Long Term Risk

With Lars Peter Hansen
Motivation

• Evaluation of economic models of preferences and technologies using asset prices.
  – Market microstructure, behavioral biases, transaction costs... may make it difficult to evaluate models using short run data.
  – Economics more revealing for modeling long-run phenomena.

• Interest in full term-structure of risk-prices, but here focus on long run (slope).

• How risk averse agents value risks of permanent shocks
  – Long run risk-return frontier

• Complementary to work on long run risk using short run data (Bansal-Yaron...)
• Use Markov formulation and martingale methods to produce factorizations of model implications.
  – Separate effect of altering the standard model into transitory and permanent components.
  – Hansen, Heaton and Li (log-normal environments with linear state dynamics and constant volatility)
Stochastic discount factor

- $X_t$ a Markov process, in a space $(\Omega, \mathcal{F}, P)$, $\mathcal{F}_t$ the associated (completed) filtration.

- A *Stochastic Discount Factor* $S$ is a strictly positive adapted process such that
  \[ E[S_t \Pi_t | \mathcal{F}_0] \]  
  is the price at time 0 of a claim to the payoff $\Pi_t$ at $t$.

  - Arrow-Debreu prices

- Frictionless trading, law of one price ⇒
  - $S_0 = 1$ and $S_{t+u} = S_t S_u$

- Semigroup property
• $\theta_t$ the shift operator:

$$(\theta_t X)_u = X_{t+u}.$$ 

• Since $S_u$ only depends on the history of the Markov process $X$ between dates 0 and $u$, $S_u(\theta_t)$ only depends on the history of $X$ between dates $t$ and $t+u$.

• Frictionless trading, law of one price $\Rightarrow$

• $S_0 = 1$ and $S_{t+u} = S_t S_u(\theta_t)$.

• $S_t$ is a **multiplicative functional**.
Example

- 

\[
\begin{align*}
    dX^f_t &= \xi_f (\bar{x}^f_f - X^f_t) dt + \sqrt{X^f_t} \sigma_f dB^f_t, \\
    dX^o_t &= \xi_o (\bar{x}^o_o - X^o_t) dt + \sigma_o dB^o_t
\end{align*}
\]

with \(\xi_i > 0, \bar{x}_i > 0\) for \(i = f, o\) and \(2\xi_f \bar{x}_f \geq \sigma^2_f\) where \(B = (B^f, B^o)\) is a bivariate standard Brownian motion.

- Per-capita consumption

\[
dc_t = X^o_t dt + \sqrt{X^f_t} \vartheta_f dB^f_t + \vartheta_o dB^o_t
\]

where \(c_t = \log(C_t)\)

- Interesting case:
  - \(\sigma_o > 0, \vartheta_0 \geq 0\) (positive \(B^o\)'s are unambiguously good)
  - \(\sigma_f < 0, \vartheta_f \geq 0\) (positive \(B^f\)'s are unambiguously good)
Breeden-Lucas model

• Representative investor preferences are given by:

\[ E \int_0^{\infty} \exp(-bt) \frac{C_t^{1-a} - 1}{1 - a} \]

for \( a \) and \( b \) strictly positive.

• With additive utility \( S_t = \frac{e^{-bt}u'(C_t)}{u'(C_0)} \)

• The stochastic discount factor in the Breeden-Lucas model is \( S_t = \exp(A_t^s) \) where

\[ A_t^s = -a \int_0^t X_s^o ds - bt - a \int_0^t \sqrt{X_s^f} \vartheta_f dB_s^f - a \int_0^t \vartheta_o dB_s^o. \]
Kreps-Porteus with unitary elasticity of intertemporal substitution

- Utility aggregator satisfies:

\[ f(C, W) = -bW \{(a - 1) \log C + \log[(1 - a)W]\} \]

- Drift of \( W_t \) equals \( f(C_t, W_t) \)
  - variance multiplier =0 (Duffie-Lions)

- Guess a continuation value process of the form:

\[ W_t = \frac{1}{1 - a} \exp \left[(1 - a)(w_f X_t^f + w_o X_t^o + \log C_t + \bar{w})\right] \]

- The coefficients satisfy:

\[
\begin{align*}
-\xi_f w_f f + \frac{(1 - a)\sigma_f^2}{2}(w_f)^2 + (1 - a)\vartheta_f \sigma_f w_f + \frac{(1 - a)\vartheta_f^2}{2} &= bw_f \\
-\xi_o w_o + \frac{1}{(1 - a)\vartheta_o^2} &= bw_o \\
\xi_f w_f f + \xi_o w_o + \frac{(1 - a)\sigma_o^2}{2}(w_o)^2 + (1 - a)\vartheta_o w_o + \frac{(1 - a)\vartheta_o^2}{2} &= bw. 
\end{align*}
\]
• \( w_0 > 0, \ w_f \) real if either \( \vartheta_f \sigma_f > 0 \) or \( \vartheta_f \sigma_f < 0 \) and \\
\( \xi_f + b \geq 2(a - 1)|\vartheta_f \sigma_f| \). In either case \( w_f < 0 \) and we are \\
interested in root with smallest absolute value.

• From Duffie-Epstein

\[
S_t = \frac{e^{\int_0^t f_W(C_s,W_s)ds} f_C(C_t,W_t)}{f_C(C_0,W_0)}
\]

• The stochastic discount factor is the product of two 
functionals. One is the exponential of:

\[
A_s^t = - \int_0^t X_s^o ds - bt - \int_0^t \sqrt{X_s^f \vartheta_f} dB_s^f - \int_0^t \vartheta_o dB_s^o.
\]

The other is a martingale that is the exponential of:

\[
A_w^t = (1 - a) \left[ \int_0^t \sqrt{X_s^f (\vartheta_f + w_f \sigma_f)} dB_s^f + \int_0^t (\vartheta_o + w_o \sigma_o) dB_s^o \right] \\
- \frac{(1 - a)^2}{2} \int_0^t X_s^f (\vartheta_f + w_f \sigma_f)^2 ds - \frac{(1 - a)^2(\vartheta_o + w_o \sigma_o)^2}{2} t
\]
The basic Markov process

- \{X_t : t \geq 0\} be a continuous time Markov process on a state space \(D_0 \subseteq \mathbb{R}^n\).
- An \(\mathcal{F}_t\) n-dimensional Brownian motion \(\{B_t\}\)
- \[
X_t = X_0 + \int_0^t \xi(X_u)du + \int_0^t \sigma(X_u)dB_u
\] (2)
- \((\xi, \Sigma), \quad \Sigma = \sigma\sigma'\)
- More generally \(X\) may also display jumps.
  - \(X = X^c + X^j\)
  - \(X^c\) as in (2)
  - \(X^j\) characterized by a finite conditional measure \(\eta(dx|X_{t-})\)
Building processes that grow or decay

- A real-valued process \( \{ M_t : t \geq 0 \} \) adapted, (with a version that is) right continuous with left limits. (A functional)

- The functional \( \{ M_t : t \geq 0 \} \) is multiplicative if \( M_0 = 1 \), and \( M_{t+u} = M_u(\theta_t)M_t \). Here \( \theta \) is the shift operator.

- Product of multiplicative processes is multiplicative.

- If \( M \) strictly positive \( \log(M) \) will satisfy an additive property.

- A functional is additive if \( A_0 = 0 \) and \( A_{t+u} = A_u(\theta_t) + A_t \), for each nonnegative \( t \) and \( u \).

- Parameterize \( \log(M) \)
Parameterization of multiplicative processes

- $(\beta, \gamma)$ that satisfies:
  
a) $\beta : D_0 \to \mathbb{R}$ and $\int_0^t \beta(X_u)du < \infty$ for every positive $t$;
  
b) $\gamma : D_0 \to \mathbb{R}^m$ and $\int_0^t |\gamma(X_u)|^2du < \infty$ for every positive $t$;

$$A_t = \int_0^t \beta(X_u)du + \int_0^t \gamma(X_u) \cdot dB_u$$

- Process $A$ may be non-stationary even if $X$ is stationary.
- $A_t = \psi(X_t) - \psi(X_0)$
- Exponential of additive processes (strictly positive multiplicative functionals).
  - Parameterized by the additive process $(\beta, \gamma)$
- When jumps are present add function of jumps
Growth in payouts

- $G$ a growth process: $G$ adapted, $G_0 = 1$ and $G_{t+u} = G_t G_u(\theta_t)$
- $M = SG$ is also multiplicative
  - $M_t \psi(x) = \mathbb{E} [D_0 M_t \psi(X_t)|X_0 = x]$, is the time-zero price of payoff $D_0 G\psi(X_t)$.
- Multiplicative $M \Rightarrow M_0 = \mathbb{I}$ and $M_{t+u} = M_t M_u$
Objective: Multiplicative decomposition

- Establish decomposition for multiplicative functionals:

\[ M_t = \exp(\rho t) \hat{M}_t \left[ \frac{\varphi(X_0)}{\varphi(X_t)} \right] \]

where

- \( \rho \) is a deterministic growth rate;
- \( \hat{M}_t \) is a multiplicative martingale;
- \( \varphi \) is a strictly positive function of the Markov state;

- If \( X \) is stationary, \( \frac{\varphi(X_0)}{\varphi(X_t)} \) stationary component, \( \hat{M} \) the martingale component of \( M \), and \( \rho \) its growth rate.

  - Not entirely correct because of possible correlation between stationary and martingale components.
Implications of multiplicative decomposition

• If $\hat{M}$ is a martingale for $F \in \mathcal{F}_t$

$$\hat{P}_r(F) = E[\hat{M}_t 1_F]$$

• $X$ remains Markovian.

• $E [M_t \psi(X_t) | X_0 = x] = \exp(\rho t) \phi(x) \hat{E} \left[ \frac{\psi(X_t)}{\phi(X_t)} | X_0 = x \right]$

• $\exp(-\rho t) \phi(X_t)$ as a numeraire. Applicable when the multiplicative process does not define a price.
• If there exists a stationary distribution $\hat{\varsigma}$ for $X$ under $\hat{P}r$, such that “stochastic stability” holds

$$
\lim_{t \to \infty} E_{\hat{P}r} [\psi(X_t) | X_0 = x] = \int \psi(y) d\hat{\varsigma},
$$

whenever $\int \psi(X_t) d\hat{\varsigma}$ is well defined, then

$$
\lim_{t \to \infty} e^{-\rho t} E[M_t \psi(X_t) | X_0 = x]
= \lim_{t \to \infty} E \left( \hat{M}_t \left[ \frac{\psi(X_t)}{\phi(X_t)} \right] | X_0 = x \right) \phi(x)
= \left( \int \frac{\psi(y)}{\phi(y)} d\hat{\varsigma} \right) \phi(x)
$$

- $\rho = \rho(M)$ is the (deterministic) growth rate
- All state dependence is given by the eigenfunction $\phi$
  * 1 factor model
- Approximation is “good” for $\psi$ with $\left| \int \frac{\psi(y)}{\phi(y)} d\hat{\varsigma} \right| < \infty$
\[
\frac{1}{t} \log E [M_t \psi(X_t) | X_0 = x] - \rho \approx \frac{1}{t} \left[ \log \phi(x) + \log \int \frac{\psi(y)}{\phi(y)} d\zeta \right]
\]

• Since \( \rho(M) \) is independent of \( \psi \), one may think of \( \psi \) as the transient contribution.

• If \( M = SG \), \( \rho(G) - \rho(GS) \) is long-run expected rate of return.

• Using \( G = 1 \) as risk-free reference, \( \rho(G) + \rho(S) - \rho(GS) \) is long-run expected excess rate of return.

• Mapping “risk in \( G \)” \( \leftrightarrow \rho(G) - \rho(SG) \) is a long-run risk-return frontier.

• Martingale cash flows.
Remainder of lecture

• Strategy to establish decomposition
  – Perron-Frobenius

• Applications

• In paper
  – Sufficient conditions stationarity, recurrence
  – Existence
  – Uniqueness
Establishing the decomposition

- Find positive solution to “eigenvalue problem”
  
  \[ E[M_t \phi(X_t) | X_0 = x] = \exp(\rho t) \phi(x) \]

- \( \exp(-\rho t) M_t \phi(X_t) \) is a martingale

- \( \hat{M}_t := \exp(-\rho t) M_t \frac{\phi(X_t)}{\phi(X_0)} \) is a martingale with \( M_0 = 1 \)

- \( M_t = \exp(\rho t) \hat{M}_t \frac{\phi(X_0)}{\phi(X_t)} \)
A local equation

- Find $\rho, \phi$ such that for each $x$

$$\lim_{t \searrow 0} \frac{E[M_t \phi(X_t) - \phi(x)|X_0 = x]}{t} = \rho \phi(x)$$

- Ito’s lemma allows you to compute $\lim_{t \searrow 0} \frac{E[M_t \phi(X_t) - \phi(x)|X_0 = x]}{t}$
  - $X_t$ real valued and $dX_t = \xi(X_t)dt + \sigma(X_t)dB_t$, $B$ a Brownian
  - $M_t = \exp(A_t), A_t = \int_0^t \beta(X_u)du + \int_0^t \gamma(X_u)dB_u$
  - For $\psi$ “sufficiently regular” $\lim_{t \searrow 0} \frac{E[M_t \psi(X_t) - \psi(x)|X_0 = x]}{t}$

$$= [\beta(x) + \frac{\gamma^2(x)}{2}]\psi(x) + [\xi(x) + \sigma(x)\gamma(x)]\psi'(x) + \frac{\sigma^2}{2}\psi''(x)$$
• Find positive $\phi$ such that:
\[
\rho \phi(x) = [\beta(x) + \frac{\gamma^2(x)}{2}] \phi(x) + [\xi(x) + \sigma(x)\gamma(x)] \phi'(x) + \frac{\sigma^2}{2} \phi''(x)
\]
• Multidimensional version also including jumps
• Non-uniqueness of solutions, but there is at most one pair $(\rho, \phi)$ such that $X$ stochastically stable after change in measure.
• $\rho$ is smallest among candidate eigenvalue
• Connection to numerical analysis literature.
Recipe for parameterized examples

• Use Ito’s to verify guess of eigenfunction.
• Compute eigenvalues.
• Typically multiple candidates, choose smallest.
• Write down candidate $\hat{M}$ and derive dynamics for $X$ under new probability (Girsanov’s theory)
• Verify stochastic stability
Application: 1. Long run risk-return frontier

\[ dX_t^f = \xi_f (\bar{x}_f - X_t^f) dt + \sqrt{X_t^f \sigma_f} dB_t^f, \]
\[ dX_t^o = \xi_o (\bar{x}_o - X_t^o) dt + \sigma_o dB_t^o \]

with \( \xi_i > 0, \bar{x}_i > 0 \) for \( i = f, o \) and \( 2\xi_f \bar{x}_f \geq \sigma_f^2 \) where 
\( B = (B^f, B^o) \) is a bivariate standard Brownian motion.

• Per-capita consumption

\[ dc_t = X_t^o dt + \sqrt{X_t^f \vartheta_f} dB_t^f + \vartheta_o dB_t^o \]

where \( c_t = \log(C_t) \)
• Choose a growth process for payouts $G = \exp(A^g)$ with

$$A^g_t = \int_0^t \sqrt{X_s^f f_t^g} dB^f_s + \int_0^t \gamma_s^g dB^o_s - \int_0^t \frac{X_s^f (\gamma_s^g)^2 + (\gamma_s^o)^2}{2} ds.$$ 

• $\gamma^g_f$ parameterizes $B^f$ risk of cash flow, $\gamma^g_o$ parameterizes $B^o$ risk.
• $M = GS$, $S$ as in Breeden-Lucas or Kreps-Porteous with $\rho = 1$
• Guess eigenfunction $\exp(c_f x_f + c_o x_o)$.
• Quadratic equation hence two candidates, the one associated with smallest eigenvalue $\rho(M)$ induces stochastic stability.
• (Long run) risk-return frontier: mapping

$$(\gamma^g_f, \gamma^g_o) \mapsto -\rho$$
• Short-run risk prices: Derivative of required instantaneous expected rate of return with respect to risk exposure.

• In Breeden-Lucas the short run risk-price to the cash flow exposure to the $B^o$ risk is: $av_o$
  - non-positive, unless consumption growth is positive correlated with the changes in the rate of growth of consumption.

• The long run risk prices are the derivatives of $\rho$ with respect to risk-exposures.

• In Breeden-Lucas the long run risk price of exposure to $B^o$ is:

  $$av_o + \frac{a}{\xi_o} \sigma_o$$

• Larger than short-run price. Difference between long-run and short-run prices is largest the slower the state variable $X^o$ mean reverts and the more this state variable is sensitive to $B^o$. 
• Short-run risk-price in Kreps-Porteus (with unitary elasticity of intertemporal substitution) is given by:

\[ a\vartheta_0 + (a - 1) \frac{\sigma_0}{b + \xi_0} > a\vartheta_0 \]

• Long-run risk-price in K-P:

\[ a\vartheta_0 + \frac{a}{b + \xi_0} \sigma_0 \]

• K-P has flatter term structure of risk prices than B-L.

• Cash flow risk exposure to \( B^f \) feeds through eigenfunction. Because volatility is state dependent, there is nonlinearity in the long-run price of the volatility risk.
Application 2: Decompositions of Stochastic Discount Factor

- \( S_t = \exp(\rho t) \hat{M}_t \frac{\phi(X_0)}{\phi(X_t)} \) (3)

- Alvarez and Jermann (2005) use a decomposition as (3) and prices of equity and risk-free long bonds to show that most of the volatility of the stochastic discount factor comes from the permanent component.
  - Suggest utility functions need to magnify importance of permanent component of consumption to fit data.
Application 3: Habit persistence models

• Bansall and Lehman show that for a variety of asset-pricing models the stochastic discount factor $S^*$ can be written as

$$S^* = S_t \frac{f(X_t)}{f(X_0)},$$

where $S$ is the stochastic discount factor of a Breeden-Lucas model

– Models with social externalities such as e.g. Campbell and Cochrane (1999) and Santos and Veronesi (2006)

• Martingale component same as Breeden-Lucas
Summary

• Use Markov formulation and martingale methods to produce factorizations of pricing operators.

• Multiplicative decomposition allows for
  – Characterization of long run risk-return tradeoff.
  – Distinguishing transient versus permanent changes in model ingredients