

for a and b strictly positive.

- With additive utility $S_t = \frac{e^{-\mathsf{b}t} u'(C_t)}{u'(C_0)}$
- The stochastic discount factor in the Breeden-Lucas model is $S_t = \exp(A_t^s)$ where

$$A_t^s = -\mathsf{a} \int_0^t X_s^o ds - \mathsf{b}t - \mathsf{a} \int_0^t \sqrt{X_s^f} \vartheta_f dB_s^f - \mathsf{a} \int_0^t \vartheta_o dB_s^o.$$

Kreps-Porteus with unitary elasticity of intertemporal substitution

- Utility aggregator satisfies:

$$f(C,W) = -bW\{(a-1)\log C + \log[(1-a)W]\}$$

.

- Drift of W_t equals $f(C_t, W_t)$
  - variance multiplier =0 (Duffie-Lions)
- Guess a continuation value process of the form:

$$W_t = \frac{1}{1-\mathsf{a}} \exp\left[(1-\mathsf{a})(\mathsf{w}_f X_t^f + \mathsf{w}_o X_t^o + \log C_t + \bar{\mathsf{w}})\right]$$

- The coefficients satisfy:

$$\begin{aligned}
-\xi_f \mathsf{w}_f + \frac{(1-\mathsf{a})\sigma_f^2}{2}(\mathsf{w}_f)^2 + (1-\mathsf{a})\vartheta_f\sigma_f\mathsf{w}_f + \frac{(1-\mathsf{a})\vartheta_f^2}{2} &= \mathsf{b}\mathsf{w}_f \\
-\xi_o\mathsf{w}_o + 1 &= \mathsf{b}\mathsf{w}_o \\
\xi_f\bar{x}_f\mathsf{w}_f + \xi_o\bar{x}_o\mathsf{w}_o + \frac{(1-\mathsf{a})\sigma_o^2}{2}(\mathsf{w}_o)^2 + (1-\mathsf{a})\vartheta_o\sigma_o\mathsf{w}_o + \frac{(1-\mathsf{a})\vartheta_o^2}{2} &= \mathsf{b}\bar{\mathsf{w}}.
\end{aligned}$$

- $\mathsf{w}_0 > 0$, w_f real if either $\vartheta_f \sigma_f > 0$ or $\vartheta_f \sigma_f < 0$ and $\xi_f + \mathsf{b} \geq 2(\mathsf{a} - 1)|\vartheta_f \sigma_f|$. In either case $\mathsf{w}_f < 0$ and we are interested in root with smallest absolute value.

- From Duffie-Epstein

$$S_t = \frac{e^{[\int_0^t f_W(C_s, W_s) ds]} f_C(C_t, W_t)}{f_C(C_0, W_0)}$$

- The stochastic discount factor is the product of two functionals. One is the exponential of:

$$A_t^s = -\int_0^t X_s^o ds - \mathsf{b}t - \int_0^t \sqrt{X_s^f} \vartheta_f dB_s^f - \int_0^t \vartheta_o dB_s^o.$$

The other is a martingale that is the exponential of:

$$\begin{aligned} A_t^w &= (1-\mathsf{a}) \left[\int_0^t \sqrt{X_s^f} (\vartheta_f + \mathsf{w}_f \sigma_f) dB_s^f + \int_0^t (\vartheta_o + \mathsf{w}_o \sigma_o) dB_s^o \right] \\ &- \frac{(1-\mathsf{a})^2}{2} \int_0^t X_s^f (\vartheta_f + \mathsf{w}_f \sigma_f)^2 ds - \frac{(1-\mathsf{a})^2 (\vartheta_o + \mathsf{w}_o \sigma_o)^2}{2} t \end{aligned}$$

The basic Markov process

- $\{X_t : t \geq 0\}$ be a continuous time Markov process on a state space $D_0 \subseteq \mathbb{R}^n$.
- An $\mathcal{F}_t$ n-dimensional Brownian motion $\{B_t\}$
- $$X_t = X_0 + \int_0^t \xi(X_u)du + \int_0^t \sigma(X_u)dB_u \tag{2}$$
- $(\xi, \Sigma), \quad \Sigma = \sigma\sigma'$
- More generally X may also display jumps.
  - $X = X^c + X^j$
  - X^c as in (2)
  - X^j characterized by a finite conditional measure $\eta(dx|X_{t-})$

Building processes that grow or decay

- A real-valued process $\{M_t : t \geq 0\}$ adapted, (with a version that is) right continuous with left limits. (**A functional**)
- The functional $\{M_t : t \geq 0\}$ is **multiplicative** if $M_0 = 1$, and $M_{t+u} = M_u(\theta_t)M_t$. Here θ is the shift operator.
- Product of multiplicative processes is multiplicative.
- If M strictly positive $\log(M)$ will satisfy an additive property.
- A functional is **additive** if $A_0 = 0$ and $A_{t+u} = A_u(\theta_t) + A_t$, for each nonnegative t and u.
- Parameterize $\log(M)$

Parameterization of multiplicative processes

- (β, γ) that satisfies:

  a) $\beta : D_0 \to \mathbb{R}$ and $\int_0^t \beta(X_u)du < \infty$ for every positive t;

  b) $\gamma : D_0 \to \mathbb{R}^m$ and $\int_0^t |\gamma(X_u)|^2 du < \infty$ for every positive t;

$$A_t = \int_0^t \beta(X_u)du + \int_0^t \gamma(X_u) \cdot dB_u$$

- Process A may be non-stationary even if X is stationary.
- $A_t = \psi(X_t) - \psi(X_0)$
- Exponential of additive processes (strictly positive multiplicative functionals).
  - Parameterized by the additive process (β, γ)
- When jumps are present add function of jumps

Growth in payouts

- G a growth process: G adapted, $G_0 = 1$ and $G_{t+u} = G_t G_u(\theta_t)$
- $M = SG$ is also multiplicative
- $$\mathbb{M}_t \psi(x) = E\left[D_0 M_t \psi(X_t) | X_0 = x\right],$$
  is the time-zero price of payoff $D_0 G \psi(X_t)$.
- Multiplicative $M \Rightarrow \mathbb{M}_0 = \mathbb{I}$ and $\mathbb{M}_{t+u} = \mathbb{M}_t \mathbb{M}_u$

Objective: Multiplicative decomposition

- Establish decomposition for multiplicative functionals:

$$M_t = \exp(\rho t)\hat{M}_t \left[\frac{\varphi(X_0)}{\varphi(X_t)}\right]$$

  where

  - ρ is a deterministic growth rate;
  - $\hat{M}_t$ is a multiplicative martingale;
  - φ is a strictly positive function of the Markov state;

- If X is stationary, $\frac{\varphi(X_0)}{\varphi(X_t)}$ stationary component, $\hat{M}$ the martingale component of M, and ρ its growth rate.
  - Not entirely correct because of possible correlation between stationary and martingale components.

Implications of multiplicative decomposition

- If $\hat{M}$ is a martingale for $F \in \mathcal{F}_t$

$$\hat{P}r(F) = E[\hat{M}_t \mathbf{1}_F]$$

- X remains Markovian.

- $$E\left[M_t \psi(X_t) | X_0 = x\right] = \exp(\rho t)\phi(x)\hat{E}\left[\frac{\psi(X_t)}{\phi(X_t)} | X_0 = x\right]$$

- $\exp(-\rho t)\phi(X_t)$ as a *numeraire*. Applicable when the multiplicative process does not define a price.

- If there exists a stationary distribution $\hat{\varsigma}$ for X under $\hat{P}r$, such that "stochastic stability" holds

$$\lim_{t\to\infty} E_{\hat{P}r}\left[\psi(X_t)|X_0 = x\right] = \int \psi(y)d\hat{\varsigma},$$

whenever $\int \psi(X_t)d\hat{\varsigma}$ is well defined, then

$$\begin{aligned}\lim_{t\to\infty} e^{-\rho t} E[M_t\psi(X_t) \quad &| \quad X_0 = x] \\ &= \lim_{t\to\infty} E\left(\hat{M}_t\left[\frac{\psi(X_t)}{\phi(X_t)}\right]|X_0 = x\right)\phi(x) \\ &= (\int \frac{\psi(y)}{\phi(y)}d\hat{\varsigma})\phi(x)\end{aligned}$$

  - $\rho = \rho(M)$ is the (deterministic) growth rate
  - All state dependence is given by the eigenfunction ϕ
    - 1 factor model
  - Approximation is "good" for ψ with $\left|\int \frac{\psi(y)}{\phi(y)}d\hat{\varsigma}\right| < \infty$

- $$\frac{1}{t}\log E\left[M_t\psi(X_t)|X_0 = x\right] - \rho \approx \frac{1}{t}\left[\log\phi(x) + \log\int\frac{\psi(y)}{\phi(y)}d\hat{\varsigma}\right]$$

- Since $\rho(M)$ is independent of ψ, one may think of ψ as the transient contribution.

- If $M = SG$, $\rho(G) - \rho(GS)$ is long-run expected rate of return

- Using $G = 1$ as risk-free reference, $\rho(G) + \rho(S) - \rho(GS)$ is long-run expected excess rate of return.

- Mapping "risk in G" $\hookrightarrow \rho(G) - \rho(SG)$ is a long-run risk-return frontier

- Martingale cash flows.

Remainder of lecture

- Strategy to establish decomposition
  - Perron-Frobenius
- Applications
- In paper
  - Sufficient conditions stationarity, recurrence
  - Existence
  - Uniqueness

Establishing the decomposition

- Find positive solution to "eigenvalue problem"

$$E[M_t \phi(X_t) | X_0 = x] = \exp(\rho t)\phi(x)$$

- $\exp(-\rho t) M_t \phi(X_t)$ is a martingale
- $\hat{M}_t := \exp(-\rho t) M_t \frac{\phi(X_t)}{\phi(X_0)}$ is a martingale with $M_0 = 1$
- $M_t = \exp(\rho t) \hat{M}_t \frac{\phi(X_0)}{\phi(X_t)}$

A local equation

- Find ρ, ϕ such that for each x

$$\lim_{t \searrow 0} \frac{E[M_t \phi(X_t) - \phi(x) | X_0 = x]}{t} = \rho \phi(x)$$

- Ito's lemma allows you to compute $\lim_{t \searrow 0} \frac{E[M_t \phi(X_t) - \phi(x) | X_0 = x]}{t}$
  - X_t real valued and $dX_t = \xi(X_t)dt + \sigma(X_t)dB_t$, B a Brownian
  - $M_t = \exp(A_t)$, $A_t = \int_0^t \beta(X_u)du + \int_0^t \gamma(X_u)dB_u$
  - For ψ "sufficiently regular" $\lim_{t \searrow 0} \frac{E[M_t \psi(X_t) - \psi(x) | X_0 = x]}{t}$ $= [\beta(x) + \frac{\gamma^2(x)}{2}]\psi(x) + [\xi(x) + \sigma(x)\gamma(x)]\psi'(x) + \frac{\sigma^2}{2}\psi''(x)$

- Find positive ϕ such that:
  $\rho\phi(x) = [\beta(x) + \frac{\gamma^2(x)}{2}]\phi(x) + [\xi(x) + \sigma(x)\gamma(x)]\phi'(x) + \frac{\sigma^2}{2}\phi''(x)$
- Multidimensional version also including jumps
- Non-uniqueness of solutions, but there is at most one pair (ρ, ϕ) such that X stochastically stable after change in measure.
- ρ is smallest among candidate eigenvalue
- Connection to numerical analysis literature.

Recipe for parameterized examples

- Use Ito's to verify guess of eigenfunction.
- Compute eigenvalues.
- Typically multiple candidates, choose smallest.
- Write down candidate $\hat{M}$ and derive dynamics for X under new probability (Girsanov's theory)
- Verify stochastic stability

Application: 1. Long run risk-return frontier

- $$\begin{aligned} dX_t^f &= \xi_f(\bar{x}_f - X_t^f)dt + \sqrt{X_t^f}\sigma_f dB_t^f, \\ dX_t^o &= \xi_o(\bar{x}_o - X_t^o)dt + \sigma_o dB_t^o \end{aligned}$$

  with $\xi_i > 0$, $\bar{x}_i > 0$ for $i = f, o$ and $2\xi_f\bar{x}_f \geq \sigma_f^2$ where $B = (B^f, B^o)$ is a bivariate standard Brownian motion.

- Per-capita consumption

  $$dc_t = X_t^o dt + \sqrt{X_t^f}\vartheta_f dB_t^f + \vartheta_o dB_t^o$$

  where $c_t = \log(C_t)$

- Choose a growth process for payouts $G = \exp(A^g)$ with

$$A_t^g = \int_0^t \sqrt{X_s^f}\gamma_f^g dB_s^f + \int_0^t \gamma_o^g dB_s^o - \int_0^t \frac{X_s^f(\gamma_f^g)^2 + (\gamma_o^g)^2}{2} ds.$$

- γ_f^g parameterizes B^f risk of cash flow, γ_o^g parameterizes B^o risk.
- $M = GS$, S as in Breeden-Lucas or Kreps-Porteous with $\rho = 1$
- Guess eigenfunction $\exp(\mathsf{c}_f x_f + \mathsf{c}_o x_o)$.
- Quadratic equation hence two candidates, the one associated with smallest eigenvalue $\rho(M)$ induces stochastic stability.
- (Long run) risk-return frontier: mapping

$$(\gamma_f^g, \gamma_o^g) \hookrightarrow -\rho$$

- Short-run risk prices: Derivative of required instantaneous expected rate of return with respect to risk exposure.
- In Breeden-Lucas the short run risk-price to the cash flow exposure to the B^o risk is: $\mathsf{a}\vartheta_\circ$
  - non-positive, unless consumption growth is positive correlated with the changes in the rate of growth of consumption.
- The *long run risk prices* are the derivatives of ρ with respect to risk-exposures.
- In Breeden-Lucas the long run risk price of exposure to B^o is:

$$\mathsf{a}\vartheta_\circ + \frac{\mathsf{a}}{\xi_\circ}\sigma_\circ$$

- Larger than short-run price. Difference between long-run and short-run prices is largest the slower the state variable X^o mean reverts and the more this state variable is sensitive to B^o.

- Short-run risk-price in Kreps-Porteus (with unitary elasticity of intertemporal substitution) is given by:

$$\mathsf{a}\vartheta_\circ + (\mathsf{a} - 1)\frac{\sigma_\circ}{\mathsf{b} + \xi_\circ} > \mathsf{a}\vartheta_\circ$$

- Long-run risk-price in K-P:

$$\mathsf{a}\vartheta_\circ + \frac{\mathsf{a}}{\mathsf{b} + \xi_\circ}\sigma_\circ$$

- K-P has flatter term structure of risk prices than B-L.
- Cash flow risk exposure to B^f feeds through eigenfunction. Because volatility is state dependent, there is nonlinearity in the long-run price of the volatility risk.

Application 2: Decompositions of Stochastic Discount Factor

- $$S_t = \exp(\rho t) \hat{M}_t \frac{\phi(X_0)}{\phi(X_t)} \tag{3}$$
- Alvarez and Jermann (2005) use a decomposition as (3) and prices of equity and risk-free long bonds to show that most of the volatility of the stochastic discount factor comes from the permanent component.
  - Suggest utility functions need to magnify importance of permanent component of consumption to fit data.

Application 3: Habit persistence models

- Bansall and Lehman show that for a variety of asset-pricing models the stochastic discount factor S^* can be written as

$$S^* = S_t \frac{f(X_t)}{f(X_0)},$$

where S is the stochastic discount factor of a Breeden-Lucas model

  – Models with social externalities such as *e.g.* Campbell and Cochrane (1999) and Santos and Veronesi (2006)

- Martingale component same as Breeden-Lucas

Summary

- Use Markov formulation and martingale methods to produce factorizations of pricing operators.
- Multiplicative decomposition allows for
  - Characterization of long run risk-return tradeoff.
  - Distinguishing transient versus permanent changes in model ingredients